Newton↔Kepler 'Logic',

and an enlightening loiter with Emmy Noether.

Peter Hoffman

Having been recently reading history (of science and other stuff), a very short rant cannot be resisted. Apologies for both it and the excessively long sentence just below. Not much more of that will occur further down!

Johannes Kepler, holding the unquestioned christian convictions ubiquitous in the central Europe of the Thirty Years War (about 400 years ago, when protestants and roman catholics were quite prepared to settle religious differences by decapitating each other, and even to occasionally cut the victim's tongue out before the execution—which makes the islamofascist thugs from modern Baghdad and Karachi almost look good by comparison—though one may fear humankind to be on the path towards the Thirty *Minutes* War, in which Tehran and Tel Aviv get incinerated), but at the same time himself admirably tolerant to religious differences and with a surprisingly modern attitude towards mathematics and science, discovered the following three laws for the motions of planets around the sun:

Kep I: Orbits are ellipses.

Kep II: For a given planet, the variable-length line interval joining it to the sun sweeps out area at a constant rate.

Kep III: The ratio, of the square of the time to complete an orbit (e.g. the "Martian year"), to the cube of the diameter of the orbit, is the same for all planets (and all asteroids, for that matter).

We'll see below that these laws are somewhat refinable, both by Kepler himself: e.g. the sun is at one focus of the ellipse, and by Newton later: e.g. no, the focus is better placed at the centre-of-mass of the sun and planet together, if we ignore the effects of all the other planets and heavenly bodies. And also Newton: Kepler's ratio from **Kep III** is closely related to a constant easily determinable by down-to-earth experiments and to the mass of the sun. So the latter has then been determined quite accurately (once the distance from the sun to at least one planet is known), fortunately contradicting neither the bible nor the koran, which have nothing to say on the matter. The mass of the earth can similarly (but not ideally!) be determined, using its distance to the moon and the length of the "lunar month" or "sidereal month".

Kepler's stunning successes were empirical rules, teased out with huge energy, ingenuity and persistence, from a substantial body of accurate observations by the Prague-based Dane, Tycho Brahe. They depended on Kepler's adoption of the still very controversial system from half a century earlier of the Polish cleric Copernicus. The latter took the sun to be the immobile centre of the universe (rather than the earth), with the planets, including Earth, travelling around the sun. But Brahe's data made it clear to the open-minded (those who were not still in the hypnotic trance spun by Aristotle and Thomas Aquinas) that these orbits are not perfect circles.

Half a century later, Isaac Newton, strongly influenced also by Kepler's Tuscan contemporary Galileo (who spent his latter years under papal house arrest for advocating Copernicanism), put forward basic laws of motion, plus an inverse-square law of gravitational force, and, from them, logically deduced (the refined versions of) Kepler's laws. Actually Newton regarded the details of these deductions at first to be too easy to bother including in his "Principia ···", though he did include details of a converse, which might be summarized as 'Kepler $\Longrightarrow$ Newton's inverse square law'. Thus calculus had some rather amazing early applications to scientific progress (even though Newton pretended not to know calculus in the Principia).

As usual, history is not as simple as above, with a few heroes doing everything important. In particular, the universal credit given to Newton needs rather to be spread around somewhat. The names Huygens and Hooke should also appear in this story. See [**Arnold2**] for many interesting remarks.

The historical facts above are indisputably the basis for our present scientific age. One of our times' striking (though not surprising) discoveries is of the existence of planets which orbit other stars (than the sun). It is easy to describe how these planets (which have never been 'seen' using any kind of telescope and perhaps never will be) are discovered :

If Newton's theory were perfectly accurate (or Einstein's, which so far it has been, to 11 significant digits in at least one aspect), and if the earth and sun were the only heavenly bodies, then Newton's refinement mentioned above would also have the sun itself 'wobbling' around an ellipse which is a few hundred miles in diameter, an ellipse which has the same (common centre-of-mass) focus mentioned above for the earth's orbit. (That common centre-of-mass is very close to the actual centre of the sun itself, relative to the size of the sun. So the motion of the sun looks much more like wobbling than orbiting!) This phenomenon is exactly what is used to detect new 'solar' systems : a large enough planet in a tight orbit around a 'visible' star can be detected by observing the wobbling of the star, the latter observation via slight frequency changes (so-called Doppler effect).

Inspired by Bob Osserman's Math Monthly article **[Osser]** explaining how the more refined deduction of Newton is no harder to derive and describe than the usual one with the sun 'nailed down', and not finding a suitable reference for good (but not necessarily sophisticated) math students on the actual deduction of Kepler's laws from the differential equation of Newton, I decided to do this write-up, which includes both. There is nothing new below except some very tiny notational details, and possibly for putting this all together in a form suitable for lecturing to, or reading by, rigorous-minded math students in their early undergraduate years. It's hard to imagine a person holding a mathematics degree who has never seen some (at least rudimentary) analysis of the motion of the planet on which he/she happens to be a passenger!

It then proved impossible to resist the temptation to add some pontification about the usefulness of 'Lie theoretic' considerations in physics, since the two basic constants of motion which more-or-less give **Kep I** and **Kep II** (essentially conservation of energy and of angular momentum respectively) are seen here easily to arise from the invariance of the DE under time translation and under space rotation, respectively. But the relevant theorem, due to Emmy Noether, has many different statements, of various levels of vagueness, and not all that clearly related to each other. So in the end, we got an entire second part to the write-up (twice the length of what precedes it), including introductions to lagrangian and hamiltonian mechanics, a number of versions of Noether's theorem, and finally the discussion of the so-called "hidden symmetry" in Kepler's problem. The general theory is presented entirely in a language which may hopefully appeal to pure mathematics undergraduates who like to see definitions, theorems and proofs, as opposed to formulas, rules, and manipulations of physically motivated symbols. One hopes both to stimulate their interest in the intellectual content of all this, and to increase their willingness to learn that latter very efficient classical language of theoretical physicists. In any case it is all written up in largely coordinate-free language which facilitates the passage to the general case using symplectic manifolds, but at a level requiring no need for knowledge of manifolds, bundles, forms, etc.

1. Some general vector calculus notation and formulae.

Consider a curve $t \mapsto \vec{w}(t)$ in either $\mathbf{R^3}$ or $\mathbf{R^2}$. In the next section, we shall easily show how to reduce from $\mathbf{R^3}$ to $\mathbf{R^2}$ for the curves which arise in the context here. Physically we are thinking of $\vec{w}(t)$ as the position of the orbiting point ("test particle", planet, etc.) in 3-space. The 2-space reduced to contains both

(i) the line joining that point to the origin (which, at our two refinement levels, would correspond to either the sun or to the earlier common centre-of-mass); and

(ii) the velocity vector of the orbiting point;

both specified at some initial time. Assume that t ranges over some open interval which contains 0 (perhaps over all of $\mathbf{R}$), and, to avoid irrelevant fuss, assume that $\vec{w}$ is C^∞ there (i.e. has all derivatives). Define

$$\vec{v} \ = \ \vec{v}(t) \ := \ \frac{d\vec{w}}{dt} \ = \ \dot{\vec{w}} \qquad\qquad \text{(velocity)} \ ,$$

and

$$\vec{a} \ = \ \vec{a}(t) \ := \ \frac{d\vec{v}}{dt} \ = \ \dot{\vec{v}} \ = \ \frac{d^2\vec{w}}{dt^2} \ = \ \ddot{\vec{w}} \qquad \text{(acceleration)} \ .$$

When dealing with $\mathbf{R^2}$, (and assuming $\vec{w}(0) \neq \vec{0}$), on the largest open interval containing 0 of those t for which $\vec{w}(t) \neq 0$, unique real-valued C^∞-functions $r(t)$ and $\theta(t)$ are defined as usual by polar coordinate formulae:

$$\vec{w} \ = \ (r\cos\theta, r\sin\theta) \ \ ; \ \ r(0) > 0 \ \ ; \ \ -\pi < \theta(0) \leq \pi \ \ .$$

The same can be done on any other component open interval of $\vec{w}^{-1}(\mathbf{R}^2 \setminus \{\vec{0}\})$. All important formulae involve either $\dot{\theta}$, or trig functions applied to θ, so the norming conditions (e.g. $-\pi < \theta(0) \leq \pi$) play no real role.

With $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$, define, for the same open set of t, a pair of vector-valued C^∞-functions by

$$\vec{\rho} \ = \ \vec{\rho}(t) \ := \ +\cos\theta \ \vec{i} \ + \ \sin\theta \ \vec{j} \ \ ,$$

$$\vec{\varphi} \ = \ \vec{\varphi}(t) \ := \ -\sin\theta \ \vec{i} \ + \ \cos\theta \ \vec{j} \ \ .$$

So

$$\vec{w} \ = \ r\vec{\rho} \ \ ; \ \ \dot{\vec{\rho}} \ = \ \dot{\theta}\vec{\varphi} \ \ ; \ \ \dot{\vec{\varphi}} \ = \ -\dot{\theta}\vec{\rho} \qquad (1)$$

The fixed $\{\vec{i}, \vec{j}\}$, and, for every t, the set $\{\vec{\rho}(t), \vec{\varphi}(t)\}$, are orthonormal pairs, with respect to the standard inner product, which will be denoted $< \vec{\ }, \vec{\ } >$.

Note that

$$\vec{v} \ = \ \dot{\vec{w}} \ = \ \frac{d(r\vec{\rho})}{dt} \ = \ \dot{r}\vec{\rho} \ + \ r\dot{\vec{\rho}} \ = \ \dot{r}\vec{\rho} \ + \ r\dot{\theta}\vec{\varphi} \qquad (2)$$

So

$$\vec{w} \ \times \vec{v} \ = \ r\vec{\rho} \ \times (\dot{r}\vec{\rho} \ + \ r\dot{\theta}\vec{\varphi}) \ = \ r^2\dot{\theta} \ \vec{\rho} \ \times \vec{\varphi} \qquad (3)$$

on each open t-interval of $\vec{w}^{-1}(\mathbf{R}^2 \setminus \{\vec{0}\})$, since $\vec{\rho} \times \vec{\rho} \ = \ \vec{0}$. (If you need to brush up on the vector cross-product, there is an appendix on that in the last part of this paper.) The latter vector $\vec{\rho} \ \times \vec{\varphi}$ is a fixed unit vector poking out along the "z-axis", despite us working essentially in 2-space, so we still have $\mathbf{R}^3$ in the back of our minds to accept our vector cross-products.

The area swept by the interval from the origin to $\vec{w}$ over a closed time interval $[t_0, t]$ inside one of these open t-intervals is

$$\int_{\theta(t_0)}^{\theta(t)} \frac{1}{2} r^2 \ d\theta \quad = \quad \int_{t_0}^{t} \frac{1}{2} r^2 \dot{\theta} \ dt \quad .$$

So the time rate of change of this area at t is $\frac{1}{2}r^2\dot{\theta}$ (4)

2. Pedestrian Reduction from 3-space to 2-space.

The result here has an alternative proof in the next section.

Suppose initially working in 3-space, and let $\vec{p}$ be a (constant) non-zero vector with inner products $< \vec{p} \ , \ \vec{w}(0) \ >= 0 =< \vec{p} \ , \ \vec{v}(0) \ >$. If we assume that $\vec{w}$ satisfies a differential equation ('DE' from now on) of the form, for some C^∞ function H,

$$H(\vec{w}, t)\ddot{\vec{w}}(t) \ = \ \vec{w}(t) \qquad (*) \quad ,$$

it is easily proved below that $< \vec{p} \ , \ \vec{w}(t) \ >= 0$ for all t. Then we just take our 2-space to be the orthogonal complement, i.e. $\vec{p}^{\ \perp}$.

For the proof, let $f(t) \ := \ < \vec{p} \ , \ \vec{w}(t) \ >$, so that $f(0) = 0$.

Then $f'(t) \; = \; < \vec{p} \; , \; \vec{v}(t) \; >$, so that $f'(0) = 0$.
Also

$$f''(t) \;=\; < \vec{p}\;,\; \ddot{\vec{w}}(t) \;> \;=\; H(\vec{w},t)^{-1} < \vec{p}\;,\; \vec{w}(t) \;> \;=\; H(\vec{w},t)^{-1} f(t) \;.$$

Clearly $f(t) = 0$ for all t is the unique solution of this last DE with the initial conditions on the two lines just above, as required. In the last display, dividing by H is no problem because the right-hand side of the original DE, and therefore H, are both non-zero on the interval of t at issue.

3. The Mathematics of Kep II.

Starting now, we work essentially in 2-space, leaving a particular choice of coordinates for later. From now on, assume that $\vec{w}$ satisfies $(*)$, the DE in the last section, with H becoming more specialized a few sections later, to bring in the inverse square law. We are working over an open interval of t as indicated earlier.

Theorem 1. (Kep II) *The rate of change,* $\frac{1}{2}r^2\dot{\theta}$ *, of area* (from (4) at the end of the 1st section) *is constant over any such interval.*

Proof. By (3) of the 1st section, it suffices to prove that $\vec{w} \times \vec{v}$ remains constant. But, using an easily proved 'cross-product rule',

$$\frac{d(\vec{w} \times \vec{v})}{dt} \;=\; \dot{\vec{w}} \times \vec{v} + \vec{w} \times \dot{\vec{v}} \;=\; \vec{v} \times \vec{v} + \vec{w} \times \ddot{\vec{w}} \;=\; \vec{v} \times \vec{v} + H(\vec{w},t)^{-1} \vec{w} \times \vec{w} \;=\; \vec{0} \;,$$

since $\vec{z} \times \vec{z} \;=\; \vec{0}$ for any vector $\vec{z}$.

Again, the appearance of H in a denominator is fine, since our interval of t is chosen to avoid a 'collision with the origin' by our 'particle' $\vec{w}$.

Remark on the obsolescence of Section 2. Since $\vec{w} \times \vec{v}$ is a vector perpendicular to any plane containing $\vec{w}$ and $\vec{v}$, and that vector remains fixed as time progresses, so does such a plane, as proved alternatively in the last section. (By "plane" we mean 2-dimensional vector subspace. Unless $\vec{v}$ and $\vec{w}$ happen to be linearly dependent, there is only one such plane above.)

Cor. 2 and Definition of L . *The number* $r^2\dot{\theta} \;=:\; L$ *is constant on any such interval.*

Theorem 3. *If* $[t_1, t_2] \subset \vec{w}^{-1}(\mathbf{R}^2 \setminus \{\vec{0}\})$ *and* $\vec{w}(t_1) = \vec{w}(t_2)$, *with* $\vec{w}\,|_{]t_1,t_2[}$ *not taking that value and being injective* (so, over the interval $[t_1, t_2]$, the 'particle' $\vec{w}$ traverses once a simple closed orbit avoiding the origin), *then, with* $T_{\rm cl} := t_2 - t_1$ *and* $A_{\rm cl}$ *being the area inside the closed orbit,*

$$A_{\rm cl} \;=\; \frac{L}{2}\, T_{\rm cl} \quad .$$

Proof.

$$A_{\rm cl} \;=\; \int_{t_1}^{t_2} \frac{1}{2} r^2 \dot{\theta}\; dt \;=\; \int_{t_1}^{t_2} \frac{L}{2}\; dt \;=\; \frac{L}{2}\, T_{\rm cl} \; .$$

Remark. We are not assuming $\dot{\vec{w}}(t_1) = \dot{\vec{w}}(t_2)$ above. So such a closed orbit would not necessarily be repeated. One of the most interesting facts about specializing to an inverse square force law, as we are about to do, is that repetition of the elliptical orbit actually happens. That could be called Kepler's 0th law, though its discovery, in the case of the earth's orbit at least, presumably predates Kepler by many thousands of years!

4. Some Physics related to Kep II.

In actual Newtonian gravitational mechanics of the less refined case where a centre of attraction (e.g. the sun) is fixed at the origin, we have

$$H(\vec{w}, t) \;=\; -\,\frac{|\vec{w}|^3}{K} \;=\; -\,\frac{r^3}{K} \; ,$$

for a positive constant K which is the product of Newton's universal gravitational constant and the mass of the central attracting object (i.e. $K = GM$, to recall a common notation in physics texts).

To see this, Newton's law,

$$\overrightarrow{\rm force} \;=\; m\,\vec{a} \quad ,$$

where m is the mass of the object whose orbit is studied, becomes (with the two sides switched)

$$m\,\ddot{\vec{w}} \;=\; -\,\frac{GMm}{|\vec{w}|^2}\,\vec{\rho} \quad .$$

Here the right-hand side is the famous inverse square law. Cancelling m on both sides (on the left, it's "inertial" mass and on the right it's "gravitational" mass—that these agree is a rather profound physical law), replacing GM by K and $\vec{\rho}$ by $\frac{\vec{w}}{|\vec{w}|}$ then yields the original DE denoted $(*)$, namely

$$\ddot{\vec{w}}(t) \;=\; H(\vec{w},t)^{-1}\vec{w}(t) \;,$$

with H as in the first display of this section.

The vector $\vec{w} \;\times \vec{v} \;=\; r^2\dot{\theta}\;\vec{\rho} \;\times \vec{\varphi}$ (see (3) of the 1st section) is angular momentum divided by m. Its length, which is $r^2\dot{\theta} \;=\; L$ (up to sign), is the significant aspect. And we are talking about angular momentum of 'rotation around the origin'—not some other point. The fact that L is conserved is very natural here: the DE says the force on the particle (planet) always acts radially, so conservation seems physically obvious. Note that this holds for any H; it is independent of specializing H to $\frac{|\vec{w}|^3}{K}$.

So we see that **Kep II** is virtually equivalent to the fact that angular momentum of a planet about the sun is conserved, because the sun 'pulls directly' on the planet, and imparts no 'rotational impetus' to it. Furthermore **Kep II** is logically independent of the inverse square dropping off of gravitational attraction with distance from the sun. It depends only on the force being an attraction (or repulsion, which of course it isn't) directly towards (resp. away from) the central attracting (resp. repulsing) object. (A pair of protons would be a repulsive example, if quantum 2-body treatment weren't needed.)

5. The Mathematics of Kep I and Kep III .

Theorem 4. (Kep I) *Assume that*

(i) *the inverse square form of the DE holds, namely* $-r^2\ddot{\vec{w}}/K = \vec{\rho}$, *usually written* $\vec{a} = -\frac{K}{r^2}\;\vec{\rho}$, *for some number* $K > 0$ *;*

(ii) (much more than necessary—see the following Appendix) *there is a point on the orbit where* $\vec{w}$ *and* $\vec{v}$ *are non-zero and perpendicular to each other.*

Then, for a suitable choice of coordinates in 2-space, the orbit is given geometrically (i.e. the path, but not the traversing of it as a function of time) by

$$r(1 + e\;\cos\theta) \;=\; D \quad ,$$

for some number $e \geq 0$ *and with* D *being the positive number* L^2/K .

Below we shall rewrite the conclusion equation in more familiar Cartesian form as a conic. The number e will be the eccentricity of the ellipse in the case of a closed orbit, and will be analyzed physically in Section 7, relating it to energy.

Proof.

$$\frac{L}{K}\,\dot{\vec{v}} \;=\; (\frac{L}{K})\,(-\frac{K}{r^2}\,\vec{\rho}) \;=\; (r^2\dot{\theta})\,(-\frac{1}{r^2}\,\vec{\rho}) \;=\; -\,\dot{\theta}\,\vec{\rho} \;=\; \dot{\vec{\varphi}} \quad .$$

The equalities from left to right come from: the DE, the definition of L, simple-mindedness, and (1) in the 1st section.

The far ends of the above display give a simple 'DE', which obviously solves as

$$\frac{L}{K}\,\vec{v} \;=\; \vec{\varphi} \;+\; \vec{C} \quad ,$$

for some "arbitrary" constant (vector) $\vec{C}$.

