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Multiplicatively dependent vectors with coordinates algebraic numbers

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In memory of Professor Alan Baker

Abstract. We shall prove that close to each point in $\mathbb{C}^n$ with coordinates of comparable size there is a point $(t_1, \dots, t_n)$ with the property that no multiplicatively dependent vector $(u_1, \dots, u_n)$ with coordinates which are algebraic numbers of height at most H and degree at most d is very close to $(t_1, \dots, t_n)$.

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1. Introduction

Let n be an integer with $n \geq 2$ and R be a ring with identity. A vector $\mathbf{v} = (v_1 \dots, v_n)$ in R^n is said to be multiplicatively dependent if all its coordinates are non-zero and there is a non-zero integer vector $\mathbf{k} = (k_1, \dots, k_n)$ for which

$$\mathbf{v}^{\mathbf{k}} = v_1^{k_1} \cdots v_n^{k_n} = 1. \tag{1.1}$$

Let S be a subset of R. We denote by $\mathcal{M}_n(S)$ the set of multiplicatively dependent vectors with coordinates in S.

In [Pap18] Pappalardi, Sha, Shparlinski and Stewart give asymptotic formulae for the number of multiplicatively dependent vectors of algebraic numbers of fixed degree, or within a fixed number field, and bounded height. For any algebraic number α, let

$$f(x) = a_d x^d + \cdots + a_1 x + a_0$$

be the minimal polynomial of α, so with content 1 and positive leading coefficient. Suppose that f factors as

$$f(x) = a_d(x - \alpha_1) \cdots (x - \alpha_d)$$

over the complex numbers. The height $H(\alpha)$ of α is given by

$$H(\alpha) = \left(a_d \prod_{i=1}^{d} \max(1, |\alpha_i|) \right)^{1/d}.$$

For positive integers n, d and H with $n \geq 2$ we denote by $M^*_{n,d}(H)$ the number of multiplicatively dependent n-tuples whose coordinates are algebraic numbers of degree d and height at most H. Pappalardi, Sha, Shparlinski and Stewart proved that

$$M^*_{n,d}(H) = C(n,d) H^{d(d+1)(n-1)} + O(H^{d(d+1)(n-1)-d/2} \log H) \tag{1.2}$$

where

$$C(n,d) = (nw_0(d) + 2n(n-1)) C_1(d)^{n-1},$$

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$w_o(d)$ is the number of roots of unity of degree d,

$$C_1(d)=\frac{d2^d}{\zeta(d+1)}\prod_{j=1}^{\lfloor (d-1)/2\rfloor}\frac{(d+1)(2j)^{d-2j}}{(2j+1)^{d-2j+1}}$$

and $\zeta(s)$ is the Riemann zeta function.

Sha, Shparlinski and Stewart [Sha] studied the distribution of elements of $\mathcal{M}_n(S)$ in $\mathbb{R}^n$ and in $\mathbb{C}^n$ when S is a number field or the ring of algebraic integers of a number field. They showed that $\mathcal{M}_n(\mathbb{Q})$ is dense in $\mathbb{R}^n$ and if K is a number field which is not contained in $\mathbb{R}$ then $\mathcal{M}_n(K)$ is dense in $\mathbb{C}^n$. Further if K is contained in $\mathbb{R}$ and of degree at least 2 over the rationals then $\mathcal{M}_n(\mathcal{O}_K)$ is dense in $\mathbb{R}^n$ where $\mathcal{O}_K$ denotes the ring of algebraic integers of K. Furthermore if K is contained in $\mathbb{C}$ but not in $\mathbb{R}$ and of degree at least 3 over the rationals then $\mathcal{M}_n(\mathcal{O}_K)$ is dense in $\mathbb{C}^n$. In addition they showed that there are significant irregularities in the distribution of the points of $\mathcal{M}_n(\mathbb{Z})$ in $\mathbb{R}^n$ and the distribution of the points of $\mathcal{M}_n(\mathcal{O}_K)$ in $\mathbb{C}^n$ when K is an imaginary quadratic field. For $\mathbf{z}=(z_1,\dots,z_n)$ in $\mathbb{C}^n$ put