Reset the clock so that condition (ii) holds at time zero. Now use condition (ii) to rechoose coordinates in $\mathbf{R}^2$ so that $\vec{w}(0)$ and $\vec{v}(0)$ are positive scalar multiples of $\vec{i}$ and $\vec{j}$, respectively. Expressed otherwise:

$$\theta(0)=0 \;\;;\;\; \dot{\theta}(0)>0,\ \text{so}\ L>0 \;\;;\;\; \vec{\rho}(0)=\vec{i} \;\;;\;\; \vec{\varphi}(0)=\vec{j} \quad .$$

Evaluating both sides of the displayed solution at 0, and since the left side is a positive multiple of $\vec{j}$, we see that $\vec{C}=e\,\vec{j}$ for some $e>-1$. The proof is completed by the following equalities, whose justifications, respectively, are the definition of $\vec{\varphi}$, linearity, the above displayed solution, (2) of the 1st section, orthonormality, and the definition of L :

$$r(1+e\,\cos\theta) \;=\; r(<\vec{\varphi},\vec{\varphi}> \;+\; e<\vec{\varphi},\vec{j}>) \;=\; r<\vec{\varphi}\,,\,\vec{\varphi}+e\vec{j}> \;=$$

$$r<\vec{\varphi}\,,\,\frac{L}{K}\vec{v}> \;=\; \frac{L}{K}r<\vec{\varphi}\,,\,\dot{r}\vec{\rho}\;+\;r\dot{\theta}\vec{\varphi}> \;=\; \frac{L}{K}r^2\dot{\theta} \;=\; \frac{L^2}{K} \quad ,$$

as required—except to note that the cases when $-1<e<0$ can be replaced by changing the sign of e, as follows. One gets an ellipse in all the cases $-1<e<1$ (see below for the Cartesian equation). But two e's differing only by sign give the same ellipse after reflecting in the origin; that is, change the signs of the Cartesian coordinates, or change θ to $\theta+\pi$ in polar coordinates.

Using the usual formulas $r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta$, the conclusion of the theorem gives the orbit in Cartesian coordinates as

$$\sqrt{x^2 + y^2} \ + \ ex \ = \ D$$

or, almost equivalently,

$$y^2 \ + \ (1 - e^2)x^2 \ + \ 2eDx \ = \ D^2 \ .$$

Let us write this quadratic in the familiar forms from high school analytic geometry:

For $0 \leq e < 1$, it quickly converts to the **ellipse**

$$\frac{[x + (eD/(1 - e^2))]^2}{a^2} \ + \ \frac{y^2}{a^2(1 - e^2)} \ = \ 1 \ ,$$

with semi-major axis $a = D/(1 - e^2)$, semi-minor axis $b = a\sqrt{1 - e^2}$, eccentricity e , centred at $(-eD/(1 - e^2) \ , \ 0)$,
and with one focus at $(0, 0)$, the source of the radial vector field.
It is of course a circle when $e = 0$.

For $e = 1$, the equation reduces to the **parabola**

$$y^2 \ + \ 2Dx \ = \ D^2 \ .$$

For $e > 1$, it reduces to the **hyperbola**

$$\frac{[x - (eD/(e^2 - 1))]^2}{a^2} \ - \ \frac{y^2}{a^2(e^2 - 1)} \ = \ 1 \ ,$$

with $a = D/(e^2 - 1)$. Actually, the original equation, before squaring, just reduces to the 'left branch', not the entire hyperbola. One mustn't engage in hyperbole.

This half-hyperbola would be the orbit of an interloper from the far reaches of the universe, which comes for a brief visit to the solar system, but doesn't stay. Actually it might not even penetrate the solar system, but should at some point be much closer to the sun than to any other star, for this to be a reasonable model.

The parabola (and the circle) presumably happen with probability zero; but that's a moot point, since all this is only an approximation (however excellent) to physical reality.

Theorem 5. (Kep III) *Above, in the only case of a* **closed** *orbit, that is, the ellipse, we have, with notation from* **Theorem 3**,

$$\frac{T_{\rm cl}^2}{a^3} \;=\; \frac{4\pi^2}{K} \;,$$

(a one-sentence theorem with six commas).

Note that the number on the right-hand side is independent of which ellipse we are referring to: all the orbits have $\text{period}^2/\text{diameter}^3 = \pi^2/2K$.

Proof. This is now direct, with justifications for the following equalities being, respectively: **Theorem 3**, Archimedes, the definitions of D and b, simple-minded, and the definition of a :

$$T_{\rm cl}^2 \;=\; (\frac{2}{L}A_{\rm cl})^2 \;=\; \frac{4}{L^2}(\pi ab)^2 \;=\; \frac{4\pi^2a^2(a\sqrt{1-e^2})^2}{DK} \;=\; \frac{4\pi^2}{K}\,\frac{a^4(1-e^2)}{D} \;=\; \frac{4\pi^2}{K}a^3 \;.$$

APPENDIX : Eliminating the extra baggage in Theorem 4 .

Hypothesis (ii) of **Theorem 4** may be replaced by a much less stringent one, which simply eliminates the case where the 'particle' moves directly towards (or away from) the central attracting object. (See the formulation of Case B below). In fact, the proof is really the same, but initially results in a messier formula for the orbit. The formula in **Theorem 4** is easier to work with and simpler to comprehend for all the subsequent work, so we did that first; but, as the result below shows, it is perfectly general. Logically, this appendix should be read before, or simultaneously with, the first part of Section 5.

Actually, we shall consider all cases (with, of course, $\vec{w}(t) \neq \vec{0}$). So assume (i) in Theorem 4 and copy the first two paragraphs of that theorem's proof verbatim. Then re-set the clock and partly re-choose $\mathbf{R}^2$-coordinates so that $\vec{w}(0)$ is a positive multiple of $\vec{i}$, giving $\theta(0) = 0$, $\vec{\rho}(0) = \vec{i}$ and $\vec{\varphi}(0) = \vec{j}$. Now let

$$\vec{v}(0) \;=\; \alpha\vec{i} + \beta\vec{j} \;\;.$$

Then, substituting, the constant $\vec{C}$ is determined as

$$\vec{C} \ = \ \gamma\vec{i} \ + \ \delta\vec{j} \qquad \text{where} \qquad \gamma := \frac{L\alpha}{K} \ , \quad \delta := \frac{L\beta}{K} \ - \ 1 \quad .$$

Going through the last part of the proof again, we have

$$r(1+\delta\ \cos\theta - \gamma\ \sin\theta) \ = \ r(1 \ + \ < -\sin\theta\vec{i} + \cos\theta\vec{j} \ , \ \gamma\vec{i} + \delta\vec{j} >)$$

$$= \ r(< \vec{\varphi}, \vec{\varphi} > \ + \ < \vec{\varphi}, \vec{C} >) \ = \ r < \vec{\varphi} \ , \ \vec{\varphi} + \vec{C} >$$

$$= \ r < \vec{\varphi} \ , \ \frac{L}{K}\vec{v} > \ = \ \frac{L}{K} r < \vec{\varphi} \ , \ \dot{r}\vec{\rho} \ + \ r\dot{\theta}\vec{\varphi} > \ = \ \frac{L}{K} r^2\dot{\theta} \ = \ \frac{L^2}{K} \ =: \ D \ .$$

So the new slightly messier polar equation is

$$r(1+\delta\ \cos\theta - \gamma\ \sin\theta) \ = \ D \ .$$

As above, this is (almost, because of squaring) equivalent to the Cartesian equation

$$(\delta^2-1)x^2 \ + \ (\gamma^2-1)y^2 \ - \ 2\gamma\delta xy \ - \ 2\delta Dx \ + \ 2\gamma Dy \ + \ D^2 \ = \ 0.$$

Case A. *If* $\{\vec{w}(0) \ , \ \vec{v}(0)\}$ *is linearly dependent—that is,* $\vec{v}(0)$ *is a scalar multiple of* $\vec{i}$—then, since

$$\vec{v} = \dot{r}\vec{\rho} + r\dot{\theta}\vec{\varphi} \ \text{ and } \ r(0) \neq 0 \ \text{ and } \ \vec{\varphi}(0) = \vec{j} \neq \vec{0} \ ,$$

we get $\dot{\theta}(0) = 0$, so $L = r(0)^2\dot{\theta}(0) = 0$, immediately yielding $D = 0 = \gamma$ and $\delta = -1$ from the definitions of those numbers.

The Cartesian equation above reduces to $-y^2 = 0$, i.e. $y = 0$. This is hardly surprising—the particle just zips out-and-back along the positive x-axis, with speed as given by the elementary gravitational theory of a ball thrown (quite vigorously!) vertically into space. That needs no further exposition here, except to note that sometimes it never comes back (escape velocity!).

Case B. *If* $\{\vec{w}(0) \ , \ \vec{v}(0)\}$ *is linearly independent; that is,* $\vec{v}(0) = \alpha\vec{i} + \beta\vec{j}$ *for some* α *and non-zero* β , then we get that $\dot{\theta}(0)$, so L, and D are all definitely non-zero.

The Cartesian equation above is then a (non-degenerate) conic which does not pass through the origin. And such a conic achieves a minimum distance from the origin. At that point of time, $|\vec{w}|^2$ being minimized, we have

$$0 \;=\; \frac{d|\vec{w}|^2}{dt} \;=\; \frac{d < \vec{w}, \vec{w} >}{dt} \;=\; 2 < \vec{w}, \dot{\vec{w}} > \quad .$$

Also $\dot{\vec{w}} \neq \vec{0}$ at that point of time, since we would otherwise have Case A, where the orbit is along a line interval, not a genuine conic. So we have now **proved** hypothesis (ii) in Theorem 4, and no more need be said.

6. Physical Interlude on Kep III .

Recalling $K = MG$ for the less refined version of Newtonian celestial mechanics, the formula in Theorem 5 'solves' to give

$$M \;=\; \frac{\pi^2 \text{ diameter}^3}{2G \text{ period}^2} \quad .$$

Let's use the earth to make a rough estimate for the mass of the sun. Using a more accurate calculation for that, plus the period of another planet, reversing the formula will give a value for the diameter of that planet's orbit, one which agrees spectacularly well with astronomers' observations, for every planet.

The gravitational constant G is often given as $6.67 \times 10^{-11} \; \frac{\text{m}^3}{\text{sec}^2\text{kg}}$, so we'll use $\frac{2}{3} \times 10^{-10}$. The number π^2 can have 10 as a pretty good estimate. My mother told me that the sun is 93 million miles away, about 150 million km., so 300×10^6 km., or 3×10^{11} m. is a good estimate for the diameter of the earth's orbit. The period is about 365 days, roughly 9000 hours, or 54×10^4 min., or 324×10^5 sec. When squaring this in the denominator of the displayed formula, estimate it as roughly $(25 \times 13 \times 10^5)^2$, about $(\frac{10^2}{4})^2 \times 160 \times 10^{10} \;=\; 10^{15}$.

So we end up estimating the mass of the sun as

$$\frac{10 \times (3 \times 10^{11})^3}{2 \times 2/3 \times 10^{-10} \times 10^{15}} \;=\; \frac{27 \times 3 \times 10^{34-5}}{4} \;=\; \frac{81}{4} \times 10^{29} \quad ,$$

or roughly 2×10^{30} kg. That's only about 1/2% more than the usual estimate by people who own calculators, so the roughities in our replacements above cancelled each other rather well.

As for the mass of the earth, one needn't rely on what mother said about the distance to the moon (but we'll check that later). Using only Newton's gravitation law and basic law of motion, at the surface of the earth, gravitational acceleration of any body would be GM/r^2, where now the mass M is that of the earth, and r is its radius. (This depends on a special case of some mathematics which made Newton work rather hard—to show that we can assume the earth has all its mass concentrated at a point, at its centre. The general, harder case, is if you worked at the bottom of a deep mine shaft. Newton would tell you that all the mass closer to the surface 'cancels', and you can concentrate, at a point at the centre of the earth, all the mass further from the surface.) So here we get

$$M = \frac{ar^2}{G} \quad .$$

The acceleration at the surface of the earth is often given as 9.8 m/sec^2, which we'll just replace by 10. We all know it's about 25,000 miles around the equator, or 40,000 km., or 4×10^7 m. So r is about $\frac{4\times10^7}{2\pi}$, and squaring it in the numerator, again replacing π^2 by 10, we'll use $\frac{2^2\times(10^7)^2}{\pi^2}$, or about 4×10^{13}. Thus the <u>mass of the earth</u> gets estimated (independently of Kepler) as $\frac{10\times4\times10^{13}}{2/3\times10^{-10}} = \underline{6 \times 10^{24}\text{ kg.}}$

Let's check this against **Kep III**. A useable estimate for the period of the moon (the "sidereal month", checking how long between successive passings of a fixed star by the moon from our viewpoint) is 27.3 days, or roughly 650 hours, or 40,000 min., or $2,500,000 = 2.5 \times 10^6$ sec. My mother tells me the moon's orbit is more egg-shaped, and that 10 times around the equator, or 4×10^8 metres, is a good estimate for the furthest distance to the moon. So **Kep III** for the moon's orbit will here estimate the earth's mass to be $\frac{\pi^2}{2G}\frac{\text{diameter}^3}{\text{period}^2}$, or about $\frac{10\times(8\times10^8)^3}{2\times2/3\times10^{-10}\times(2.5\times10^6)^2}$, which is

$$\frac{3 \times 8^3 \times 10^{23}}{4 \times (2.5)^2} = \frac{3 \times 8^3 \times 10^{23}}{10 \times 2.5} = \frac{24}{25} \times 8^2 \times 10^{23} \approx 62 \times 10^{23}\text{ kg. },$$

which is reasonably close to the previous 6×10^{24} kg.

One could wield a calculator and easily get a more accurate estimate of the earth's mass by the first (terrestrial) method. And then use that and **Kep III** to estimate the furthest distance to the moon. The answer agrees extremely well with independent estimates by astronomers

Having checked all this evidence for him or herself, the skeptical (even paranoid) mind is then ready to accept just how extraordinarily accurate Newton's theory from 350 years ago really is.

In any case, the sun has the mass of roughly 1/3 of a million objects, each with the mass of the earth. And the earth, estimating its volume as

$$\frac{4}{3}\pi r^3 \;=\; \frac{4}{3}\pi(\frac{4\times 10^7}{2\pi})^3 \;=\; \frac{4^4\times 10^{21}}{3\times 2^3\times\pi^2} \;\approx\; \frac{64}{6}\frac{10^{21}}{10} \;\approx\; 10^{21}\ \mathrm{m}^3 \quad ,$$

or $10^{21}\times(10^2)^3 \;=\; 10^{27}$ cc , with mass roughly 6×10^{27} gm , has average density of about 6 gm/cc , or 6 times that of water. This seems a bit surprising at first, till one learns that it's pretty well solid iron (or should I say liquid iron?) all the way down from somewhere not too far below the surface. And iron is easily over 7 gm/cc . The earth is the densest of all the solar system's planets, and more than 4 times denser than the sun, on average.

7. Some Physics and Mathematics related to Kep I .

The first line of the proof of Theorem 4 (which came out of the blue, but which we shall dwell on later with some indication of systematic Lie theory methods) tells us that the time derivative of $\frac{L}{K}\vec{v}-\vec{\varphi}$ is zero, and so the latter is a constant, i.e. a conserved (vector) quantity.

We had another conserved quantity, namely $\vec{w}\times\vec{v}$, a vector whose length, multiplied by the mass of the orbiting object, was interpreted physically as angular momentum.

So let's calculate the length of the conserved vector from the first paragraph above above in two different ways.

On the one hand, in the proof after picking our coordinate system, that vector became $e\vec{j}$, so it's length is e, that being non-negative once the cases where $-1 < e < 0$ have been finessed away.

On the other hand, let's calculate directly with the expression above multiplied by the constant $\frac{K}{L}$, using the orthonormality of $\{\vec{\rho},\vec{\varphi}\}$:

$$|\vec{v} - \frac{K}{L}\vec{\varphi}|^2 \;=\; < \vec{v} - \frac{K}{L}\vec{\varphi} \,,\, \vec{v} - \frac{K}{L}\vec{\varphi} >$$

$$= |\vec{v}|^2 - 2\frac{K}{L} < \vec{v} \,,\, \vec{\varphi} > + \frac{K^2}{L^2}|\vec{\varphi}|^2 \;=\; |\vec{v}|^2 - 2\frac{K}{L} < \dot{r}\vec{\rho} + r\dot{\theta}\vec{\varphi} \,,\, \vec{\varphi} > + \frac{K^2}{L^2}$$

$$= |\vec{v}|^2 \;-\; 2\frac{K}{L}r\dot{\theta} \;+\; \frac{K^2}{L^2} \;=\; |\vec{v}|^2 \;-\; 2\frac{K}{L}\frac{L}{r} \;+\; \frac{K^2}{L^2} \;=\; J \;+\; \frac{K^2}{L^2} \;,$$

where we define

$$J \;:=\; |\vec{v}|^2 \;-\; 2\frac{K}{r} \;,$$

which is conserved also, since $\frac{K^2}{L^2}$ is just a number. The constant J is physically interpreted and re-expressed below.

But first note that we now have an expression for e:

$$e^2 \;=\; |e\vec{j}|^2 \;=\; |\frac{L}{K}(\vec{v} - \frac{K}{L}\vec{\varphi})|^2 \;=\; (\frac{L}{K})^2(J \;+\; \frac{K^2}{L^2}) \;=\; 1 + \frac{JL^2}{K^2} \;.$$

So we have

$$e \;=\; \sqrt{1 + \frac{JL^2}{K^2}} \;=\; \sqrt{1 + \frac{JD}{K}} \;=\; \sqrt{1 + \frac{JD^2}{L^2}} \quad .$$

The first expresses it in terms of the constant K from the DE, and conserved quantities angular momentum and energy (as we'll see momentarily!); the second is rather simpler, using the right-hand side from the polar quadratic solution given by Theorem 4; and the third probably should be erased.

Remark. I do not know whether the conserved vector $\frac{L}{K}\vec{v} - \vec{\varphi}$ has any famous name (though, up to a constant, its length certainly has, as we see below). The name "Runge-Lenz vector" occurs (see [**Guill-Stern**]) for a vector which differs by a constant scalar multiple and a rotation of 90 degrees, so is effectively equivalent in its information content. The latter can be written in our notation as

$$\dot{\vec{w}} \times \vec{L} \;-\; K\vec{\rho} \quad , \quad \text{where} \;\; \vec{L} = \vec{w} \times \vec{v} = r^2\dot{\theta}\;\vec{\rho} \times \vec{\varphi}$$

is our conserved angular momentum vector. Express both in terms of $\{\vec{\rho}, \vec{\varphi}\}$ to see this connection between the two vectors.

Now for some more physics: with m still the mass of the orbiting 'particle', the constant

$$\frac{1}{2}mJ \;=\; \frac{1}{2}m|\vec{v}|^2 \;-\; \frac{mK}{r} \;=\; \frac{1}{2}m|\vec{v}|^2 \;-\; \frac{mMG}{r}$$

will look familiar to anyone who remembers their first course in physics. It is total energy, conservation of which is about as fundamental as anything in physics. The $\frac{1}{2}m|\vec{v}|^2$ is kinetic energy; the $-\frac{mMG}{r}$ is gravitational potential energy. Since $\vec{v} = \dot{r}\vec{\rho} + r\dot{\theta}\vec{\varphi}$, the $|\vec{v}|^2$ half of J is expressible as $\dot{r}^2 + r^2\dot{\theta}^2$. Multiplying by $\frac{1}{2}m$ chops the kinetic energy into two pieces which could be non-standardly termed 'radial' and 'angular', respectively.

Much more is sayable and has been said about this "2-body problem", much of it motivated by attempts to get a handle on the 3-body problem. In the next section we go through the simple vector arguments which will allow all our analysis of the DE $(*)$ to be used again in the more refined case where we take the origin to be the centre of mass of the two bodies. (K will not be MG there.) This case must be used when dealing, say, with binary stars, where m and M have the same order of magnitude; and also for the analysis of the detecting of extra-solar planets via the "wobble" of their star.

Here we finish with a few remarks about the question of explicitly solving $(*)$; that is, writing $\vec{w}$ explicitly in terms of t; equivalently, expressing r and θ explicitly in terms of t. The equation

$$r(1 + e\,\cos\theta) \;=\; D \qquad\qquad (**)$$

allows them to be expressed in terms of each other, so, generically, getting one gets the other. (That won't work when $e = 0$, for example.) Actually, depending on what "explicit" is taken to mean, this question seems quite hard. What we do just below is to express t in terms of r [or of θ by $(**)$ or other means]. So the question is reduced to finding the inverse function explicitly in some sense.