$$\|\mathbf{z}\|=\sqrt{|z_1|^2+\cdots+|z_n|^2}$$

and for a real number H greater than 1 put

$$\mu_n(H;\mathcal{O}_K)=\sup_{\substack{\mathbf{x}\in\mathbb{C}^n\\ \|\mathbf{x}\|\le H}}\ \inf_{\mathbf{v}\in\mathcal{M}_n(\mathcal{O}_K)}\|\mathbf{x}-\mathbf{v}\|$$

when K is an imaginary quadratic field. Sha, Shparlinski and Stewart proved there is a positive number $C_0(n)$, which is effectively computable in terms of n, such that for $n\ge 3$

$$\mu_n(H,\mathcal{O}_K)\gg H/(\log H)^{C_0(n)} \tag{1.3}$$

and

$$\mu_2(H,\mathcal{O}_K)\gg H.$$

Furthermore, for $n\ge 4$, there exists a positive number $C_1(K)$, which is effectively computable in terms of K, such that

$$\mu_n(H;\mathcal{O}_K)\ll H/(\log H)^{C_1(K)}.$$

The purpose of this note is to show, extending estimate (1.3), that the multiplicatively dependent vectors in $\mathbb{C}^n$ with coordinates which are algebraic numbers of degree at most d and height at most H are distributed in a very irregular manner. We shall prove that throughout $\mathbb{C}^n$ there are many large polydiscs which have no elements of $\mathcal{M}_n(\mathbb{C})$ whose coordinates are algebraic numbers of degree at most d and height at most H. In particular we shall show that close to every point of $\mathbb{C}^n$, with $n\ge 2$, with coordinates which are not too small there is a point which does not have any very close points from $\mathcal{M}_n(\mathbb{C})$ whose coordinates are algebraic numbers of degree at most d and height at most H. We shall prove the following result.

Theorem 1.1. *Let n and d be positive integers with $n\ge 2$. Let ε and H be real numbers with $0<\varepsilon<1$ and $H\ge 3$. There exist positive numbers c, c_0 and C, which are effectively computable in terms of n, of n and ε, and of n, d and ε respectively, such that if H exceeds C and $\mathbf{z}=(z_1,\dots,z_n)$ in $\mathbb{C}^n$ satisfies*

$$e^{(\log H)^{\varepsilon}}<|z_j|\le H \tag{1.4}$$

for $j=1,\dots,n$ then there exists $\mathbf{t}=(t_1,\dots,t_n)$ in $\mathbb{C}^n$ with

$$|z_j-t_j|<|z_j|/(\log|z_j|)^{c} \tag{1.5}$$

for $j = 1, \ldots, n$ and such that there is no element $\mathbf{u} = (u_1, \ldots, u_n)$ in $\mathcal{M}_n(\mathbb{C})$ with coordinates which are algebraic numbers of height at most H and degree at most d for which

$$|u_j - t_j| < |t_j|/(\log |t_j|)^{c_0} \tag{1.6}$$

for $j = 1, \ldots, n$.

For the proof of Theorem 1.1 we shall require a result of Loxton and van der Poorten [Lox83, van77] which states that if $\alpha_1, \ldots, \alpha_n$ are algebraic numbers which are multiplicatively dependent then we can find a dependence relation where the exponents are not too large. In addition we need two results of Stewart [Ste18] which generalize earlier work of Tijdeman [Tij73, Tij74] on the sequence of integers generated by a finite set of primes. The results [Ste18, Tij73, Tij74] depend on the fundamental work of Baker on estimates for linear forms in the logarithms of algebraic numbers.

2. Preliminary lemmas

We first state the result of Loxton and van der Poorten [Lox83, van77].

Lemma 2.1. *Let $n \geq 2$ and let $\alpha_1, \ldots, \alpha_n$ be multiplicatively dependent non-zero algebraic numbers of degree at most d which are not roots of unity. Then there is a positive number c, which depends only on n and d, and there are rational integers $k_1, \ldots, k_n$, not all zero, such that*

$$\alpha_1^{k_1} \cdots \alpha_n^{k_n} = 1$$

and

$$|k_j| \leq c \prod_{m=1, m \neq j}^{n} \log H(\alpha_j), \quad j = 1, \ldots, n.$$

Next we shall need Theorem 1 of [Ste18].