The displayed line a page or two back beginning with e^2 yields

$$\frac{K^2(e^2-1)}{L^2} \;=\; J \;=\; |\vec{v}|^2 \;-\; \frac{2K}{r} \;.$$

Replacing $|\vec{v}|^2$ by $\dot{r}^2 + r^2\dot{\theta}^2$ we get

$$\dot{r}^2 + r^2\dot{\theta}^2 \;=\; \frac{2K}{r} \;+\; B \;, \quad \text{where} \;\; B := \frac{K^2(e^2-1)}{L^2} = \frac{K(e^2-1)}{D} \;.$$

Replacing $\dot\theta$ by $\frac{L}{r^2}$ and moving that term to the right-hand side, we get

$$\dot r^2 \ = \ -\frac{L^2}{r^2} \ + \ \frac{2K}{r} \ + \ B \ \ .$$

Thus

$$t \ = \ \int \frac{r\ dr}{\sqrt{Br^2+2Kr-L^2}} \quad ,$$

and the indefinite integral on the right is certainly expressible in terms of elementary functions by elementary calculus. So that seems to do the job. However it's not clear how useful it is. For example, for an orbit which is a circle, so $e=0$, the time t is hardly determined by r !

To deal with θ similarly, we can write $\dot\theta = \frac{L}{r^2}$, and then replace r by $\frac{D}{1+e\ \cos\theta}$. Then blind manipulation produces

$$t \ = \ \frac{D^2}{L} \int \frac{d\theta}{(1+e\ \cos\theta)^2} \quad .$$

8. The More Refined Case

(relevant to 'sunless' solar systems and binary stars).

. . . where Alph, the sacred river ran,
Through caverns measureless to man,
Down to a sunless sea.

Coleridge

In this section, we are going to maintain that all the preceding is
(i) clearly mathematically correct, but
(ii) physically incorrect; yet
(iii) the mathematics is perfectly usable as is, once we fix the physics.

To begin with, an *inertial frame* is a concept from physics. Here, where we are assuming the perfect accuracy of Newtonian mechanics, an inertial frame may be roughly defined as a way of setting up measurements of distances and times so that Newton's second law, $\vec f = m\vec a$, holds. In general, less is needed ("object at rest stays that way in absence of force"), and also more elaboration is needed—e.g. which forces count as $\vec f$'s?—is the choice of distance units (metres or light-years) relevant?—does it include setting up coordinate axes?—etc.... But for our simple purposes, namely the two-body problem, where only the gravitational force which each body exerts on the other is to be considered, it will be unnecessary to get *too* involved in discussing inertial frames.

Another aspect of this, which we'll largely be able (and wise) to avoid, is that of sharply separating physical objects from mathematical objects. For example, our previous $\vec{w}(t)$ in many places would be referred to as the *particle* [physical object]; in others as the *position in space of the particle* [physical or mathematical?]; and in yet others, here particularly, as simply *an ordered triple of real numbers* (or *an ordered pair*, once we've reduced to $\mathbf{R}^2$ from $\mathbf{R}^3$)[certainly a mathematical object]. Besides philosophers, perhaps it is only a few pure mathematicians like me (and possibly some of my readers) who might want to fuss about this distinction. But there are times, when reading works by applied mathematicians/mathematical physicists, that I myself find things mildly confusing. So if we were considering a more complicated subject (such as general relativity), I'd probably feel the need for a much more elaborate discussion than we now give.

Here we need only Newton's basic assumption that there is at least one inertial frame : i.e.

firstly, a bijection between the set of instants of time and the set $\mathbf{R}$;

secondly, an injection from the set of point objects in space being considered, mapping to $\mathbf{R}^3$, such that any acceptable way of measuring distances, speeds, accelerations, etc. 'obeys' the laws of euclidean geometry and behaves smoothly (C^∞) as a function of time;

thirdly, a force, exerted by an object at $\vec{W}$ of mass M on an object at $\vec{w}$ of mass m, which is

$$-\frac{mMG}{|\vec{w}-\vec{W}|^3}(\vec{w}-\vec{W}) \quad ,$$

in the units chosen;

and finally, that this force does act so that the 2nd law of motion is valid, yielding

$$\ddot{\vec{w}} \;=\; \vec{a} \;=\; -\,\frac{MG}{|\vec{w}-\vec{W}|^3}(\vec{w}-\vec{W})$$

for the acceleration of the mass m object, in the absence of any other gravitation.

Actually, Newton took 'absolute space' ("...*the parts of that immovable space in which these motions are performed do by no means come under the observations of our senses"*) as a given. So, assuming this space to have all the structure of an abstract Euclidean 3-dimensional geometry (for example, relations of incidence, in-betweenness and congruence in Hilbert's axiomatization), one would only be left with choosing a particular point-object in space as 'origin' in order to feel that the technical notation above makes perfect sense (and very elementary algebra assures us that the choice of that origin point is irrelevant to the correctness of the equation).

Now Newton was certainly aware that using instead as origin some object, "$\vec{p}(t)$", which was in 'uniform motion' with respect to his "absolute space" would also do perfectly well to produce another inertial frame [i.e. his 2nd law would still apply when all objects had their position vectors $\vec{w}(t)$ changed to $\vec{w}(t)-\vec{p}(t)$]. Such an origin-object would have position vector of the form $\vec{p}(t) = \vec{c} + t\vec{b}$ for fixed $\vec{c}$ and $\vec{b}$, with respect to his "absolute space"; or equivalently $\ddot{\vec{p}}(t) = 0$ for all t. And he was furthermore well aware that using an object for which $\ddot{\vec{p}}(t) \neq 0$ for some t would cause the 2nd law to fail. (All this was largely Galileo's discovery.)

There is no need really to refer to "absolute space"—our one genuine assumption is for the existence of at least one inertial frame, in the sense above, most basically here, that Newton's 2nd law holds (and for us, only *gravitational* forces need be considered for the left-hand side). In particular, questions about the introduction of units of physical measurement, or mathematical coordinate axes, are all irrelevant and essentially trivial in this discussion. The main point is that, if every position vector $\vec{w}$, measured in some fixed inertial frame, is replaced by $\vec{w} - \vec{p}$ for a fixed $\vec{p}(t) = \vec{c} + t\vec{b}$, then you have another inertial frame.

Now one can reverse the roles of $\vec{W}, M$ and $\vec{w}, m$ above, and say that the object $\vec{w}$ of mass m exerts on the object $\vec{W}$ of mass M a force which is

$$-\frac{MmG}{|\vec{W} - \vec{w}|^3}(\vec{W} - \vec{w}) \quad ,$$

exactly the negative of the force acting on the actor of mass m. (Note also how the expression is independent of choice of origin.) And so, that 'upper case' object must accelerate according to

$$\ddot{\vec{W}} \ = \ \vec{A} \ = \ -\frac{mG}{|\vec{W}-\vec{w}|^3}(\vec{W}-\vec{w}) \ \ ,$$

when measured in our given frame (or any inertial frame). We just interchanged upper and lower case in the previous, of course. We'll consistently use $\vec{W}, \vec{V}, \vec{A}, M$ for the 'big' object.

In particular, $\ddot{\vec{W}} = \vec{A} \neq \vec{0}$ for every relevant t. And so **it is apparently ludicrous**, in view of our immediately previous discussion, **to start working in a coordinate system based on having the 'large' object of mass M as fixed at the origin**. But that is **exactly what we were doing for most of the previous seven sections** !

Let

$$\mu \ = \ \frac{m}{M} \ \ .$$

Then $\vec{A} = -\mu\vec{a}$. This is very small, if μ is. For example, having the objects as the sun and earth gives $\mu < 1/300,000$, by the estimates of Section 6, so it's not really all that "ludicrous" sometimes; those teaching students unfamiliar with vector calculus should be given some slack (though we'll see that very, very little such knowledge is needed to sort this out).

And if we wish to study binary star systems of masses m and M, we might very well have $\mu = 1/2$ or 1.2 . . . , so the approximation fixing one of them as the origin is very bad. Formulas below will measure how bad. Furthermore, if we wish to speak about the "wobble" of the star orbited by an extra-solar planet, or indeed of the Sun-Earth or Sun-Jupiter systems (using a different sort of approximation which assumes no other planets in the solar system), then clearly we can't have the star fixed at the origin, or we won't see much wobble at all in the resulting mathematics !

So let us start with **any** inertial frame, as per our basic assumption, plus two gravitating objects as described above. We plan to show :

(A) the centre-of-mass, $\vec{\mathsf{cm}}$, of the two objects can be taken as the origin in a new, more convenient, and, most importantly, **inertial** frame; and

(B) with respect to this new frame, the DE which we studied so much in the previous sections, for somewhat modified values of the number K, will apply for all of the following four situations :

(i) describing the motion of either body for an observer fixed at that mutual centre-of-mass origin;
(ii) describing the motion of either body for an observer positioned at the other body (at its individual centre-of-mass).

The extra mathematics will just amount to very simple vector algebra.

In the end, we get four ellipses which are all of the same *shape* (same *eccentricities*) in the case of closed orbits, and similarly for non-closed orbits. The *sizes* of these ellipses differ, of course.

Define, in the initially given frame (i.e. maybe in absolute space, if your name is Isaac)

$$\vec{\mathsf{cm}} \;:=\; \frac{m}{m+M}\vec{w} \;+\; \frac{M}{M+m}\vec{W} \;=\; \frac{1}{1+\mu^{-1}}\vec{w} \;+\; \frac{1}{1+\mu}\vec{W} \;.$$

Thus

$$\begin{aligned}\ddot{\vec{\mathsf{cm}}} \;&:=\; \frac{m}{m+M}\ddot{\vec{w}} \;+\; \frac{M}{M+m}\ddot{\vec{W}} \\ &=\; \frac{m}{m+M}\Big[-\frac{MG}{|\vec{w}-\vec{W}|^3}(\vec{w}-\vec{W})\Big] \;+\; \frac{M}{M+m}\Big[-\frac{mG}{|\vec{W}-\vec{w}|^3}(\vec{W}-\vec{w})\Big] \\ &=\; -\frac{mMG}{(m+M)|\vec{w}-\vec{W}|^3}(\vec{w}-\vec{W}) + \frac{mMG}{(m+M)|\vec{w}-\vec{W}|^3}(\vec{w}-\vec{W}) \;=\; \vec{0}\;,\end{aligned}$$

as required to show that $\vec{\mathsf{cm}}$ can be taken as the origin for a new inertial frame.

Next, mathematically define [with physical interpretations—for binary star systems, one of the two stars will have to swallow its pride and be willing to be called a "planet"!]:

$$\vec{w}_- \;:=\; \vec{w} - \vec{\mathsf{cm}}$$

[the "planet"'s orbit in the new inertial frame, i.e. viewed from its origin];

$$\vec{W}_- \;:=\; \vec{W} - \vec{\mathsf{cm}}$$

[the "star"'s orbit in the new frame];

$$\vec{w}_+ \;:=\; \vec{w} - \vec{W} \;=\; \vec{w}_- - \vec{W}_-$$

[the “planet”’s orbit viewed from the “star” (Copernicus’ preference)];

$$\vec{W}_+ \ := \ \vec{W} - \vec{w} \ = \ \vec{W}_- - \vec{w}_-$$

[the “star”’s orbit viewed from “planet” (the Pope’s preference)].

Proposition. *All four of these (i.e. $\vec{y} = \vec{w}_\pm$ or $\vec{W}_\pm$) satisfy a DE of the form*

$$\ddot{\vec{y}} \ = \ -\frac{K}{|\vec{y}|^3}\,\vec{y} \quad ,$$

(such as we studied in earlier parts of this write-up) *for positive constants K as follows :*

(i) $K \ = \ \frac{MG}{(1+\mu)^2}$ *when* $\vec{y} = \vec{w}_-$;

(i)′ $K \ = \ \frac{mG}{(1+\mu^{-1})^2}$ *when* $\vec{y} = \vec{W}_-$;

(ii) $K \ = \ (1+\mu)MG$ *when* $\vec{y} = \vec{w}_+$;

(ii)′ $K \ = \ (1+\mu^{-1})mG$ *when* $\vec{y} = \vec{W}_+$.

So having K equal to exactly MG or to mG as before simply doesn’t happen—only as an approximation. The last two K are actually equal; in fact, both are just $(m+M)G$. In (ii) we have Copernicus’ natural question answered: What orbit would you see, viewed from a star, of a single planet, if those two constituted the entire Newtonian universe? But perhaps (i) and (i)′ are in the end more fundamental, namely description of the orbits by an observer actually in the natural inertial frame.

Proof. It is clear by symmetry that (i)′ and (ii)′ follow from (i) and (ii) respectively. But first we’ll prove (i) in the second display below, and then give a single argument which gets all the other three from it, alternative proofs for the primed ones. All this is just very easy algebra.

Firstly, freshman manipulations give us

$$\vec{W}_- \ = \ -\,\mu\vec{w}_- \quad \text{and} \quad \vec{w}_+ \ = \ \vec{w}_- - \vec{W}_- \ = \ (1+\mu)\vec{w}_- \quad .$$

[Notice that these, plus the symmetrical fact that

$$\vec{W}_+ \ = \ (1+\mu^{-1})\vec{W}_- \ = \ -\,(1+\mu)\vec{w}_- \ ,$$

show that the orbits for all four are *similar* in the precise geometric sense—obtainable from each other by a dilation of the plane by a positive factor. That justifies the earlier claim about four ellipses with the same eccentricities, once we know that the orbit is in fact an ellipse in the closed-orbit case. Because of the negative factor, $-\mu$, relating $\vec{W}_-$ to $\vec{w}_-$, we see that the "big star" traverses its 'little' orbit 'on the opposite side', but in the same direction, to that of the "little planet" in its 'big' orbit.]

Now, to see (i), just calculate easily:

$$\ddot{\vec{w}}_- = \frac{d^2(\vec{w} - \vec{\mathsf{cm}})}{dt^2} = \ddot{\vec{w}} = -\frac{MG}{|\vec{w} - \vec{W}|^3}(\vec{w} - \vec{W})$$

$$= -\frac{MG}{|\vec{w}_+|^3}(\vec{w}_+) = -\frac{MG}{|(1+\mu)\vec{w}_-|^3}(1+\mu)\vec{w}_-$$

$$= -\frac{MG/(1+\mu)^2}{|\vec{w}_-|^3}\vec{w}_- \quad .$$

To deduce the other three, we use the following rather obvious fact: Assume that ν is a non-zero number, and let $\vec{z} := \nu\vec{y}$, where $\vec{y}$ satisfies the DE displayed in the theorem. Then $\vec{z}$ clearly satisfies the DE

$$\ddot{\vec{z}} = -\frac{|\nu|^3 K}{|\vec{z}|^3}\,\vec{z} \quad .$$

Starting with $K = \frac{MG}{(1+\mu)^2}$, and setting ν successively equal to

$$-\mu \quad , \quad (1+\mu) \quad , \quad -(1+\mu)$$

then yields the other three claims from (i) by easy algebra, completing the proof.

PART II : Noether's Theorem, Lagrangians, Hamiltonians, Lie Algebras, and the Runge-Lenz Vector.

The remaining sections of this write-up have one specialized aim, and a few general ones. The specialized aim is to explain (as simply as I know how) the extra, so-called "hidden", *symmetry* in the Kepler problem. This symmetry gives rise to the *conserved quantity*, called the Runge-Lenz vector earlier, sometimes also associated with Laplace, and even with Hermann, Bernoulli and Pauli.

One of the general aims is to present various versions of Emmy Noether's theorem which connects symmetries to conserved quantities in a general and beautiful way. The number of times this theorem of Noether has been referred to in a vague, offhand way seems to exceed by several orders of magnitude the number of times some sort of precise statement has been offered. Also, various versions, particularly lagrangian versus hamiltonian versions, appear to be rather unconnected to each other in the expository literature, again except for offhand remarks. So I'll attempt to relate three or four different versions of the theorem.

A second general aim will be to introduce portions of both lagrangian and hamiltonian mechanics (classical, that is, non-quantum), using notation which might be more appealing to students of pure mathematics. Frequently ([**Arnold**] being an exception) the subject is formulated in the language of the academic discipline which could be called manipulation of symbols with physical associations. That is the lingua franca of physicists. It has an appealing efficiency, but can sometimes turn off students who have come to appreciate the definition/theorem/proof style of post-19th century mathematics. Hopefully the write-up below will help some such students get started on the subject, and to realize that it's usually not hard to translate back-and-forth between the languages, and to convert ambiguously or vaguely stated claims into theorems.

A reader seriously interested in thoroughly learning this (and the hamiltonian) version of mechanics should consult the references for lots of good examples of mechanical systems, which we have left out almost entirely here. Start with [**Tong**] and [**XXXXXX**], then [**YYYYYYY**] and [**Arnold**] perhaps.

To be a bit more specific with examples of this "lingua franca", encountering phrases like the following might then become like "water off a duck's back", in that one soon sees what is being talked about. (These are verbatim quotes from courses offered in the 'best' universities, ones not unconnected to Isaac himself!):

> One must be careful to note the distinction between $\frac{dF}{dt}$ and $\frac{\partial F}{\partial t}$.
>
> Even though $t = s$, $\frac{\partial}{\partial t}$ is not the same as $\frac{\partial}{\partial s}$. This is an example of the second fundamental confusion of calculus.
>
> When using this notation, one must take care not to confuse the two roles of the $\dot{q}_a$'s: at one stage, $\dot{q}_a$ represents a generalized velocity coordinate

independent of q_a and t; at another it denotes the time derivative of q_a.

$\frac{\partial \dot{x}}{\partial \dot{s}} = \frac{\partial x}{\partial s}$ by canceling the dots.

And perhaps a third hope is that after gaining more appreciation of classical mechanics, the study of quantum mechanics will seem somewhat less to be a collection of unmotivated calculational recipes (which happen to have a magical connection to micro-reality!).

And finally, a fourth aim will be to present hamiltonian mechanics on phase spaces which are open subsets of $\mathbf{R}^k \times \mathbf{R}^k$, not symplectic manifolds in general, but in the form of a very firm stepping stone to that general case (which is more than adequately presented in [**Arnold**]). We shall formulate it in an almost entirely coordinate-free manner with notation which makes passage to the general case just a matter of presenting the definitions of *manifold* and the various types of vector/form/tensor bundles/fields on them. All the formulas here in the mechanics are notationally identical to the ones in the general case.

9. Emmy Noether+symmetry $\Longrightarrow$ conserved quantities

(or, why, if she actually had any taste, Martha would opine that lagrangians are a good thing).

First we shall prove two ultimately redundant propositions, generalizing partways towards a most general (here!) version of conservation of energy.

Proposition 9.1. *For a differentiable function* $P : \mathbf{R} \to \mathbf{R}$*, define* $T : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ *by*

$$T(\vec{y}, \vec{z}) \;:=\; |\vec{z}|^2 - P(|\vec{y}|^2) \;.$$

Then, for each fixed $\vec{w}$ *satisfiying*

$$\ddot{\vec{w}} \;=\; P'(|\vec{w}|^2)\vec{w} \;,$$

the function $t \mapsto T[\vec{w}(t), \dot{\vec{w}}(t)]$ *is constant.*

Thus any DE of the form $\ddot{\vec{w}} \;=\; Q(r)\vec{w}$, where Q has an anti-derivative, 'obeys' energy conservation, total energy being $\frac{1}{2}mT[\vec{w}(t), \dot{\vec{w}}(t)]$ for any particular t.

Proof. We have $\quad \frac{d}{dt}T[\vec{w}(t), \dot{\vec{w}}(t)]$

$$= \frac{d}{dt}[|\dot{\vec{w}}(t)|^2 - P(|\vec{w}(t)|^2)] \ = \ \frac{d}{dt} < \dot{\vec{w}}(t), \dot{\vec{w}}(t) > -P'(|\vec{w}|^2)\frac{d}{dt}(|\vec{w}(t)|^2)$$

$$= \ 2 < \dot{\vec{w}}(t), \ddot{\vec{w}}(t) > -P'(|\vec{w}|^2)2 < \dot{\vec{w}}, \vec{w} >$$

$$= \ 2 < \dot{\vec{w}}, P'(|\vec{w}|^2)\vec{w} > \ - \ 2 < \dot{\vec{w}}, P'(|\vec{w}|^2)\vec{w} > \ = \ 0,$$

as required.

That deals with a general system with a radial force, but actually is far from full generality.

Suppose given a C^∞-map $\vec{Q} : \mathbf{R}^3 \to \mathbf{R}^3$, and choose a smooth function $R : \mathbf{R}^3 \to \mathbf{R}$ such that for all (x, y, z) we have

$$\vec{Q}(x, y, z) \ = \ [\ \frac{\partial R}{\partial x}(x, y, z) \ , \ \frac{\partial R}{\partial y}(x, y, z) \ , \ \frac{\partial R}{\partial z}(x, y, z) \ ] \ ,$$

i.e. $\vec{Q} \ = \ \vec{\nabla} R$, an occasional notation for the so-called *gradient*.

Proposition 9.2. *Let*

$$T(\vec{y}, \vec{z}) \ := \ |\vec{z}|^2 - 2R(\vec{y}) \ .$$

If $\vec{w}$ satisfies

$$\ddot{\vec{w}}(t) \ = \ \vec{Q}[\vec{w}(t)] \ ,$$

then $T[\vec{w}(t), \dot{\vec{w}}(t)]$ is independent of t.

This takes care of energy conservation for most DEs which at all resemble what we've seen in the Kepler problem.

Proof. We have $\dfrac{d}{dt}T[\vec{w}(t), \dot{\vec{w}}(t)] \ = \ \dfrac{d}{dt}|\dot{\vec{w}}(t)|^2 - 2\dfrac{d}{dt}R[\vec{w}(t)] \ =$

$$= \ 2 < \dot{\vec{w}}(t), \ddot{\vec{w}}(t) > \ - 2[\ \frac{\partial R}{\partial x}[\vec{w}(t)]\dot{w}_1(t) + \frac{\partial R}{\partial y}[\vec{w}(t)]\dot{w}_2(t) + \frac{\partial R}{\partial z}[\vec{w}(t)]\dot{w}_3(t)\ ]$$

$$= \ 2 < \dot{\vec{w}}, \vec{Q}(\vec{w}) > \ - 2[\ \frac{\partial R}{\partial x}(\vec{w})\ \dot{w}_1 + \frac{\partial R}{\partial y}(\vec{w})\ \dot{w}_2 + \frac{\partial R}{\partial z}(\vec{w})\ \dot{w}_3\ ]$$

$$= \ 2 < (\ \dot{w}_1, \dot{w}_2, \dot{w}_3\ )\ ,\ (\ \frac{\partial R}{\partial x}(\vec{w}), \frac{\partial R}{\partial y}(\vec{w}), \frac{\partial R}{\partial z}(\vec{w})\ )\ > \ - \ \text{itself} \ = \ 0,$$

as required.

It is rather easy to see that **9.1** may obtained as a special case of **9.2**; that will be left as an exercise. We shall see below that the only crucial thing to getting conservation of energy is that the lagrangian function, which starting now will become ubiquitous, does not involve t ‘explicitly’.

Lagrangians.

> The 200 year interval from the discoveries of Huygens and Newton to the geometrization of mathematics by Riemann and Poincaré seems a mathematical desert, filled only by calculations.
>
> V.I. Arnold

To explain a much more general method for discovering conserved quantities, one which will reproduce all the above on energy and angular momentum conservation (in fact, just about everything possible when the method is re-interpreted in various contexts), it is necessary to sketch the lagrangian approach to classical mechanics (and later the hamiltonian). We'll write it as pure mathematics, not as manipulation of symbols with strong physical associations. Then we'll give some standard examples with interpretations in physics.

In many applications, one has a positive integer N which would normally correspond physically to the number of particles in a mechanical system. The system at any particular time is considered to be completely given by

(i) three position coordinates for each particle, making $3N$ in all, which would be denoted $y_1, \cdots, y_{3N}$ in the following; and

(ii) a total of $3N$ velocity coordinates $z_1, \cdots, z_{3N}$.

Not merely to sanitize the notation a bit, but also for other purposes below, we replace $3N$ by any integer k, not necessarily divisible by 3.

A Lagrangian $L = L(\vec{y};\vec{z}) = L(y_1, \cdots, y_k \; ; \; z_1, \cdots, z_k)$ is a smooth function $\mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}$ (where the domain is more likely to be a proper open subset of $\mathbf{R}^k \times \mathbf{R}^k$, but that will be suppressed). Here is a non-standard notation for two other functions, $\vec{\nabla}_* L$ and $\vec{\nabla}_{**} L$, both from $\mathbf{R}^k \times \mathbf{R}^k$ to codomain $\mathbf{R}^k$ this time :

$$\vec{\nabla}_* L(\vec{y};\vec{z}) \;:=\; [\; \frac{\partial L}{\partial y_1}(\vec{y};\vec{z}) \;,\; \frac{\partial L}{\partial y_2}(\vec{y};\vec{z}) \;,\; \cdots \;,\; \frac{\partial L}{\partial y_k}(\vec{y};\vec{z}) \;] \;,$$

and similarly for $\vec{\nabla}_{**}$ using the $\frac{\partial L}{\partial z_i}$ for components. (Sometimes the physicists' notation for these two are $\frac{\partial L}{\partial \vec{y}}$ and $\frac{\partial L}{\partial \vec{z}}$ respectively. Really, we're just splitting the gradient into its two halves.) In fact, written in the unambiguous style of the relatively elementary text **[Spiv]**, the definition would be

$$\vec{\nabla}_* L(\vec{y};\vec{z}) \;:=\; [\; D_1 L(\vec{y};\vec{z}) \;,\; D_2 L(\vec{y};\vec{z}) \;,\; \cdots \;,\; D_k L(\vec{y};\vec{z}) \;] \;,$$

which becomes very clean, more complete and hardly ambiguous as

$$\vec{\nabla}_* \;:=\; [\; D_1 \;,\; D_2 \;,\; \cdots \;,\; D_k \;] \;, \quad \text{and} \quad \vec{\nabla}_{**} \;:=\; [\; D_{k+1} \;,\; D_{k+2} \;,\; \cdots \;,\; D_{2k} \;] \;.$$

In this notation, the *Lagrange* (or *Euler-Lagrange*) *equation* is a DE for the 'unknown' $\vec{w} : T \to \mathbf{R}^k$, i.e. the path $\vec{w} \in \mathcal{P} := C^\infty(T, \mathbf{R}^k)$, where T is an open interval of reals, generically named t (and conceived as 'physical time', just as with our notation in previous sections with $N = 1, \; k = 3$).

The Lagrange equation is

$$\frac{d}{dt}(\vec{\nabla}_{**} L[\vec{w}(t), \dot{\vec{w}}(t)]) \;=\; \vec{\nabla}_* L[\vec{w}(t), \dot{\vec{w}}(t)] \quad .$$

This equation is often written simply $\frac{d}{dt}(\frac{\partial L}{\partial \dot{w}}) = \frac{\partial L}{\partial w}$ or even as $\frac{d}{dt}(\frac{\partial L}{\partial \dot{y}}) = \frac{\partial L}{\partial y}$.

[By thinking about physical units, it is easy to remember which way round this goes : $\vec{\nabla}_{**}$ is 'differentiation with respect to the right-hand set of variables', which, after being replaced by the components of $\dot{\vec{w}}(t)$, have units $\frac{\text{distance}}{\text{time}}$. So $\frac{1}{\text{time}} \cdot \frac{\text{units of } L}{\text{dist./time}}$, that is, $\frac{\text{units of } L}{\text{distance}}$ are the units for the the left-hand side of Lagrange's equation. And that is also the units for the right-hand side, since $\vec{\nabla}_*$ is 'differentiation with respect to the left-hand set of variables', which are replaced by the components of $\vec{w}(t)$, whose physical units would normally be distance. And the whole thing wouldn't work if one mistakenly wrote Lagrange backwards—known as the lagrundge equation.]

What has Lagrange's equation to do with physics? The rough description of what one does with a classical mechanical system to see what L should be is this: Figure out what kinetic energy should be and write it as a function, $K(\dot{\vec{w}})$, of velocities. Then do the same for potential energy, as a function, $V(\vec{w})$, of positions. [Maybe *both* $\vec{w}$ *and* $\dot{\vec{w}}$ are needed for K, though not likely for V. And sometimes t will also occur explicitly as a variable in the expression for kinetic energy.] Now replace $\vec{w}$ by variables $\vec{y}$, and $\dot{\vec{w}}$ by variables $\vec{z}$, and take L to be the difference; i.e. in the simplest case as indicated above

$$L(\vec{y}\ ;\ \vec{z}) \ := \ K(\vec{z}) - V(\vec{y}) \ .$$

[Thus the "units of L" above are actually units of energy, that is, $\frac{\text{mass}\times\text{dist}^2}{\text{time}^2}$, but that detail was not needed earlier.] Note the *minus* sign in L (which is *not* total energy). That sign sometimes gets converted into a plus sign in specific cases where the 'potential energy' expression happens to have a negative sign.

Referring to the earlier part of this write-up, with $N = 1$, a single particle of mass m, in the central gravitational field of a "nailed down" ("less refined") glob of mass M would have

$$K(\dot{\vec{w}}) \ = \ \frac{1}{2}m|\dot{\vec{w}}|^2 \qquad \text{and} \qquad V(\vec{w}) \ = \ -\frac{mMG}{|\vec{w}|} \quad ,$$

so

$$L(\vec{y}\ ;\ \vec{z}) \ = \ \frac{1}{2}m|\vec{z}|^2 \ + \ \frac{mMG}{|\vec{y}|} \quad .$$

What does Lagrange's equation turn out to be in this example? Well,

$$\vec{\nabla}_* L(\vec{y}\ ;\ \vec{z}) \ = \ [\ \frac{\partial L}{\partial y_1}\ ,\ \frac{\partial L}{\partial y_2}\ ,\ \frac{\partial L}{\partial y_3}]$$

$$= \ [\ \frac{(-1/2)(2y_1)mMG}{(y_1^2+y_2^2+y_3^2)^{3/2}}\ ,\ \frac{(-1/2)(2y_2)mMG}{(y_1^2+y_2^2+y_3^2)^{3/2}}\ ,\ \frac{(-1/2)(2y_3)mMG}{(y_1^2+y_2^2+y_3^2)^{3/2}}\ ]$$

$$= \ -\frac{mMG}{|\vec{y}|^3}\ \vec{y}\ .$$

On the other hand

$$\vec{\nabla}_{**}L(\vec{y}\ ;\ \vec{z}) \ = \ [\ \frac{\partial L}{\partial z_1}\ ,\ \frac{\partial L}{\partial z_2}\ ,\ \frac{\partial L}{\partial z_3}] \ = \ \frac{1}{2}m(2z_1, 2z_2, 2z_3) \ = \ m\vec{z}\ .$$

After cancelling m on both sides, Lagrange's equation becomes

$$\frac{d}{dt}\dot{\vec{w}}(t) \ = \ -\frac{MG}{|\vec{w}(t)|^3}\ \vec{w}(t)\ ,$$

which we've seen plenty of, in the preceding!

However it is motivated, the 'physical truth' of the Lagrange approach is at least somewhat mysterious. (But after all, so are Newton's equations to begin with!) One can give a kind of pseudo-proof of [Newton's laws] $\Longleftrightarrow$ [Lagrange holds] , generalizing what is just above. There is a more abstract proof that Lagrange's equation comes from a principle of Hamilton asserting that the system should follow a path in "velocity space" making the "action" stationary. Many treatments of this, and details concerning the calculus of variations, may be found elsewhere. A great thing about the Lagrange approach is that any reasonable change of variables makes no difference to the Lagrange equation in its abstract form. By using vector notation, we are de-emphasizing this (in some psychological sense). There is a statement and proof in the appendix just below. It is important, among other things, in freeing one from a dependence on expressing everything in an inertial frame, as is needed with Newton's original formulation. None of the material in the present paragraph nor in the appendix below is essential for what follows.

Appendix : Mathematical Meaning and Proof of :
"Lagrange's equations remain the same under any change of coordinates."

The notion of "change of coordinates" is a good example of a concept which is really from the language of 'physical symbol manipulation', not strictly the language of mathematics. The corresponding mathematical concept is that of a diffeomorphism α from $\mathbf{R}^k$ to itself (that is, a smooth bijective map whose inverse is smooth). More generally, one might deal with diffeomorphisms between open subsets of k-manifolds or just $\mathbf{R}^k$, but let us not sweat the domain/codomain of α. Diffeomorphisms have plenty of other applications in mathematical physics (such as our 1-parameter families later) and elsewhere in mathematics.

One (most famous) example should suffice to illustrate this. Take $k = 2$ and think about polar coordinates. We get a couple of open subsets of $\mathbf{R}^2$ below, with the relevant diffeomorphism :

$$\alpha_{\text{polar}} \ : \ \mathbf{R}_{>0} \ \times \ ]-\frac{\pi}{2} \ , \ \frac{\pi}{2}[\quad \longrightarrow \quad \mathbf{R}_{>0} \times \mathbf{R}$$

$$(r, \theta) \ \mapsto \ (r\text{cos}\theta, r\text{sin}\theta)$$

$$[\sqrt{x^2+y^2} \ , \ \text{Tan}^{-1}(y/x)] \ \hookleftarrow\!\!| \ (x, y)$$

Given such a diffeomorphism α, its Jacobian matrix (map) is

$$J_\alpha : \mathbf{R}^k \to \mathbf{R}^{k \times k} \qquad \text{(the set of } k \times k \text{ matrices)}$$

given by

$$[J_\alpha(\vec{x})]_{ij} \ := \ (D_i\alpha_j)(\vec{x}) \ .$$

The right-hand side in the physicists' language is $\frac{\partial \alpha_j}{\partial x_i}$, where α_j is the jth component of α. The matrix $J_\alpha(\vec{x})$ is invertible with inverse equal to $J_{\alpha^{-1}}(\alpha(\vec{x}))$. But of course the Jacobian is definable for any smooth map, not necessarily bijective, and then non-singularity isn't likely.

Given a lagrangian $L : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}$ and a diffeomorphism α of $\mathbf{R}^k$, we first must define an appropriate corresponding $L_\alpha : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}$. We cannot just use $L \circ (\alpha \times \alpha)$, simply because

$$L[\alpha(\vec{w}(t)); \alpha(\dot{\vec{w}}(t))] \ \neq \ L[(\alpha \circ \vec{w})(t)); \dot{\overline{(\alpha \circ \vec{w})}}(t))] \ .$$

We do want to use composition with α in the 1st slot (see the definition below), and arrange things so that we get the correct thing (the right-hand side above) in the 2nd slot. In fact

$$\dot{\overline{(\alpha \circ \vec{w})}}(t)) \ = \ \dot{\vec{w}}(t) \ J_\alpha(\vec{w}(t)) \ ,$$

the right-hand side being of the form (row vector)·(matrix). This follows from the chain-rule calculation

$$[\frac{d(\alpha \circ \vec{w})(t)}{dt}]_i \;=\; \frac{d(\alpha_i \circ \vec{w})(t)}{dt} \;=\; \sum_j (D_j \alpha_i)[\vec{w}(t)] \;\cdot\; \dot{w}_j(t)$$

$$= \;\dot{\vec{w}}(t) \begin{pmatrix} J_\alpha[\vec{w}(t)]_{1i} \\ J_\alpha[\vec{w}(t)]_{2i} \\ \bullet \\ \bullet \\ J_\alpha[\vec{w}(t)]_{ki} \end{pmatrix} \;=\; \dot{\vec{w}}(t)\;(J_\alpha[\vec{w}(t)]_{\text{ith col}}) \;.$$

Definition.

$$L_\alpha(\vec{y}\;;\;\vec{z}) \;:=\; L[\alpha(\vec{y})\;;\;\vec{z}J_\alpha(\vec{y})] \;.$$

So, by the previous display, we now have

$$L_\alpha[\vec{w}(t)\;;\;\dot{\vec{w}}(t)] \;=\; L[(\alpha \circ \vec{w})(t))\;;\;\dot{\overline{(\alpha \circ \vec{w})}(t))}] \;.$$

As well as applying to smooth scalar-valued functions L, both the definition and the display just above will be regarded as applying to vector-valued functions, such as $\vec{\nabla}_* L$ and $\vec{\nabla}_{**} L$. So, for example,

$$(\vec{\nabla}_* L)_\alpha \;:=\; \vec{\nabla}_* L[\alpha(\vec{y})\;;\;\vec{z}J_\alpha(\vec{y})] \;.$$

The main identities below, $(*)$ and $(**)$, relate $\vec{\nabla}_* L_\alpha$ and $\vec{\nabla}_{**} L_\alpha$, which maybe should be written $\vec{\nabla}_*(L_\alpha)$ and $\vec{\nabla}_{**}(L_\alpha)$, to $(\vec{\nabla}_* L)_\alpha$ and $(\vec{\nabla}_{**} L)_\alpha$.

The result we want is this :

Lemma. *The Lagrange equation for L_α, which we'll denote as Lag(L_α), has the curve $\vec{w}$ as a solution if and only if the Lagrange equation for L has $\alpha \circ \vec{w}$ as a solution.*

Symmetry shows that we need only prove half of this, namely :

$$\alpha \circ \vec{w} \text{ solves Lag}(L) \;\;\overset{*}{\Longrightarrow}\;\; \vec{w} \text{ solves Lag}(L_\alpha) \;.$$

To see why half suffices, prove the converse of the above as follows :

$$\vec{w} \text{ solves Lag}(L_\alpha) \;\;\Longrightarrow\;\; \alpha^{-1} \circ (\alpha \circ \vec{w}) \text{ solves Lag}(L_\alpha)$$

$$\Longrightarrow\;\; \alpha \circ \vec{w} \text{ solves Lag}((L_\alpha)_{\alpha^{-1}}) \;\;\Longrightarrow\;\; \alpha \circ \vec{w} \text{ solves Lag}(L) \;.$$

The first and last implications are simply because, respectively,

$\alpha^{-1} \circ \alpha \circ \vec{w} = \vec{w}$ (obvious), and $(L_\alpha)_{\alpha^{-1}} = L$ (easy exercise).

The middle one is $\stackrel{*}{\Longrightarrow}$ with $[\alpha, \vec{w}, L]$ changed to $[\alpha^{-1}, \alpha \circ \vec{w}, L_\alpha]$.

To prove $\stackrel{*}{\Longrightarrow}$ is a somewhat tedious calculation for the mathematically scrupulous; the corresponding manipulation in the physicists' language with lots of symbols ∂ only takes 5 or 6 equal signs and less ink (though some of these equalities may make many of their insightful graduate students nervous, at least until they have full command of which manipulations can be used when, and which cannot—e.g. "cancel the dots").

Define

$$J_\alpha^+ : \mathbf{R}^k \to \mathbf{R}^{k\times k\times k} \qquad \text{(the set of } k \times k \times k \text{ tensors)}$$

by

$$[J_\alpha^+(\vec{x})]_{ijp} \;:=\; (D_i D_j \alpha_p)(\vec{x}) \ .$$

The right-hand side in the other language is $\frac{\partial^2 \alpha_p}{\partial x_i \partial x_j}$, bringing to mind the word "Hessian".

I claim

$$(\vec{\nabla}_* L_\alpha)(\vec{y}, \vec{z}) \;=\; (\vec{\nabla}_* L)_\alpha(\vec{y}, \vec{z}) \cdot J_\alpha(\vec{y}) \;+\; (\vec{\nabla}_{**} L)_\alpha(\vec{y}, \vec{z}) \cdot [\vec{z} \cdot J_\alpha^+(\vec{y})] \qquad (*)$$

and

$$(\vec{\nabla}_{**} L_\alpha)(\vec{y}, \vec{z}) \;=\; (\vec{\nabla}_{**} L)_\alpha(\vec{y}, \vec{z}) \cdot J_\alpha(\vec{y}) \qquad (**)$$

Before proving $(*)$ and $(**)$, complete $\stackrel{*}{\Longrightarrow}$'s proof as follows :

$$\frac{d}{dt}(\vec{\nabla}_{**} L_\alpha[\vec{w}(t), \dot{\vec{w}}(t)]) \;=\; \frac{d}{dt}(\vec{\nabla}_{**} L[(\alpha \circ \vec{w})(t) \ ; \ \dot{\overline{(\alpha \circ \vec{w})}}(t)] \cdot J_\alpha[\vec{w}(t)])$$

$$= \; (\vec{\nabla}_* L)[(\alpha\circ\vec{w})(t) \ ; \ \dot{\overline{(\alpha \circ \vec{w})}}(t)] \cdot J_\alpha[\vec{w}(t)] + (\vec{\nabla}_{**} L)[(\alpha\circ\vec{w})(t) \ ; \ \dot{\overline{(\alpha \circ \vec{w})}}(t)] \cdot \frac{d}{dt}(J_\alpha[\vec{w}(t)]) \ . \ .$$

The equalities are justified respectively by $(**)$, and by the product rule plus the fact that $\alpha \circ \vec{w}$ solves $\mathrm{Lag}(L)$.