Lemma 2.2. *Let $\alpha_1, \ldots, \alpha_r$ be multiplicatively independent algebraic numbers with $|\alpha_i| > 1$ for $i = 1, \ldots, r$. Put*

$$T = \{\alpha_1^{h_1} \cdots \alpha_r^{h_r} \mid h_i \geq 0 \text{ for } i = 1, \ldots, r\}.$$

There exists a positive number c, which is effectively computable in terms of $\alpha_1, \ldots, \alpha_r$, such that if t and t' are in T with $|t| \geq 3$ then

$$|t - t'| > |t|/(\log |t|)^c.$$

We shall also require Theorem 3 of [Ste18].

Lemma 2.3. *Let α_1, α_2 and α_3 be multiplicatively independent algebraic numbers with $|\alpha_i| > 1$ for $i = 1, 2, 3$. Suppose that α_1 and α_2 are positive real numbers and that $\alpha_3/|\alpha_3|$ is not a root of unity. Put*

$$T = \{\alpha_1^{h_1}\alpha_2^{h_2}\alpha_3^{h_3} \mid h_i \geq 0 \text{ for } i = 1, 2, 3\}.$$

There exists a positive number c_1, which is effectively computable in terms of α_1, α_2 and α_3, such that for any complex number z with $|z| \geq 3$ there exists an element t of T with

$$|z - t| \leq |z|/(\log |z|)^{c_1}.$$

3. Proof of Theorem 1.1

Let $r_1, r_2, \dots,$ be the increasing sequence of primes congruent to 1 modulo 4. Let

$$r_j = a_j^2 + b_j^2$$

with a_j and b_j positive integers for $j = 1, 2, \dots$ and put

$$\pi_j = a_j + ib_j$$

for $j = 1, 2, \dots$. Notice that

$$|\pi_j| > 1 \tag{3.7}$$

and that $\pi_j/|\pi_j|$ is not a root of unity since the only rational prime which ramifies in $\mathbb{Q}(i)$ is 2. Let $q_1, q_2, \dots$ be the increasing sequence of primes congruent to 3 modulo 4. Let H and ε be real numbers with $0 < \varepsilon < 1$ and $H \geq 3$ and suppose that $\mathbf{z} = (z_1, \dots, z_n)$ in $\mathbb{C}^n$ satisfies (1.4). Assume that C is sufficiently large that $(\log C)^\varepsilon \geq \log 3$ and that H exceeds C. Then, by Lemma 2.3, for each integer j with $1 \leq j \leq n$ there exists a positive number c_j, which is effectively computable in terms of q_{2j-1}, q_{2j} and π_j, and non-negative integers $\lambda_{j,1}, \lambda_{j,2}, \lambda_{j,3}$ such that

$$\left| z_j - q_{2j-1}^{\lambda_{j,1}} q_{2j}^{\lambda_{j,2}} \pi_j^{\lambda_{j,3}} \right| < \frac{|z_j|}{(\log |z_j|)^{c_j}}. \tag{3.8}$$

Put

$$t_j = q_{2j-1}^{\lambda_{j,1}} q_{2j}^{\lambda_{j,2}} \pi_j^{\lambda_{j,3}} \tag{3.9}$$

for $j = 1, \dots, n$ and

$$\mathbf{t} = (t_1, \dots, t_n).$$

Then, by (3.8),

$$|z_j - t_j| < |z_j|/(\log |z_j|)^c \tag{3.10}$$

where

$$c = \min_{1 \leq j \leq n} c_j.$$

Observe that c is effectively computable in terms of n and so (1.5) holds.

Let $\mathbf{u} = (u_1, \dots, u_n)$ be an element of $\mathcal{M}_n(\mathbb{C})$ with coordinates algebraic numbers of height at most H and degree at most d. We shall prove that $\mathbf{u}$ is not close to $\mathbf{t}$. Further we note that $\mathbf{t}$ is not in $\mathcal{M}_n(\mathbb{C})$.