On the other hand, substituting $[\vec{w}(t), \dot{\vec{w}}(t)]$ for $[\vec{y}, \vec{z}]$ in $(*)$ gives the same answer for $(\vec{\nabla}_* L_\alpha)[\vec{w}(t), \dot{\vec{w}}(t)]$, as required to verify Lagrange's equation, modulo checking the far right-hand ends of the two expressions agree, namely :

$$\frac{d}{dt}(J_\alpha[\vec{w}(t)]) \;=\; \dot{\vec{w}}(t) \cdot J_\alpha^+[\vec{w}(t)] \;.$$

The (i,j)th component of this is simply the following instance of the chain rule :

$$\frac{d}{dt}(D_i\alpha_j[\vec{w}(t)]) \;=\; \sum_p \dot{\vec{w}}_p(t) \cdot D_p D_i \alpha_j[\vec{w}(t)] \;.$$

To prove $(**)$, check that the ith components agree, for $i \le k$:

$$D_{i+k}L_\alpha(\vec{y}, \vec{z}) \;=\; \frac{\partial L_\alpha}{\partial z_i}(\vec{y}, \vec{z}) \;=\; \frac{\partial}{\partial z_i} L[\alpha\vec{y}, \vec{z}\cdot J_\alpha(\vec{y})]$$

$$= \sum_j \frac{\partial L}{\partial y_j}[\alpha\vec{y}, \vec{z}\cdot J_\alpha(\vec{y})] \frac{\partial}{\partial z_i}[(\vec{y}, \vec{z}) \mapsto \alpha_j(\vec{y})] + \sum_j \frac{\partial L}{\partial z_j}[\alpha\vec{y}, \vec{z}\cdot J_\alpha(\vec{y})] \frac{\partial}{\partial z_i}[(\vec{y}, \vec{z}) \mapsto [\vec{z}\cdot J_\alpha(\vec{y})]_j]$$

$$= \quad 0 \quad + \vec{\nabla}_{**}L[\alpha\vec{y}, \vec{z}\cdot J_\alpha(\vec{y})] \cdot J_\alpha(\vec{y})_{i\text{th col.}} \;=\; (\vec{\nabla}_{**}L)_\alpha(\vec{y}, \vec{z}) \cdot J_\alpha(\vec{y})_{i\text{th col.}}$$

Similarly for the ith component of $(*)$:

$$D_i L_\alpha(\vec{y}, \vec{z}) \;=\; \frac{\partial L_\alpha}{\partial y_i}(\vec{y}, \vec{z}) \;=\; \frac{\partial}{\partial y_i} L[\alpha\vec{y}, \vec{z}\cdot J_\alpha(\vec{y})]$$

$$= \sum_j \frac{\partial L}{\partial y_j}[\alpha\vec{y}, \vec{z}\cdot J_\alpha(\vec{y})] \frac{\partial}{\partial y_i}[(\vec{y}, \vec{z}) \mapsto \alpha_j(\vec{y})] + \sum_j \frac{\partial L}{\partial z_j}[\alpha\vec{y}, \vec{z}\cdot J_\alpha(\vec{y})] \frac{\partial}{\partial y_i}[(\vec{y}, \vec{z}) \mapsto [\vec{z}\cdot J_\alpha(\vec{y})]_j]$$

$$= \vec{\nabla}_* L[\alpha\vec{y}, \vec{z}\cdot J_\alpha(\vec{y})] \cdot J_\alpha(\vec{y})_{i\text{th col.}} \;+\; \vec{\nabla}_{**}L[\alpha\vec{y}, \vec{z}\cdot J_\alpha(\vec{y})] \cdot [\vec{z}J_\alpha^+(\vec{y})]_{i\text{th col.}}$$

$$= (\vec{\nabla}_* L)_\alpha(\vec{y}, \vec{z}) \cdot J_\alpha(\vec{y})_{i\text{th col.}} \;+\; (\vec{\nabla}_{**}L)_\alpha(\vec{y}, \vec{z}) \cdot [\vec{z}J_\alpha^+(\vec{y})]_{i\text{th col.}}$$

The disciple of fastidiously unambiguous mathematical notation can, everywhere above, replace $\frac{\partial}{\partial y_i}$ by D_i , and replace $\frac{\partial}{\partial z_i}$ by D_{k+i} , and perhaps start the last two displays with $D_i(L_\alpha)(\vec{y}, \vec{z})$, to distinguish it from $(D_iL)_\alpha(\vec{y}, \vec{z})$, and with $D_{i+k}(L_\alpha)(\vec{y}, \vec{z})$.

This completes the proof and the appendix.

Noether's general construction of conserved quantities.

> Noether's proof ... is commonly regarded to be difficult to present in university texts.
>
> N.H. Ibragimov

Let us get back to the topic of conserved quantities (AKA "constants of the motion"; AKA "first integrals"), and Emmy Noether's deep insight into relating these to continuous symmetries of the system. At first below there will appear not to be anything involving groups or representations or Lie theory; but be patient.

Let $\mathcal{Q} := C^\infty(T \times \mathcal{P}, \mathbf{R})$, the algebra of quantities. (I might as well use the word, since it's so popular in the mathematical physics literature!) Fix a Lagrangian L. Define such a quantity q to be an L-conserved quantity if and only if, for all solutions $\vec{w}$ of the Lagrange equations, the number $q(t, \vec{w})$ is independent of t ; i.e., $q(t_1, \vec{w}) = q(t_2, \vec{w})$ for all t_1, t_2 in T; i.e., for each such $\vec{w}$, we find that $\frac{dq(t,\vec{w})}{dt}$ is the zero function.

(Later, in hamiltonian mechanics, we shall follow tradition and use "$\vec{q}$" to denote position coordinates. This has nothing to do with 'q for quantity' above.)

There is a map $C^\infty(\mathbf{R}^k \times \mathbf{R}^k, \mathbf{R}) \to \mathcal{Q}$ given by

$$F \mapsto [(t, \vec{w}) \mapsto F(\vec{w}(t), \dot{\vec{w}}(t)] \ .$$

Conserved quantities are normally discussed ab initio as if they all came from such F; that is, could be regarded as functions on velocity space, namely those F for which $\frac{d}{dt}[F(\vec{w}(t), \dot{\vec{w}}(t)]$ is the zero function for every 'orbit' $\vec{w}$ of the L-theory (nothing to do with surgery!). However, I can see no easy way (at least not in the lagrangian version of the subject) without the (apparently, not really) more general notion in the definition above, to discuss Noether's theorem in such a way that conservation of energy, linear momentum and angular momentum all drop out easily as special cases.

Interrupt to reconcile the two approaches. Many students of physics will (and all should) look askance at the penultimate paragraph above. They ought to insist to me that a conserved quantity be a function on velocity space, as in the last paragraph (or equivalently as a function on *phase space*, discussed below in the section on hamiltonian mechanics). After all, a physical quantity is called 'an observable', and should be directly observable or computable from such. This is entirely reasonable, but it is worth pointing out that this reasonableness depends on a strong and well-justified belief that all the fundamental laws of macroscopic physics are given by 2nd order DE's (at most—not *higher* order). Not just conserved quantities, but the entire edifice of the lagrangian and hamiltonian approaches rests on that belief.

For example, if some physical law turned out to be $\frac{d}{dt}\ddot{\vec{w}} = \vec{0}$, then a conserved quantity would be $(\vec{w}, t) \mapsto |\ddot{\vec{w}}(t)|$, since $\ddot{\vec{w}}$ itself is a conserved 'vector quantity'. But there is no function F of six real variables such that $|\ddot{\vec{w}}(t)| = F(\vec{w}(t), \dot{\vec{w}}(t))$ for all solutions $\vec{w}$.

However we can see as follows that all conserved quantities for lagrangian systems really are coming from functions on velocity space. For this, the rather formal details in the next several paragraphs must be supplemented by something more geometric/analytic, namely a basic existence/uniqueness/smoothness theorem for DEs. *For a given $[\vec{y}, \vec{z}]$ in velocity space and a given lagrangian and t_0, there is a unique smooth path $\vec{w}$ satisfying Lagrange's equation and for which $[\vec{w}(t_0), \dot{\vec{w}}(t_0)] = [\vec{y}, \vec{z}]$.* Then, given a conserved quantity q as defined above, one can specify F as in the paragraph just above this small print by $F(\vec{y}, \vec{z}) = q(t_0, \vec{w})$ for the unique $\vec{w}$ above. It is immediate to see that this F is well-defined (depends only on q), and that q comes from this F as in the paragraph above this small print. Thus, after-the-fact, all the conserved quantities do come from such F, though not in the simple sense of the theorem Noether-Version II (several paragraphs ahead), which is less general than Noether-Version I (just ahead). By the time we're done with first specializing the mildly objectionable theorem just below to the case of symmetries of configuration space, and then generalizing the latter in hamiltonian mechanics to symmetries of phase space, we shall be safely back in the traditional camp, and also will have 'captured' all the conserved quantities known to classical (non-quantum) man.

Recall that $\mathcal{P}$ is the space of paths in $\mathbf{R}^k$, i.e. smooth $\vec{w} : T \to \mathbf{R}^k$ for an open interval T of reals. We shall denote by $\mathcal{P}^{\mathcal{P}}$ a set of functions from $\mathcal{P}$ to itself. Just below we introduce "smooth" functions $\vec{G}$ from an interval of reals into $\mathcal{P}^{\mathcal{P}}$. There *smooth* (for the time anyway) means that the derivatives, $\frac{d\vec{G}}{ds} : S \to \mathcal{P}^{\mathcal{P}}$ and $\dot{\vec{G}} : S \to \mathcal{P}^{\mathcal{P}}$, (and their properties used just below) exist (and hold). These derivatives are defined by limits, just as in elementary calculus, after evaluation, so everything is taking place in Euclidean space, and we don't need to get into any structure on $\mathcal{P}^{\mathcal{P}}$, such as its topology. See also the discussion of the 'interchange of limits' just after the next proof.

Let S be an open interval of reals, generically named s, with $0 \in S$. (The domain, S, of s on which the following $\vec{G}$ is defined may be thought of as a kind of abstract 'Lie theoretic time', as we see much later.) Suppose given $\vec{G} : S \to \mathcal{P}^{\mathcal{P}}$ with $\vec{G}(0) =$ the identity map of $\mathcal{P}$; that is $\vec{G}(0)(\vec{w}) = \vec{w}$ for all $\vec{w} \in \mathcal{P}$.

Theorem. (Noether-Version I.) *Given L and $\vec{G}$, suppose that E is a function in $C^\infty(\mathbf{R}^k \times \mathbf{R}^k, \mathbf{R})$ such that, for all orbits $\vec{w}$ of L-theory, we have*

$$\frac{d}{dt}[E(\vec{w}(t), \dot{\vec{w}}(t)] \;=\; \frac{d}{ds}(L[\vec{G}(s)(\vec{w})(t), \dot{\vec{G}}(s)(\vec{w})(t)])|_{s=0} \quad .$$

Then the following defines an L-conserved quantity:

$$q(t, \vec{w}) \;:=\; < \; \vec{\nabla}_{**}L[\vec{w}(t), \dot{\vec{w}}(t)] \; , \; \frac{d\vec{G}}{ds}(0)(\vec{w})(t) \; > \; - \; E[\vec{w}(t), \dot{\vec{w}}(t)] \quad .$$

Explicating and 'expliciting':
The inner product on the right-hand side of the formula is the standard one on $\mathbf{R}^k$.

Define $\frac{d\vec{G}}{ds} : S \to \mathcal{P}^{\mathcal{P}}$ by fixing $(t, \vec{w})$ in $\vec{G}(s)(\vec{w})(t)$ and differentiating.

Define $\dot{\vec{G}} : S \to \mathcal{P}^{\mathcal{P}}$ by fixing $(s, \vec{w})$ in $\vec{G}(s)(\vec{w})(t)$ and differentiating.

In a later example with $\vec{G}(s)(\vec{w})(t) = \vec{w}(t) + s\vec{b}$ for a fixed vector $\vec{b}$, we can easily check that

$$\dot{\vec{G}}(s)(\vec{w})(t) = \dot{\vec{w}}(t) \quad ; \quad \vec{G}(s)(\dot{\vec{w}})(t) = \dot{\vec{w}}(t) + s\vec{b} \quad ; \text{ and } \quad \frac{d\vec{G}}{ds}(s)(\vec{w})(t) = \vec{b} \; .$$

So the three left-hand sides are usually distinct.

Note that $\dot{\vec{G}}(s)(\vec{w}) = \dot{\overline{\vec{G}(s)(\vec{w})}}$, where the overline is just to indicate the range of the overdot. And so the 2nd variable in L in the hypothesis of the theorem is the 'correct' one.

Proof. Fix a $\vec{w}$ which satisfies Lagrange's equation. We need only to differentiate, with respect to t, firstly the inner product half of $q(t, \vec{w})$ [using a suitable product rule followed by Lagrange's equation]; and then the E-half [using its assumed property followed by a chain rule]; and finally to see that the answers agree:

Firstly

$$\frac{d}{dt} < \vec{\nabla}_{**}L[\vec{w}(t);\dot{\vec{w}}(t)] \ , \ \frac{d\vec{G}}{ds}(0)(\vec{w})(t) \ >$$

$$= \ < \frac{d}{dt}(\vec{\nabla}_{**}L[\vec{w}(t);\dot{\vec{w}}(t)]) \ , \ \frac{d\vec{G}}{ds}(0)(\vec{w})(t) \ > \ + \ < \vec{\nabla}_{**}L[\vec{w}(t);\dot{\vec{w}}(t)] \ , \ \frac{d}{dt}[\frac{d\vec{G}}{ds}(0)(\vec{w})(t)] \ >$$

$$= \ < \vec{\nabla}_{*}L[\vec{w}(t);\dot{\vec{w}}(t)]) \ , \ \frac{d\vec{G}}{ds}(0)(\vec{w})(t) \ > \ + \ < \vec{\nabla}_{**}L[\vec{w}(t);\dot{\vec{w}}(t)] \ , \ \frac{d\dot{\vec{G}}}{ds}(0)(\vec{w})(t)] \ > \ .$$

On the other hand

$$\frac{d}{dt}[E(\vec{w}(t),\dot{\vec{w}}(t)] \ = \ \frac{d}{ds}(L[\vec{G}(s)(\vec{w})(t);\dot{\vec{G}}(s)(\vec{w})(t)])|_{s=0}$$

$$= \ < \ \vec{\nabla}_{*}L[\vec{G}(s)(\vec{w})(t);\dot{\vec{G}}(s)(\vec{w})(t)]) \ , \ \frac{d\vec{G}}{ds}(s)(\vec{w})(t) \ >_{s=0} \ +$$

$$< \ \vec{\nabla}_{**}L[\vec{G}(s)(\vec{w})(t);\dot{\vec{G}}(s)(\vec{w})(t)] \ , \ \frac{d\dot{\vec{G}}}{ds}(s)(\vec{w})(t)] \ >_{s=0} \ ,$$

which, upon substituting 0 for s (except in $\frac{d}{ds}$!!) agrees with the previous, as required, since $\vec{G}(0)(\vec{w}) = \vec{w}$ is assumed, and therefore $\dot{\vec{G}}(0)(\vec{w}) = \dot{\vec{w}}$.

Note that the very last (huge!) displayed expression would need one more summand if we had a Lagrangian that "depends explicitly on t", as the experts like to say.

Notice also that the last equality in the first calculation uses interchange of the order of differentiation with respect to s and t, which essentially we take as part of our definition of "smooth" for $\vec{G}$. But let's get quite explicit about this, since the present write-up is supposed to be entirely 'unassuming'. The interchange referred to above reduces to the equality of the limit, first as $s \to 0$ then as $u \to t$ of the expression

$$\frac{\vec{G}(s)(\vec{w})(u) - \vec{G}(0)(\vec{w})(u) - \vec{G}(s)(\vec{w})(t) + \vec{G}(0)(\vec{w})(t)}{s(u-t)} \ ,$$

with the same limit, except that it is taken in the other order. But this amounts to precisely the equality of the so-called mixed partial derivatives,

$$D_2 D_1 \vec{F}(0,t) \ = \ D_1 D_2 \vec{F}(0,t) \ ,$$

i.e. the good old '$\frac{\partial}{\partial x \partial y} = \frac{\partial}{\partial y \partial x}$', where $\vec{F}$ is the function mapping (s, u) to $\vec{G}(s)(\vec{w})(u)$ for the fixed $\vec{G}$ and $\vec{w}$. See, for example, [**Spivak**], Theorem 2-5, p. 26, whose hypothesis is much weaker than the smoothness we are assuming. Since $\vec{F}$ is vector-valued, we are applying that theorem to each coordinate.

Next come the three most famous examples in mechanics of conserved quantities. Later we shall deal with the Runge-Lenz phenomena of the Kepler problem in the context of hamiltonian mechanics.

Time translation symmetry produces conservation of energy.

Define $\vec{G}$ by $\vec{G}(s)(\vec{w})(t) := \vec{w}(t+s)$, which gives $\vec{w}(t)$ when $s = 0$, as needed. So $\dot{\vec{G}}(s)(\vec{w})(t) := \dot{\vec{w}}(t+s)$. Then we find that $E = L$ works nicely:

$$\frac{d}{ds}(L[\vec{w}(t+s); \dot{\vec{w}}(t+s)])|_{s=0} \;=\; \frac{d}{dt}[L(\vec{w}(t); \dot{\vec{w}}(t)] \quad ,$$

as required. (We have just applied the 'rule'

$$\frac{d}{ds}f(t+s)|_{s=0} \;=\; \frac{d}{dt}f(t)$$

from high school calculus.) So here the conserved quantity is

$$q(t, \vec{w}) \;:=\; <\; \vec{\nabla}_{**}L[\vec{w}(t); \dot{\vec{w}}(t)]\;,\; \dot{\vec{w}}(t) \;>\; - \; L[\vec{w}(t), \dot{\vec{w}}(t)] \quad .$$

In the earlier specific example with $N = 1$ and

$$L(\vec{y}\;;\;\vec{z}) \;=\; \frac{1}{2}m|\vec{z}|^2 \;+\; \frac{mMG}{|\vec{y}|} \quad ,$$

we got

$$\vec{\nabla}_{**}L(\vec{y}\;;\;\vec{z}) \;=\; m\vec{z} \quad .$$

So the conserved quantity is

$$q(t, \vec{w}) \;:=\; <\; m\dot{\vec{w}}(t)\;,\; \dot{\vec{w}}(t) \;>\; - \left(\frac{1}{2}m|\dot{\vec{w}}(t)|^2 + \frac{mMG}{|\vec{w}(t)|}\right) \;=\; \frac{1}{2}m|\dot{\vec{w}}(t)|^2 - \frac{mMG}{|\vec{w}(t)|} \,,$$

our good old formula for the energy.

Actually, in our highest level of generality here (which only works because we didn't allow Langrangians, $L(\vec{y}\ ;\ \vec{z}\ ;\ t)$, which depend explicitly on t), define the pre-Hamiltonian by

$$H^{\mathrm{pre}}(\vec{y}\ ;\ \vec{z})\ :=\ <\ \vec{\nabla}_{**}L(\vec{y};\vec{z})\ ,\ \vec{z}\ >\ -\ L(\vec{y};\vec{z})\ .$$

Then $H^{\mathrm{pre}}[\vec{w}(t);\dot{\vec{w}}(t)]$ is exactly our conserved quantity $q(t,\vec{w})$ above, and is usually identified physically with the total energy of the system.

Later we shall define the hamiltonian 'as a function of $(\vec{q},\vec{p})$' in such a way that when q and p are changed to $\vec{w}(t)$ and $\vec{\nabla}_{**}L(\vec{w}(t);\dot{\vec{w}}(t))$ respectively, it agrees with H^{pre} above.

The results in **9.1** and **9.2** can be quickly obtained as special cases of this first application of Noether-Version I.

There seems to be a sort of 'cannibalism' here, in that, to define L, energy must somehow be identified and decomposed into kinetic and potential; but then L is used to help identify energy as a particular conserved quantity.

Space translation gives conservation of linear momentum.

In this application we'll definitely have $k = 3N$ for some N, and fix a $\vec{b} \in \mathbf{R}^k$. Define $\vec{G}$ by $\vec{G}(s)(\vec{w})(t) := \vec{w}(t) + s\vec{b}$, which gives $\vec{w}(t)$ when $s = 0$, as needed. So $\dot{\vec{G}}(s)(\vec{w})(t) := \dot{\vec{w}}(t)$. Then we find that $E = 0$ works nicely sometimes:

We want to consider an arbitrary finite number of particles. Introduce subscripts with generic name α which range from 1 to N. Then the αth particle will have mass m_α; and the physical positions of the particles will correspond mathematically to $\vec{y}_{(1)} = (y_1, y_2, y_3)$, $\vec{y}_{(2)} = (y_4, y_5, y_6)$, etc.; and their velocities to $\vec{z}_{(1)} = (z_1, z_2, z_3)$, $\vec{z}_{(2)} = (z_4, z_5, z_6)$, etc.

Consider a Lagrangian

$$L(\vec{y}\ ;\ \vec{z})\ :=\ \sum_\alpha \frac{1}{2} m_\alpha |\vec{z}_{(\alpha)}|^2\ -\ V(\vec{y})\ .$$

The potential half will be semi-general; more specifically, V is assumed to satisfy :

(i) it is independent of $\vec{z}$, as usual;

(ii) its dependence physically is only on the *relative* positions of the particles—mathematically, assume there is a function V_1 such that

$$V(\vec{y})\ =\ V_1(y_i - y_j\ :\ i - j = 3 \text{ or } 6 \text{ or } 9 \cdots)\ .$$

More succinctly, the condition is $i > j$ and $i \equiv j(\text{mod}3)$.

Now assume also that $\vec{b}$ has the form $(b_1, b_2, b_3, b_1, b_2, b_3, b_1, b_2, b_3, \cdots, b_1, b_2, b_3)$. That is, via $\vec{G}$, all the different particles are physically translated by the same (sb_1, sb_2, sb_3)—different particles don't have different translations. But then $V(\vec{w}(t) + s\vec{b}) = V(\vec{w}(t))$ since, for $i > j$ and $i \equiv j(\text{mod}3)$, we have

$$(y_i + sb_i) - (y_j + sb_j) = y_i - y_j + s(b_i - b_j) = y_i - y_j + s \cdot 0 = y_i - y_j \ .$$

Thus we get no dependence on s in the function to be differentiated by s in the assumptions of Noether's theorem:

$$L[\vec{G}(s)(\vec{w})(t), \dot{\vec{G}}(s)(\vec{w})(t)] \ = \ L[\vec{w}(t) + s\vec{b}, \dot{\vec{w}}(t)]$$

$$= \ \frac{1}{2}\sum_{\alpha} m_\alpha |\dot{\vec{w}}_{(\alpha)}(t)|^2 \ - \ V(\vec{w}(t) + s\vec{b}) \ = \ \frac{1}{2}\sum_{\alpha} m_\alpha |\dot{\vec{w}}_{(\alpha)}(t)|^2 \ - \ V(\vec{w}(t)) \ ,$$

as required. So we may actually take E to be the zero function in this example.

Just as before,

$$\vec{\nabla}_{**} L(\vec{y} \ ; \ \vec{z}) \ = \ \bigoplus_{\alpha} m_{(\alpha)} \vec{z}_{(\alpha)} \quad ,$$

where the $\bigoplus$ indicates stringing the $\mathbf{R}^3$-vectors together to get an $\mathbf{R}^{3N}$-vector.

So here the conserved quantity is

$$q(t, \vec{w}) \ := \ < \ \bigoplus_{\alpha} m_{(\alpha)} \dot{\vec{w}}_{(\alpha)}(t) \ , \ \vec{b} \ >_{\mathbf{R}^{3N}} \ = \ < \ \sum_{\alpha} m_{(\alpha)} \dot{\vec{w}}_{(\alpha)}(t) \ , \ (b_1, b_2, b_3) \ >_{\mathbf{R}^3} \ .$$

Since this is true for every $(b_1, b_2, b_3) \in \mathbf{R}^3$, we get a conserved vector, namely linear momentum, which is

$$\sum_{\alpha} m_{(\alpha)} \dot{\vec{w}}_{(\alpha)}(t) \ .$$

N.B. The Kepler problem with the sun nailed down, that is, the less refined version, does not satisfy the hypothesis just above about potential energy depending only on relative position. For a single particle system, the potential energy would have to be constant for that to hold.

Even more refined Kepler. Analyze the two-body problem as a 2-particle system, that is, $N = 2$ above. We *do* get conservation of momentum, which in much earlier notation, is simply

$$m\dot{\vec{w}}(t) \ + \ M\dot{\vec{W}}(t) \ .$$

We have the two $m_{(\alpha)}$'s as m and M, and $\vec{w}$ just above is $(w_1, w_2, w_3, W_1, W_2, W_3)$ from the much earlier notation. The dependence of potential energy only on relative position is clear here from

$$V(\vec{w}, \vec{W}) \ = \ \frac{MG}{|\vec{w} - \vec{W}|} \ + \ \frac{mG}{|\vec{W} - \vec{w}|} \ .$$

Adding (sb_1, sb_2, sb_3) to both $\vec{W}$ and $\vec{w}$ does not change this number, as required.

Physically one imagines all the mass, $m + M$, concentrated at the centre of mass, and it is that imaginary glob whose momentum is being conserved, as we already know anyway.

Rotational symmetry gives conservation of angular momentum.

Let $s \mapsto A_s$ be a smooth path of matrices A_s in $SO(3)$. The latter is the group of 3×3 matrices, $\{A \ : \ A^{\rm tr}A = I \ ; \ \det A = 1\}$, of 'rotations' in $\mathbf{R}^3$. For example, we could have

$$A_s \ = \ \begin{pmatrix} \cos(s) & 0 & -\sin(s) \\ 0 & 1 & 0 \\ \sin(s) & 0 & \cos(s) \end{pmatrix}$$

Or we could concentrate the 2×2 rotation matrix into the north-west corner (or the south-east). We shall assume that $A_0 = I$, as in the three examples above. So if we define

$$\vec{G}(s)(\vec{w})(t) \ := \ A_s \vec{w}(t) \quad \text{[matrix times column vector]} \ ,$$

then $\vec{G}(0)$ is the identity function, as needed. We are taking $N = 1$ here. Note that $\dot{\vec{G}}(s)(\vec{w})(t) \ := \ A_s \dot{\vec{w}}(t)$.

A property of (which almost characterizes) matrices A in $SO(3)$ is that $|A\vec{u}| = |\vec{u}|$ for all $\vec{u} \in \mathbf{R}^3$. Here we shall assume that

$$L(\vec{y} \ ; \ \vec{z}) \ = \ L_1(|\vec{y}| \ ; \ |\vec{z}|)$$

for some function L_1 . That is, the Lagrangian depends only on the *lengths* of the two vector inputs (the position and velocity, physically). Then

$$L[\vec{G}(s)(\vec{w})(t), \dot{\vec{G}}(s)(\vec{w})(t)] \;=\; L[A_s\vec{w}(t), A_s\dot{\vec{w}}(t)]$$

$$= \; L_1(|A_s\vec{w}(t)|, |A_s\dot{\vec{w}}(t)|) \;=\; L_1(|\vec{w}(t)|, |\dot{\vec{w}}(t)|) \;,$$

which is independent of s. So here again we may take E to be the zero function.

Now $A_s^{\mathrm{tr}} A_s = I$ for all s. Let

$$B_s := \frac{dA_s}{ds} \;,$$

also a 3×3-matrix, but most unlikely to be in $SO(3)$, as we see below. Easily,

$$\frac{dA_s^{\mathrm{tr}}}{ds} \;=\; \left(\frac{dA_s}{ds}\right)^{\mathrm{tr}} \;=\; B_s^{\mathrm{tr}} \;,$$

since differentiation is term-by-term. By an easily verified product rule,

$$0 \;=\; \frac{dI}{ds} \;=\; \frac{d(A_s^{\mathrm{tr}} A_s)}{ds} \;=\; B_s^{\mathrm{tr}} A_s + A_s^{\mathrm{tr}} B_s \;.$$

Putting $s = 0$ gives $B_0^{\mathrm{tr}} = -B_0$, so B_0 is a skew symmetric matrix, examples of which, corresponding to the previous three simple rotation matrix paths are

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \;, \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \;.$$

Now

$$\frac{d\vec{G}}{ds}(s)(\vec{w})(t) \;=\; B_s\vec{w}(T) \;.$$

Thus the conserved quantity is

$$q(t, \vec{w}) \;:=\; < \; \vec{\nabla}_{**}L[\vec{w}(t), \dot{\vec{w}}(t)] \;, \; B_0\vec{w}(t) \; > \;.$$

If we specialize as usual the kinetic energy half of L, so

$$L(\vec{y} \;;\; \vec{z}) \;=\; \frac{1}{2}m|\dot{\vec{z}}|^2 \;-\; V(\vec{y}) \;\;,$$

keeping the proviso that V depends only on $|\vec{y}|$, as before we get

$$\vec{\nabla}_{**}L(\vec{y}\ ;\ \vec{z}) \ = \ m\vec{z}\ ,$$

so

$$q(t,\vec{w}) \ = \ m < \ \dot{\vec{w}}(t)\ ,\ B_0\vec{w}(t) \ > \ .$$

For example, using

$$A_s \ = \ \begin{pmatrix} \cos(s) & -\sin(s) & 0 \\ \sin(s) & \cos(s) & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \text{ so } \ B_0 \ = \ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ ,$$

the conserved quantity is

$$m < \ \vec{v}(t)\ ,\ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \ >$$

$$= \ m < \ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \ , \ \begin{pmatrix} -w_2 \\ w_1 \\ 0 \end{pmatrix} \ > \ = \ m(v_2w_1 - v_1w_2)\ .$$

Using the other two examples, you will of course find that the two other conserved quantities are the other two slots in the vector cross product of the momentum and position vectors. So this shows that $m\vec{v}\times\vec{w}$ is conserved, the angular momentum vector, as we saw very early in the particular case of the Kepler problem. Since our three examples of skew-symmetric matrices form a basis for the space of all such matrices, nothing new is obtained by trying any other B_s-examples. (No B.S. in saying that!)

Simplifying to the case of a symmetry of configuration space. The simplest lagrangian version of Noether's theorem.

The latter two of the three general examples of version I of Noether's theorem can be captured by the following 2nd version, one which is often seen in texts. Let us assume a 1-parameter continuous local action on configuration space itself, say $\vec{g} : S \to C^\infty(\mathbf{R}^k, \mathbf{R}^k)$, which is smooth, with $\vec{g}(0)$ being the identity diffeomorphism of $\mathbf{R}^k$. Adjointly, this is equivalent to a map $S \times \mathbf{R}^k \to \mathbf{R}^k$, so smoothness is particularly straightforward to formulate.

Theorem. (Noether-Version II.) *Assume*

$$\frac{d}{ds} L[\vec{g}(s)(\vec{w}(t))\ ;\ \dot{\overline{\vec{g}(s)(\vec{w}(t))}}]\ |_{s=0}\ =\ 0$$

Then the function $r : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}$ *given by*

$$r(\vec{y}, \vec{z})\ :=\ <\ \vec{\nabla}_{**} L(\vec{y}, \vec{z})\ ,\ \frac{d\vec{g}}{ds}(0)(\vec{y})\ >$$

is a conserved quantity.

The proof is to specialize the previous theorem—see the paragraph just before the interrupt which discusses what a conserved quantity should be in general. One can alternatively easily prove this last theorem directly, imitating the proof of the earlier more general version. The previous examples of linear and angular momentum are the cases of the above where we take

$$\vec{g}(s)(\vec{y}) = \vec{y} + sb \quad \text{and} \quad \vec{g}(s)(\vec{y}) = A_s \vec{y} \quad \text{respectively.}$$

Sometimes, e.g.[**Tong**] p.29, this simpler version is apparently supposed to have conservation of energy as an immediate special case, though I have been unable to see how. On the other hand, there is a mention in [**Arnold**], pp.90-91, Problem 4—"Extend Noether's theorem to non-autonomous lagrangian systems", of how to see energy conservation as an example of the last theorem, using a trick which alters the concept of configuration space. Since there is something a little odd about this, here is a brief description of the trick and the oddity.

Given any lagrangian $L : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}$ as usual, define a new one

$$L_1 : (\mathbf{R}^k \times T) \times (\mathbf{R}^k \times T) \to \mathbf{R}$$

by $$(\vec{y}, u, \vec{z}, v) \quad \mapsto \quad vL[\vec{y}\ ;\ v^{-1}\vec{z}]\ .$$

Here T ("time") is just another copy of (or, more likely, open subset of) $\mathbf{R}$, so we have changed k to $k+1$ (Note that they can't *both* be divisible by 3; and we continue not to sweat the actual domains used—cleaning up that aspect is completely trivial and almost entirely irrelevant until we get to something which hinges on global topological questions).

Now, assuming that

$$\vec{w} \text{ solves } \mathrm{Lag}(L) \quad \stackrel{?}{\Longrightarrow} \quad [t \mapsto (\vec{w}(t), t)] \text{ solves } \mathrm{Lag}(L_1)$$

(as discussed below), take our 1-parameter group on $\mathbf{R}^k \times T$ to be

$$\vec{g}_1 : S \to C^\infty(\mathbf{R}^k \times T\ ,\ \mathbf{R}^k \times T)$$

given by $$s \quad \mapsto \quad [(\vec{x}, t) \mapsto (\vec{x}, t+s)]\ .$$

(So this *is* time translation, as in the first example after Version I of Noether's theorem.) Then

$$\vec{g}_1(s)(\vec{w}(t), t) \ = \ (\vec{w}(t), t+s) \quad \text{and} \quad \dot{\overline{\vec{g}_1(s)(\vec{w}(t), t)}} \ = \ (\dot{w}(t), 1)\ .$$

And so

$$L_1[\vec{g}_1(s)(\vec{w}(t), t)\ ;\ \dot{\overline{\vec{g}_1(s)(\vec{w}(t), t)}}] \ = \ L[\vec{w}(t)\ ;\ \dot{\vec{w}}(t)]\ .$$

The latter is independent of s, so the condition in the last theorem holds. The conserved quantity is quickly seen to be

$$L[\vec{w}(t)\ ;\ \dot{\vec{w}}(t)] \ - \ < \vec{\nabla}_{**} L[\vec{w}(t)\ ;\ \dot{\vec{w}}(t)]\ ,\ \dot{\vec{w}}(t) >\ ,$$

which is the negative of the conserved energy in the first example of Version I of Noether's theorem.

So *apparently* the simpler 2nd version can be tricked into yielding energy conservation. But we must deal with $\stackrel{?}{\Longrightarrow}$. Now the dimension, k, has changed to $k+1$, so Lagrange's (vector) DE becomes "$k+1$" DEs for scalar functions. Applied to $(\vec{w}(t), t)$, all but the last turn out to be the "k" DEs in the system $\mathrm{Lag}(L)$. However the last equation states precisely that the time derivative of the conserved energy expression referred to above should be zero ! So it appears that one cannot apply the last theorem, using this particular trick, to establish conservation of energy (for any lagrangian system L

which is "not explicitly a function of t"), without first establishing the exact same conservation of energy by some other means. So Arnold had something different in mind, something which hasn't occurred yet to my geriatric mind.

Hamiltonian Mechanics.

The passage from lagrangian to hamiltonian mechanics can be largely motivated by some standard DE theory in which higher order DEs are converted into a larger number of 1st order DEs. We won't go into this, but just write down 'the answer', so to speak, and give a straightforward connection between the two. As mentioned earlier when starting lagrangian mechanics, we have not dwelt much at all on specific examples of mechanical systems, which is the best way to learn the nuts and bolts. Our purposes are different. For these examples, there are a number of excellent sources, such as [**Tong**], [**XXXXXXX**], [**YYYYYYY**].

The hamiltonian 'conversion' of lagrangian mechanics is said to be :

(i) often less convenient for dealing with particular problems concerning subtle mechanical systems;

(ii) more elegant mathematically;

(iii) more helpful for most theoretical questions;

(iv) essential for formulating the passage from classical mechanics to quantum mechanics.

We'll need it at the end to discuss the Runge-Lenz vector. And in any case, it seems desirable to discuss Noether's theorem in its hamiltonian version. There, it seems to become almost tautological; and not all that obviously related to the two theorems just discussed. These points are elaborated below.

The passage from lagrangian to hamiltonian mechanics (and back again, for that matter) is usually achieved via the *Legendre transform.* (There is no truth to the rumour that this is connected to sex-change operations.)

We can avoid a lengthy geometric (and perhaps illuminating!) discussion of the Legendre transform by the following simple-minded abstractions. (Proofs are left as exercises.) In [**Arnold**], pp. 61-64, there is a rather more involved set-up, likely needed because of working on general symplectic manifolds and their tangent and cotangent bundles.

Lpsins.

The word is an acronym for "left pseudo-inverse".

Definition. Let S be any set, and $F : S \times S \to S$ a function. Then $G : S \times S \to S$ is a lpsin for F if and only if

$$G(y, F(y,z)) \;=\; z \qquad \text{for all } (y,z) \in S \times S \;.$$

Assuming F and G are so related, the first two propositions below hold.

Prop 1. *$G|_{\{y\}\times S}$ is surjective for all $y \in S$.*

Prop 2. *If $F|_{\{y\}\times S}$ is surjective for all $y \in S$, then F is also an lpsin for G.*

Prop 3. *If $F|_{\{y\}\times S}$ is not injective for some $y \in S$, then F has no lpsin.*

Prop 4. *$F|_{\{y\}\times S}$ is bijective for all $y \in S$ if and only if F has a unique lpsin. And then, with G denoting that lpsin, $G|_{\{y\}\times S}$ is also bijective for all y, and F is G's unique lpsin.*

Actually, this should all be generalized to F whose domain might be a *proper* subset of $S \times S$. The formulation of that will be left to the reader. It is just standard fussy details concerning domains/codomains.

The hamiltonian of a lagrangian. Suppose given $L : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}$ as usual, yielding $\vec{\nabla}_{**}L : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^k$. Assume that $\vec{\nabla}_{**}L|_{\{\vec{y}\}\times \mathbf{R}^k}$ is bijective for all $\vec{y} \in \mathbf{R}^k$.

Let $G : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^k$ be the lpsin of $\vec{\nabla}_{**}L$. Implicit function theory guarantees smoothness for G. Define

$$H_L : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}$$

by $$(\vec{q}, \vec{p}) \;\mapsto\; < \vec{p} \;,\; G(\vec{q}, \vec{p}) > - \; L(\vec{q}, G(\vec{q}, \vec{p})) \;.$$

(As always here, $<\;,\;>$ is the standard inner product on $\mathbf{R}^k$.)

Definition. Given a curve $\vec{w}$ in $\mathbf{R}^k$ (think: "$\vec{y}$-space" or "configuration space" and "$(\vec{y}, \vec{z})$-space" or "velocity space"), define another curve $\vec{\sigma}_{\vec{w}}$ in $\mathbf{R}^k \times \mathbf{R}^k$ (think: "$(\vec{q}, \vec{p})$-space" or "phase space") by

$$\vec{\sigma}_{\vec{w}}(t) \;=:\; (\vec{q}_{\vec{w}}(t), \vec{p}_{\vec{w}}(t)) \;:=\; (\; \vec{w}(t) \;,\; \vec{\nabla}_{**}L[\vec{w}(t); \dot{\vec{w}}(t)] \;).$$

Thus $\vec{q}_{\vec{w}}$ is just another name for $\vec{w}$; whereas $\vec{p}_{\vec{w}}$ is $\vec{\nabla}_{**}L$ applied to $\vec{w}$ and its time derivative. In contradistinction to our former practice in configuration space and velocity space, we shall often both use $(\vec{q}, \vec{p})$ for a point in phase space and also use it for a curve in phase space. One reason for joining the sinners in this is that, in discussion of a fixed system, various such curves in phase space are disjoint. So a point in phase space determines a unique curve through it in phase space, in the presence of a hamiltonian function; whereas a point in configuration space does not determine a *unique* curve through it in configuration space, in the presence of a lagrangian function on velocity space.