Let $C_1, C_2, \dots$ be positive numbers which are effectively computable in terms of n, d and ε. By Lemma 2.1 there exist integers $k_1, \dots, k_n$, not all 0, with

$$|k_j| \leq C_1 (\log H)^{n-1} \tag{3.11}$$

for $j = 1, \dots, n$ for which

$$u_1^{k_1} \cdots u_n^{k_n} = 1. \tag{3.12}$$

We may assume, without loss of generality, that $k_1, \dots, k_i$ are non-negative and that $k_{i+1}, \dots, k_n$ are negative. Put

$$t = t_1^{k_1} \cdots t_i^{k_i} \quad \text{and} \quad t' = t_{i+1}^{-k_{i+1}} \cdots t_n^{-k_n}. \tag{3.13}$$

Notice that $q_1, \dots, q_{2n}$, $\pi_1, \dots, \pi_n$ are multiplicatively independent and that (3.7) holds. Further by (1.4) and (3.10), $|t| \geq 3$ provided that H exceeds C_2. Therefore, by Lemma 2.2, there is a positive number c', which is effectively computable in terms of n, such that

$$|t - t'| > |t|/(\log |t|)^{c'}. \tag{3.14}$$

Plainly we may suppose that

$$c' \geq 1. \tag{3.15}$$

We may assume that

$$|u_j - t_j| < |t_j|/(\log|t_j|)^{c'9(n-1)\varepsilon^{-1}} \tag{3.16}$$

for $j = 1, \ldots, n$ since if there is no $\mathbf{u}$ in $\mathcal{M}_n(\mathbb{C})$ with coordinates algebraic numbers of height at most H and degree at most d for which (3.16) holds then the result follows. Define θ_j by

$$u_j = t_j(1+\theta_j) \tag{3.17}$$

for $j = 1, \ldots, n$. Then, by (3.16),

$$|\theta_j| \leq \frac{1}{(\log|t_j|)^{c'9(n-1)\varepsilon^{-1}}} \tag{3.18}$$

for $j = 1, \ldots, n$. Now

$$|z_j - t_j| < |z_j|/(\log|z_j|)^c$$

for $j = 1, \ldots, n$ and so for $H > C_3$, by (1.4),

$$|t_j| \geq |z_j|/2 \geq |z_j|^{1/2} \tag{3.19}$$

for $j = 1, \ldots, n$. By (1.4), (3.18) and (3.19)

$$|\theta_j| \leq \left(\frac{2}{(\log H)^{\varepsilon}}\right)^{c'9(n-1)\varepsilon^{-1}}$$

hence, for $H > C_4$,

$$|\theta_j| \leq \frac{1}{(\log H)^{c'8(n-1)}} \tag{3.20}$$

for $j = 1, \ldots, n$. By (3.17) and (3.12)

$$1 = t_1^{k_1} \cdots t_n^{k_n} \prod_{j=1}^{n} (1+\theta_j)^{k_j}$$

and so

$$t_i^{-k_{i+1}} \cdots t_n^{-k_n} = t_1^{k_1} \cdots t_i^{k_i} \prod_{j=1}^{n} (1+\theta_j)^{k_j}. \tag{3.21}$$

By (3.11) and (3.20)

$$\left(1 - \frac{1}{(\log H)^{8c'(n-1)}}\right)^{nC_1(\log H)^{n-1}} \leq \left|\prod_{j=1}^{n}(1+\theta_j)^{k_j}\right| \leq \left(1 + \frac{1}{(\log H)^{7c'(n-1)}}\right)^{nC_1(\log H)^{n-1}}$$

and so, for $H > C_5$,

$$1 - \frac{1}{(\log H)^{5c'(n-1)}} \leq \left|\prod_{j=1}^{n}(1+\theta_j)^{k_j}\right| \leq 1 + \frac{1}{(\log H)^{5c'(n-1)}}. \tag{3.22}$$

By (3.13), (3.21) and (3.22)

$$|t - t'| \leq \frac{|t|}{(\log H)^{5c'(n-1)}}. \tag{3.23}$$

Notice that by (1.4), (3.11) and (3.10)

$$|t| \le (2H)^{nC_1(\log H)^{n-1}}$$

so

$$\log |t| \le C_6(\log H)^n \le C_6(\log H)^{2(n-1)}.$$

Thus, for $H > C_7$,

$$\log |t| \le (\log H)^{(5/2)(n-1)}. \tag{3.24}$$

It follows from (3.23) and (3.24) that

$$|t - t'| \le |t|/(\log |t|)^{2c'}. \tag{3.25}$$

Now from (3.14) and (3.25) we see that

$$2c' < c'$$

and this is false. Our result now follows.

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