Definition. Given a smooth function $H : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}$, which we call a hamiltonian now, define $\mathrm{Ham}(H)$, known as Hamilton's equations, to be the system of "$2k$" DEs for scalar functions, or equivalently the following pair of linear vector DEs for a curve $[\vec{q}(t), \vec{p}(t)]$:

$$\frac{d\vec{p}(t)}{dt} = -\vec{\nabla}_* H[\vec{q}(t), \vec{p}(t)] \ ;$$

$$\frac{d\vec{q}(t)}{dt} = \vec{\nabla}_{**} H[\vec{q}(t), \vec{p}(t)] \ .$$

The usual version in the physics literature is $\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}$ and $\frac{dq_i}{dt} = -\frac{\partial H}{\partial p_i}$.

Hamilton's equations may be easily written as a single vector equation (which also avoids having to remember what these interlopers, $\vec{\nabla}_{**}$ and $\vec{\nabla}_*$, mean; but now you have to remember a matrix!). Let I be the $k \times k$ identity matrix, and let J be the $2k \times 2k$ matrix

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \ .$$

Then $\mathrm{Ham}(H)$ becomes the following system, where the right-hand side has the form of a (row) vector in $\mathbf{R}^{2k}$ (the 'every-day gradient') multiplied into a square matrix :

$$\frac{d[\vec{q}(t), \vec{p}(t)]}{dt} = \vec{\nabla} H[\vec{q}(t), \vec{p}(t)] \cdot J \ ;$$

or just

$$\frac{d\vec{\sigma}(t)}{dt} = \vec{\nabla} H[\vec{\sigma}(t)] \cdot J \ ;$$

or sometimes just

$$\frac{d\vec{\sigma}}{dt} = \vec{\nabla} H \cdot J \ ; \quad \text{and later} \quad \frac{d\vec{\sigma}}{dt} = X_H \quad (\text{see } \mathbf{Prop.8}).$$

Theorem. *Given L, a curve $\vec{w}$ ("in configuration space") solves Lag(L) if and only if the corresponding curve $\vec{\sigma}_{\vec{w}}$ ("in phase space") solves Ham(H_L).*

This is another result which has an (apparently) much shorter proof (really the same one—i.e. just calculate) using the physicists' efficient, throw-caution-to-the-winds language.)

Proof. Firstly, there are a couple of basic connections between the functions L, H_L and G which follow from the definitions, and the chain and product rules. All functions in the next few paragraphs are 'evaluated at the generic point' $(\vec{q}, \vec{p})$ of "phase space", $\mathbf{R}^k \times \mathbf{R}^k$.

First define, for a function $\vec{F} : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^k$, a couple of square matrix 'half-Jacobians'

$$J_{*\vec{F}} \text{ and } J_{**\vec{F}} : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^{k \times k} \quad \text{by} \quad [J_{*\vec{F}}]_{ij} := D_i F_j \text{ and } [J_{**\vec{F}}]_{ij} := D_{k+i} F_j \ .$$

<u>Next consider the function $(\vec{q}, \vec{p}) \mapsto < \vec{p} \ , \ G(\vec{q}, \vec{p}) >$</u> :

Applying $\vec{\nabla}_*$, the product rule immediately yields $\vec{p} \cdot J_{*G}$.

Applying $\vec{\nabla}_{**}$, again using the product rule, we get $G + \vec{p} \cdot J_{**G}$.

<u>Then consider the function $(\vec{q}, \vec{p}) \mapsto L(\vec{q}, G(\vec{q}, \vec{p})$</u> :

Applying $\vec{\nabla}_*$, the chain rule and then using the fact that $\vec{\nabla}_{**} L$ is the lpsin of G yields

$$\vec{\nabla}_* L(\vec{q}, G(\vec{q}, \vec{p})) \cdot I \ + \ \vec{\nabla}_{**} L(\vec{q}, G(\vec{q}, \vec{p})) \cdot J_{*G} \ = \ \vec{\nabla}_* L(\vec{q}, G(\vec{q}, \vec{p})) \ + \ \vec{p} \cdot J_{*G} \ .$$

Applying $\vec{\nabla}_{**}$, the same two yield

$$\vec{\nabla}_* L(\vec{q}, G(\vec{q}, \vec{p})) \cdot 0 \ + \ \vec{\nabla}_{**} L(\vec{q}, G(\vec{q}, \vec{p})) \cdot J_{**G} \ = \ \vec{p} \cdot J_{**G} \ .$$

But, recalling that H_L is the difference of the two functions just considered, subtraction of the answers above in pairs then produces the basic connections we want :

$$\vec{\nabla}_* (H_L)(\vec{q}, \vec{p}) \ = \ -\vec{\nabla}_* L(\vec{q}, G(\vec{q}, \vec{p})) \qquad \text{and} \qquad \vec{\nabla}_{**}(H_L) \ = \ G \qquad (*)$$

Next start 'substituting curves into our functions'. Note that, by the definitions of $\vec{q}_{\vec{w}}$ and $\vec{p}_{\vec{w}}$, and the fact that G is the lpsin of $\vec{\nabla}_{**}L$, we have

$$G[\vec{q}_{\vec{w}}(t), \vec{p}_{\vec{w}}(t)] \;=\; G(\vec{w}(t), \vec{\nabla}_{**}L[\vec{w}(t); \dot{\vec{w}}(t)]) \;=\; \dot{\vec{w}}(t) \qquad (**)$$

Now to prove the theorem.

In one direction, assume that $\vec{w}$ solves Lag(L), and we'll check the two equations of Ham(H_L) in its original form. As for the first one, the equalities below respectively use the left half of $(*)$; then $(**)$ and that $q_{\vec{w}} = \vec{w}$; then Lag(L); and finally the definition of $\vec{p}_{\vec{w}}$.

$$-\vec{\nabla}_*(H_L)(\vec{q}_{\vec{w}}(t), \vec{p}_{\vec{w}}(t)) \;=\; \vec{\nabla}_*L(\vec{q}_{\vec{w}}(t)\ ;\ G[\vec{q}_{\vec{w}}(t), \vec{p}_{\vec{w}}(t)])\ ;$$

$$=\; (\vec{\nabla}_*L)[\vec{w}(t), \dot{\vec{w}}(t)] \;=\; \frac{d}{dt}(\vec{\nabla}_{**}L)[\vec{w}(t), \dot{\vec{w}}(t)] \;=\; \frac{d\vec{p}_{\vec{w}}(t)}{dt}\ ,$$

as required. As for the second one, use the right half of $(*)$; then $(**)$; then the definition of $\vec{q}_{\vec{w}}$.

$$\vec{\nabla}_{**}(H_L)(\vec{q}_{\vec{w}}(t), \vec{p}_{\vec{w}}(t)) \;=\; G(\vec{q}_{\vec{w}}(t), \vec{p}_{\vec{w}}(t)) \;=\; \dot{\vec{w}}(t) \;=\; \frac{d\vec{q}_{\vec{w}}(t)}{dt}\ ,$$

as required.

The other direction of the theorem is similarly a formal consequence of the definitions, and the equalities $(**)$ and $(*)$ above. Assume that $[\vec{q}_{\vec{w}}(t), \vec{p}_{\vec{w}}(t)]$ solves Ham(H_L). To see that $\vec{w}$ solves Lag(L), the following equalities are successively justified by the definition of $\vec{p}_{\vec{w}}$; the 1st Hamilton equation; the 1st equality in $(*)$; and finally $(**)$.

$$\frac{d}{dt}\vec{\nabla}_{**}L[\vec{w}(t); \dot{\vec{w}}(t)] \;=\; \frac{dp_{\vec{w}}(t)}{dt} \;=\; -\,\vec{\nabla}_*(H_L)[q_{\vec{w}}(t), \vec{p}_{\vec{w}}(t)]$$

$$=\; \vec{\nabla}_*L(q_{\vec{w}}(t), G[\vec{q}_{\vec{w}}(t), \vec{p}_{\vec{w}}(t)]) \;=\; \vec{\nabla}_*L[\vec{w}(t), \dot{\vec{w}}(t)]\ ,$$

which is the right-hand side of Lag(L), as required. (This display is really just a re-cycling of the 2nd last one.)

Related to each other are the facts that
(i) we didn't use the 2nd Hamilton equation in this last paragraph;
(ii) nor did we use Lagrange's equation in deducing that Hamilton equation.

In fact the latter holds for $[\vec{q}_{\vec{w}}, \vec{p}_{\vec{w}}]$ for *any* curve $\vec{w}$, so perhaps its proof should have been placed further up.

Conserved Quantities in Hamiltonian Mechanics.

Let M be an open subset of $\mathbf{R}^k \times \mathbf{R}^k$. This will be our name for "phase space" from now on. The general case of a symplectic manifold is done in [**Arnold**], a superb treatment. We have no intention to go to that generality. We shall however try to keep notation as coordinate-free as possible, though coordinates are needed to begin [and local ones are of course needed in the general (manifold) case]. These will be denoted in the standard way as $(\vec{q}, \vec{p})$ [and some curves as $(\vec{q}(t), \vec{p}(t))$], though [**Arnold**] uses $(\vec{p}, \vec{q})$.

If $H : M \to \mathbf{R}$ is a hamiltonian and $F : M \to \mathbf{R}$ is another smooth function, we call F a conserved quantity, (AKA 1st integral; AKA constant of the motion) of the hamiltonian system if and only if, for all $(\vec{q}(t), \vec{p}(t))$ which solve Ham(H), the number $F(\vec{q}(t), \vec{p}(t))$ is independent of t ; i.e. $\frac{d}{dt}F(\vec{q}(t), \vec{p}(t)) = 0$.

Note that, for *any curve* $(\vec{q}(t), \vec{p}(t))$ and any F, by the chain rule

$$\frac{d}{dt}F(\vec{q}(t), \vec{p}(t)) \;=\; <\vec{\nabla}_* F(\vec{q}(t), \vec{p}(t))\;,\;\frac{d\vec{q}}{dt}> \;+\; <\vec{\nabla}_{**} F(\vec{q}(t), \vec{p}(t))\;,\;\frac{d\vec{p}}{dt}> \;,$$

(where $<\;\;,\;\;>$ continues to be the standard inner product on $\mathbf{R}^k$). [In contrast to the lagrangian case, where Lag(L) is not a 1st order DE directly expressing $\frac{d\vec{w}}{dt}$] for *all solution curves* $(\vec{q}(t), \vec{p}(t))$ above, one concludes, using Hamilton's equations, that, for any smooth function F,

$$\frac{d}{dt}F(\vec{q}(t), \vec{p}(t)) \;=\; (<\vec{\nabla}_* F\;,\;\vec{\nabla}_{**} H> \;-\; <\vec{\nabla}_{**} F\;,\;\vec{\nabla}_* H>)(\vec{q}(t), \vec{p}(t)) \;.$$

Thus we have

Lemma 1. *A function F is a conserved quantity of the system defined by H if and only if, for all $(\vec{q}(t), \vec{p}(t))$ which solve Ham(H),*

$$<\vec{\nabla}_* F\;,\;\vec{\nabla}_{**} H> (\vec{q}(t), \vec{p}(t)) \;=\; <\vec{\nabla}_{**} F\;,\;\vec{\nabla}_* H> (\vec{q}(t), \vec{p}(t)) \;.$$

The kind of symmetry we see between H and F in this leads to the following definitions.

Definition. A vector field on M is a smooth map $X : M \to \mathbf{R}^k \times \mathbf{R}^k$.

Such an X we rewrite (decompose) as $X(m) \;=:\; [X_*(m), X_{**}(m)]$, for X_* and X_{**} both mapping to $\mathbf{R}^k$.

Let $\mathcal{VF}(M)$ be the set of all vector fields on M. Define a mapping

$$\omega \ : \ \mathcal{VF}(M) \times \mathcal{VF}(M) \ \longrightarrow C^\infty(M, \mathbf{R}) \qquad \text{by}$$

$$\omega(X, Y)(m) \ := \ < X_{**}(m), Y_*(m) > \ - \ < X_*(m), Y_{**}(m) > \ .$$

[It is the version of this ω in the general case which is the basic piece of structure required on a manifold M to make it into a symplectic manifold. There, vector fields need a more elaborate definition.]

Proposition 2. *The sets $\mathcal{VF}(M)$ and $C^\infty(M)$ have the obvious structures as real vector spaces. With respect to those structures, ω is bilinear and* alternating *[in that $\omega(X, Y) = -\omega(Y, X)$ for all X, Y].*

The proof is a trivial calculation.

Proposition 3. *The bilinear map ω is* non-degenerate *in that*

$$\forall X \ [\omega(X, Y) = 0 \ \forall Y \quad \Longrightarrow \quad X = 0 \ ] \ .$$

Proof. Take $Y = (X_{**}, -X_*)$; that is, $Y_* = X_{**}$ and $Y_{**} = -X_*$. Then

$$0 \ = \ \omega(X, Y) \ = \ < X_{**}, Y_* > - < X_*, Y_{**} > \ = \ < X_{**}, X_{**} > + < X_*, X_* > \ .$$

By the positivity of the standard inner product we get $X_* = 0 = X_{**}$, i.e. we have $X = 0$, as required.

Definition. The Poisson bracket

$$\{\ ,\ \} \ : \ C^\infty(M) \times C^\infty(M) \ \longrightarrow \ C^\infty(M)$$

is defined by $\{ \ F \ , \ G \ \} \ := \ \omega(\vec{\nabla} F, \vec{\nabla} G)$. That is, $\{\ ,\ \}$ is the composite

$$C^\infty(M) \times C^\infty(M) \ \stackrel{\vec{\nabla}\times\vec{\nabla}}{\longrightarrow} \mathcal{VF}(M) \times \mathcal{VF}(M) \ \stackrel{\omega}{\longrightarrow} \ C^\infty(M) \ .$$

Proposition 4. *The Poisson bracket $\{\ ,\ \}$ is bilinear and alternating.*

Proof. $\vec{\nabla}$ is linear, and ω is bilinear and alternating.

Theorem 5. *If $\{ \ H \ , \ F \ \} = 0$ then F is a conserved quantity for the system Ham(H) ; and indeed, H is a conserved quantity for Ham(F).*

Proof. The first part is immediate from **Lemma 1** and the definitions, and the second from the alternating property.

Theorem 6. *H itself is a conserved quantity for the system Ham(H).*

Proof. $\{ H , H \} = 0$ by the alternating property.

Remarks. (i) This is conservation of energy; it seems to have arisen without any work at all!

(ii) Coordinate formulas for the Poisson bracket are immediate from the definitions:

$$\{ F , G \} = < \vec{\nabla}_{**}F, \vec{\nabla}_{*}G > - < \vec{\nabla}_{*}F, \vec{\nabla}_{**}G > = \sum_{i=1}^{k}(D_{k+i}F \cdot D_iG - D_iF \cdot D_{k+i}G) \; ;$$

in the physicists' language, $\sum_1^k (\frac{\partial F}{\partial p_i} \cdot \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \cdot \frac{\partial G}{\partial p_i})$.

Earlier we had asked about a converse to Noether's theorem. For this and other matters, we need to go beyond the formal algebraic manipulations which have dominated the hamiltonian discussion so far, and to apply some mathematical/geometrical analysis, which is to establish a basic existence/uniqueness/smoothness theorem concerning DEs.

Results **7, 9** and **10** just ahead need more detail. The impatient reader may consult [**Arnold**] pp. 201-218; the patient one can wait a month or so and consult the updated version of this write-up.

Theorem 7. *There are correspondences between the following sets which, after suitably identifying flows, are bijective :*

$$\mathcal{VF}(M) \leftrightarrow \mathcal{FLOW}(M) \leftrightarrow \mathcal{DIFFOP}[C^\infty(M, \mathbf{R})] \; .$$

The middle set, of flows on *M, consists of 1-parameter groups of diffeomorphisms $\{a_t : M \to M\}$ The one on the right is the set of 1st order differential operators on the set of smooth functions on M.*

The correspondence on the left associates the vector field $\frac{da_t}{dt}|_{t=0}$ to the flow above.

On the right, the flow gives the operator taking F to $\frac{d(F \circ a_t)}{dt}|_{t=0}$.

This double way of getting an alter-ego for a vector field corresponds to our double geometric/physical intuition of a geometric vector at a point as a 'weighted direction' which is telling us both

(i) the tangent to the curve taken by some point (particle) tracing out a path on M;

(ii) the direction and weighting to use in differentiating a real-valued smooth function defined in an M-neighbourhood of the point.

Definition. The hamiltonian vector field associated to H is the unique vector field X_H such that, for all vector fields Y, we have

$$\omega(X_H, Y) \;=\; dH(Y) \;:=\; <\vec{\nabla}_* H\;,\; Y_*> \;+\; <\vec{\nabla}_{**} H\;,\; Y_{**}> \;=\; <\vec{\nabla} H\;,\; Y>_{\mathbf{R}^{2k}} \;.$$

The non-degeneracy of ω guarantees the existence and uniqueness of X_H.

Proposition 8. *Hamilton's system may be re-written*

$$\frac{d\vec{\sigma}}{dt} \;=\; X_H \;.$$

This follows since the calculation below shows $\vec{\nabla} H \cdot J \;=\; X_H$ by the non-degeneracy of ω : For all Y,

$$\omega(\vec{\nabla} H \cdot J, Y) \;=\; \omega((\vec{\nabla}_* H, \vec{\nabla}_{**} H) \cdot J, (Y_*, Y_{**}))$$
$$=\; \omega((\vec{\nabla}_{**} H, -\vec{\nabla}_* H), (Y_*, Y_{**})) \;=\; <\vec{\nabla}_{**} H, Y_{**}> - <-\vec{\nabla}_* H, Y_*>$$
$$=\; <\vec{\nabla}_{**} H, Y_{**}> + <\vec{\nabla}_* H, Y_*> \;=\; \omega(X_H, Y)\;.$$

Corollary 9. *If $\{h_t\}$ is the* Hamiltonian flow*, that is, the flow associated with X_H in the previous theorem, then*

$$\{\; F \;,\; H \;\} \;=\; \frac{d(F \circ h_t)}{dt}\Big|_{t=0}$$

Theorem 10. *F is a Ham(H) conserved quantity if and only if $\{\; F \;,\; H \;\} = 0$, i.e. that Poisson bracket yields the zero function on M.*

In one direction, this is just a repeat of **Theorem 5**. Conversely, to show that the Poisson bracket must be zero for F to be conserved, appeal again to **Lemma 1**, this time knowing from **Theorem 7** that every point in M lies on a solution to Hamilton's equations.

Lemma 11. *The Jacobi identity holds for the Poisson bracket; that is, for all functions E, F and G, we have*

$$\{\; E \;, \{\; F \;,\; G \;\}\;\} \;+\; \{\; G \;, \{\; E \;,\; F \;\}\;\} \;+\; \{\; F \;, \{\; G \;,\; E \;\}\;\} \;=\; 0 \;\;.$$

To remember this, just note the cyclic permutation. It doesn't matter which way you cycle, nor which way the brackets are inserted (as long as they are the same for each of the three summands). The proof is an unilluminating calculation from the definition, with 24 terms, half of which have negative signs and cancel the rest. It is done just below in an appendix.

Corollary 12. (Poisson) *If E and F are both conserved quantities for Ham(H), the so is* $\{\ E\ ,\ F\ \}$. (In the **Lemma 11** display, just take $G = H$, the hamiltonian. The 1st and 3rd terms are zero by assumption and linearity, so the 2nd one is as well, as required.)

A Lie algebra is any vector space over the field being considered (for us, the field $\mathbf{R}$), together with a binary operation, $[\ ,\ ]$, which is bilinear, alternating and satisfies the Jacobi identity. (If the field has $1+1=0$, *alternating* must be defined to include $[l,l]=0$, which actually implies $[l,m]=-[m,l]$ over any field—consider $[l+m,l+m]$.)

The corollary says that the smooth functions on M form a Lie algebra over $\mathbf{R}$ when the Poisson bracket is used as the binary operation.

Lemma 13. (Leibniz' rule) *For all functions E,F and G, we have*

$$\{\ E\cdot F\ ,\ G\ \}\ =\ E\cdot\{\ F\ ,\ G\ \}\ +\ F\cdot\{\ E\ ,\ G\ \}\ .$$

This says the 'operator', $\{\ -\ ,\ G\ \}$, behaves like differentiation, in that it has a product rule. Such an operator is known in algebra as a derivation.

The proof is easy and uninteresting, and the result produces no new conserved quantities, since we already knew that the product of two of them is also one of them. A Lie algebra also satisfying Leibniz' rule for an extra piece of structure which is a binary operation called multiplication is termed a **Poisson algebra**, as long as the new operation, with the vector space structure, also produces an associative algebra over $\mathbf{R}$. And so the smooth functions on any M do give a Poisson algebra.

We really must relate our hamiltonian form of Noether's theorem (results **1,5,10** here) to the two theorems which were our lagrangian versions.

Then finally we will return to Kepler's problem and explain the Runge-Lenz conserved vector in terms of lie algebras and this hamiltonian formalism.

Appendix : Proof of the Jacobi Identity for the Poisson Bracket.

Let us tediously calculate the first of the three summands : (Due to having originally done this appendix at a time when the Poisson bracket was reversed from the present definition here, there is a 'silly' double-negation in the first step to avoid re-doing the work!)

$$\{ E , \{ F , G\}\} = -\{ E , -\{ F , G\}\} = \sum_1^k [D_iE{\cdot}D_{k+i}(-\{ F , G\}) - D_{k+i}E{\cdot}D_i(-\{ F , G\})]$$

$$= \sum_{1\le i,j\le k} D_iED_{k+i}(D_jFD_{k+j}G - D_{k+j}FD_jG) - D_{k+i}ED_i(D_jFD_{k+j}G - D_{k+j}FD_jG)$$

$$= \sum_{1\le i,j\le k} D_iED_{k+i}D_jFD_{k+j}G + D_iED_jFD_{k+i}D_{k+j}G - D_iED_{k+i}D_{k+j}FD_jG$$
$$- D_iED_{k+j}FD_{k+i}D_jG - D_{k+i}ED_iD_jFD_{k+j}G - D_{k+i}ED_jFD_iD_{k+j}G$$
$$+ D_{k+i}ED_iD_{k+j}FD_jG + D_{k+i}ED_{k+j}FD_iD_jG \quad .$$

Now nobody wants to write out essentially the same thing twice more! So here is an abbreviation of the above display, the terms with a +-sign appearing first, but otherwise preserving the order—note that the "C" corresponds to subscripts larger than k; that the lack of a dot indicates composition of operators as opposed to multiplication; and that below, this proof will depend on exchanging names of variables of summation in suitable places :

$$D \cdot CD \cdot C + D \cdot D \cdot CC + C \cdot DC \cdot D + C \cdot C \cdot DD$$
$$-D \cdot CC \cdot D - D \cdot C \cdot CD - C \cdot DD \cdot C - C \cdot D \cdot DC$$

Because the 2nd and 3rd summands in the identity to be proved are obtained by cyclic permutation of the functions, E, F, G, of the previous summand, we need only do the corresponding cyclic permutations on the display just above to get the other two summands. (Thank heavens, or allah, or somebody, for cut and paste!) So now we're imagining E, F, G always being in that order, but the operators getting cycled around. The other two are

$$CD \cdot C \cdot D + D \cdot CC \cdot D + DC \cdot D \cdot C + C \cdot DD \cdot C$$

$$-CC \cdot D \cdot D - C \cdot CD \cdot D - DD \cdot C \cdot C - D \cdot DC \cdot C$$

and

$$C \cdot D \cdot CD + CC \cdot D \cdot D + D \cdot C \cdot DC + DD \cdot C \cdot C$$
$$-D \cdot D \cdot CC - CD \cdot D \cdot C - C \cdot C \cdot DD - DC \cdot C \cdot D$$

These 24 summands cancel each other in pairs as follows—always the (i, j)th positive term cancels a (j, i)th negative term, and we of course use that the operators commute under composition :

1st-16th 2nd-21st 3rd-14th 4th-23rd 5th-10th 6th-19th
7th-12th 8th-17th 9th-24th 11th-22nd 13th-18th 15th-20th

So that's the proof. If you find a more readable complete proof, modulo that silly double-negation, let me know!

Lie Morphisms and Runge-Lenz.

As we've just seen, the set of all conserved quantities $F : M \to \mathbf{R}$ for a given hamiltonian system (M, ω, H) has a lot of algebraic structure : it is a Poisson subalgebra (in particular, a Lie subalgebra) of the set of all smooth functions on M. It is also closed under composing with smooth functions $\beta : \mathbf{R} \to \mathbf{R}$. This is trivial to see from the definition; alternatively it's fun and easy to find a formula for $\{ G \ , \ \beta \circ F \}$ yielding another proof.

So far, I am completely ignorant of how one goes about proving that some given set of conserved quantities is a set of generators for this rich algebraic object—that is, how one justifies a statement of the form "these such-and-such functions are 'all' the conserved quantities for the system".

In the opposite direction, an interesting discussion coming out of the latter part of the previous section concerns a 'natural' way to seek, or at least to study, conserved quantities. Roughly speaking, this consists of specifying some finite-dimensional Lie algebra ℓ, and then mapping its generators into $C^\infty(M, \mathbf{R})$, by ϕ say, and proving (as we do in the 2×2 theorems below)

(i) $\{ H \ , \ \phi(A) \} = 0$ for each generator $A \in \ell$; and

(ii) for any 'Lie relation' involving the generators A in the specification of ℓ, the 'same' relation holds, among the $\phi(A)$, in $C^\infty(M, \mathbf{R})$, using the Poisson bracket $\{ \ , \ \}$ as Lie operation.

By (ii), one would get a Lie morphism $\phi : \ell \to C^\infty(M, \mathbf{R})$, and by (i) its image is a Lie subalgebra of the *conserved* quantities.

This can often be done in a 'natural geometrical/physical' way, roughly as follows. The Lie algebra ℓ will be the Lie algebra of a Lie group which might have been seen to act in a natural way on M, or even on some configuration space which gave rise to M. The action could possibly only be local. But at any rate it would correspond to a (local) 1-parameter group of diffeomorphisms on M, getting us back to ideas similar to ones in the lagrangian version of Noether's theorem. There are many good expositions of the Lie algebra/Lie group connection; suffice to say here that Lie algebra elements can be identified with certain vector fields on the Lie group, and so also with certain 1-parameter groups of diffeomorphisms of the Lie group.

Let us start by defining several hamiltonian systems, and then several Lie algebras. These will be paired up with each other further down.

Definition of $(M_{\text{free}}, H_{\text{free}})$ **.** This system is given by

$$M_{\text{free}} := \mathbf{R}^3 \times \mathbf{R}^3 \qquad \text{and} \qquad H_{\text{free}}(\vec{q}, \vec{p}) := \frac{1}{2m} < \vec{p}, \vec{p} > \; = \frac{|\vec{p}|^2}{2m} \; .$$

It is the hamiltonian description of a single 'free' particle (free from any force fields). It corresponds to the lagrangian system which is also $\mathbf{R}^3 \times \mathbf{R}^3$ with

$$L_{\text{free}}(\vec{y}, \vec{z}) := \frac{m}{2} < \vec{z}.\vec{z} > \; = \frac{1}{2} m |\vec{z}|^2 \; .$$

Explicit solutions to $\text{Ham}(H_{\text{free}})$ and $\text{Lag}(L_{\text{free}})$ are very easy to come by. $\text{Lag}(L_{\text{free}})$ is equality of the right-hand sides in the next two displays:

$$\frac{d}{dt}(\vec{\nabla}_{**} L[\vec{w}(t), \dot{\vec{w}}(t)]) = \frac{d}{dt}(m\dot{\vec{w}}(t)) = m\ddot{\vec{w}}(t)$$

$$\vec{\nabla}_{*} L[\vec{w}(t), \dot{\vec{w}}(t)] = \vec{0} \; .$$

So $\vec{w}(t) = \vec{r} + t\vec{s}$ for fixed $\vec{r}$ and $\vec{s}$.

$\text{Ham}(H_{\text{free}})$ is
$$\frac{d\vec{p}}{dt} = -\vec{\nabla}_{*} H_{\text{free}} = \vec{0} \; ,$$

so $\vec{p}(t) = \vec{a}$, a fixed vector; and

$$\frac{d\vec{q}}{dt} = \vec{\nabla}_{**} H_{\text{free}} = \frac{1}{m}\vec{p} = \frac{1}{m}\vec{a} \; ,$$

so $\vec{q}(t) = \vec{b} + \frac{t}{m}\vec{a}$, for fixed vectors $\vec{a}$, from above, and $\vec{b}$.

(So, Ham agrees with Lag, as he must. And, as we all learned in grade school, a free particle moves at uniform speed in a straight line. Seems like this should have appeared 99 pages earlier, if it is going to appear at all!)

Definition of $(M_{\text{Kepler}}, H_{\text{Kepler}})$ **.** This system is given by

$$M_{\text{Kepler}} := (\mathbf{R}^3 \setminus \{\vec{0}\}) \times \mathbf{R}^3 \qquad \text{and} \qquad H_{\text{Kepler}}(\vec{q},\vec{p}) := \mu|\vec{p}|^2 - \frac{\nu}{|\vec{q}|} \ ,$$

for some positive real numbers μ and ν. (We removed $\{\vec{0}\} \times \mathbf{R}^3$ in defining the phase space because of the $|\vec{q}|$ in the denominator of the hamiltonian function.) It is the hamiltonian description of a single particle under a central inverse square force field, as studied earlier here. It corresponds to the lagrangian system which is

$$L_{\text{Kepler}}(\vec{y},\vec{z}) := \frac{1}{2}m|\vec{z}|^2 + \frac{mMG}{|\vec{y}|}$$

from the first half of this write-up, with $\mu = \frac{1}{2m}$ and $\nu = mMG$. And that is where discussion of solutions of this system is discussed, albeit without the words "hamiltonian" and "lagrangian".

Definition of $(M_{\text{closed}}, H_{\text{closed}})$ **.** This system is given by

$$M_{\text{closed}} := \{m \in M_{\text{Kepler}} \mid H_{\text{Kepler}}(m) < 0\} \ , \qquad \text{and} \qquad H_{\text{closed}} := H_{\text{Kepler}}|_{M_{\text{closed}}} \ .$$

The discussion referred to just above shows that M_{closed} consists of exactly those points in phase space which correspond to a closed (elliptical) orbit for the particle.

Now we define some Lie algebras. Firstly

$$\ell := \{ \ B \in \mathbf{R}^{3\times 3} \mid B^{\text{tr}} = -B \ \} \ ,$$

the 3-dimensional space of all anti-symmetric real 3×3-matrices; i.e.

$$\ell = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} : a,b,c \text{ in } \mathbf{R} \right\} = \{B_{\vec{v}} : \vec{v} = (a,b,c) \in \mathbf{R}^3 \} \ .$$

From here on, $\vec{v}$ denotes a 'constant' vector in $\mathbf{R}^3$; it is no longer velocity—I'm running out of sensible notation! This Lie algebra of skew-symmetric matrices occurred earlier in our discussion of conservation of angular momentum. The vector space ℓ is given its Lie algebra operation $[\ ,\ ]$ as a Lie subalgebra of the commutator Lie product,

$$[B,C] \ := \ BC - CB \ , \quad \text{on } \mathbf{R}^{3\times 3} \ .$$

As a Lie algebra, ℓ is easily seen to be isomorphic to the vector cross-product Lie algebra structure on $\mathbf{R}^3$. Specifically, with notation in the penultimate display above, we have

$$[B_{\vec{u}}\ ,\ B_{\vec{v}}] \ = \ -B_{\vec{u}\times\vec{v}} \ ,$$

so that $\vec{v} \mapsto -B_{\vec{v}}$ defines a Lie algebra isomorphism $(\mathbf{R}^3, \times) \to (\ell, [\ ,\ ])$.

Now denote the 6-dimensional vector space $\ell \times \ell$ as $\ell \oplus \ell$, and make it into a Lie algebra in three different ways as follows :

$$\ell_+ \ = \ (\ell \oplus \ell\ ,\ [\ ,\ ]_+) \qquad \text{where } [\ ,\ ]_+ \text{ is determined by}$$

$$[(B,0),(C,0)]_+ \ := \ ([B,C],0) \ =: \ [(0,B),(0,C)]_+ \ ; \ \text{ and } \ [(B,0),(0,C)]_+ \ := \ (0,[B,C]) \ .$$

$$\ell_0 \ = \ (\ell \oplus \ell\ ,\ [\ ,\ ]_0) \qquad \text{where } [\ ,\ ]_0 \text{ is determined by}$$

$$[(B,0),(C,0)]_0 \ := \ ([B,C],0) \text{ but } [(0,B),(0,C)]_0 \ := \ 0 \ ; \ \text{ and } \ [(B,0),(0,C)]_0 \ := \ (0,[B,C]) \ .$$

$$\ell_- \ = \ (\ell \oplus \ell\ ,\ [\ ,\ ]_-) \qquad \text{where } [\ ,\ ]_- \text{ is determined by}$$

$$[(B,0),(C,0)]_- \ := \ ([B,C],0) \ =: \ -[(0,B),(0,C)]_- \ ; \ \text{ and } \ [(B,0),(0,C)]_- \ := \ (0,[B,C]) \ .$$

Lemma. *Each of these (i.e. for $\epsilon \ = \ +\ ,\ 0\ ,\ -$) extends uniquely to a bilinear map $\ell_\epsilon \times \ell_\epsilon \to \ell_\epsilon$ satisfying anti-symmetry and the Jacobi identity* (and so producing a Lie algebra).

In lists classifying Lie algebras, the objects ℓ , ℓ_+ , ℓ_0 and ℓ_- are often denoted, respectively, $o(3)$, $o(4)$, $e(3)$ and $o(3,1)$, at least up to isomorphism. We shall say no more about ℓ_- , except to presage what follows with an exercise :

Exercise. *In analogy to the relation between ℓ_+ and $M_{\rm closed}$ presented just below, let $M_{\rm hyper}$ be the points in $M_{\rm Kepler}$ where $H_{\rm Kepler}$ is positive. Show that these are the points in phase space corresponding to hyperbolic orbits. Find an interesting injective Lie morphism*

$$\phi_- \ : \ \ell_- \ \longrightarrow \ C^\infty(M_{\rm hyper}, \mathbf{R}) \ ,$$

whose image lies in the Lie subalgebra of conserved quantities for hyperbolic orbits, including a conserved vector analogous to Runge-Lenz.

Definition. A Lie morphism

$$\phi_0 \ : \ \ell_0 \longrightarrow C^\infty(M_{\rm free}, \mathbf{R})$$

will be determined by

$$(B_{\vec v}, 0) \ \mapsto \ [(\vec q, \vec p) \mapsto < \vec q \times \vec p, \vec v >] \ ,$$

and

$$(0, B_{\vec v}) \ \mapsto \ [(\vec q, \vec p) \mapsto < \vec p, \vec v >] \ .$$

As we'll see below, these two specifications correspond, for the free particle, to conservation respectively of angular and of linear momentum.

Definition. A Lie morphism

$$\phi_+ \ : \ \ell_+ \longrightarrow C^\infty(M_{\rm closed}, \mathbf{R})$$

will be determined by

$$(B_{\vec v}, 0) \ \mapsto \ [(\vec q, \vec p) \mapsto < \vec q \times \vec p, \vec v >] \ ,$$

and

$$(0, B_{\vec v}) \ \mapsto \ [\ (\vec q, \vec p) \mapsto \frac{1}{\sqrt{-\mu^{-1} H_{\rm closed}(\vec q, \vec p)}} < \vec\Lambda, \vec v > \ ] \ ,$$

where $\vec\Lambda \ := \ \vec p \times (\vec q \times \vec p) \ - \ \frac{\nu}{2\mu|\vec q|} \ \vec q$.

The next 2×2 theorems with $\epsilon = +$ tell us that
(i) the last formula is 'Runge-Lenz obtained by Noetherianism'; and
(ii) the first is again conservation of angular momentum, this time for the Kepler particle.

Theorems. (Conservation$_\epsilon$) *The two functions on the right-hand sides in the definition of* ϕ_ϵ *above are conserved quantities for the corresponding hamiltonian systems—namely* $(M_{\rm free}, H_{\rm free})$ *when* $\epsilon = 0$ *, and* $(M_{\rm closed}, H_{\rm closed})$ *when* $\epsilon = +$ *.*

Theorems. (Lie$_\epsilon$) *The three bracket relations defining* $[\ ,\ ]_\epsilon$ *also hold for the functions* $\phi_\epsilon(B_{\vec v}, 0)$ *and* $\phi_\epsilon(0, B_{\vec v})$*, using the Poisson bracket in the corresponding phase spaces—namely* $(M_{\rm free}, \omega)$ *when* $\epsilon = 0$ *, and* $(M_{\rm closed}, \omega)$ *when* $\epsilon = +$ *.*

The proofs are given after the appendix just below.

At this point one can speculate less vaguely as to what has been in physicists' minds when they refer to the Runge-Lenz conserved vector as coming from a hidden symmetry : it is *hidden* partly because it is hard to discover, partly because it doesn't come from a symmetry of configuration space itself (more-or-less the physical space of Newton), and partly because it doesn't apply to the entire phase space of the Kepler problem, only the part corresponding to elliptical orbits. Perhaps there are other aspects of "hidden" here, which themselves are hidden from me??—such as obscure cross-product identities?

Appendix : The Vector Cross-Product.

We'll accept without proof its easier Lie algebra properties : bilinearity and anti-symmetry.

Two further identities, (1) and (3) below, may be obtained at a single stroke from its connection with quaternionic multiplication and the standard inner product on $\mathbf{R}^3$:

$$\vec a * \vec b \;=\; - < \vec a, \vec b > \oplus\ (\vec a \times \vec b) \quad .$$

Here $*$ is the quaternionic multiplication on $\mathbf{R}^4 \;=\; \mathbf{R} \oplus \mathbf{R}^3$, though we are applying it only to pure imaginary $\vec a$ and $\vec b$ in $\mathbf{R}^3$ "=" $\{0\} \oplus \mathbf{R}^3$.

We have

$$\vec a*(\vec b*\vec c) \;=\; \vec a*[- < \vec b, \vec c > \oplus (\vec b\times\vec c)] \;=\; - < \vec a, \vec b\times\vec c > \oplus\, [\vec a\times(\vec b\times\vec c) - \ < \vec b, \vec c > \vec a\,] \ ;$$

and

$$(\vec a*\vec b)*\vec c \;=\; [- < \vec a, \vec b > \oplus (\vec a\times\vec b)]*\vec c \;=\; - < \vec a\times\vec b, \vec c > \oplus\, [(\vec a\times\vec b)\times\vec c - \ < \vec a, \vec b > \vec c\,] \ .$$

Also used here is that the quaternions form an algebra over $\mathbf{R} \;=\; \mathbf{R} \oplus \{\vec{0}\}$.

Since $*$ is associative, equating real parts gives

$$< \vec{a} \times \vec{b}, \vec{c} > \;\;=\;\; < \vec{a}, \vec{b} \times \vec{c} > \qquad\qquad (1) ,$$

yielding

$$< \vec{b}, \vec{b} \times \vec{c} > \;\;=\;\; 0 \qquad\qquad (2) ;$$

and equating pure imaginary parts gives

$$\vec{a} \times (\vec{b} \times \vec{c}) \;-\; (\vec{a} \times \vec{b}) \times \vec{c} \;=\; < \vec{b}, \vec{c} > \vec{a} \;-\; < \vec{b}, \vec{a} > \vec{c} \qquad (3) .$$

Using that $\vec{a} \times \vec{a} \;=\; \vec{0}$, identity (3) specializes to

$$\vec{a} \times (\vec{a} \times \vec{c}) \;=\; < \vec{c}, \vec{a} > \vec{a} \;-\; |\vec{a}|^2 \vec{c} \qquad (4) .$$

Now one may easily prove the more general

$$\vec{a} \times (\vec{b} \times \vec{c}) \;=\; < \vec{a}, \vec{c} > \vec{b} \;-\; < \vec{a}, \vec{b} > \vec{c} \qquad (5)$$

as follows : When $\{\vec{b}, \vec{c}\}$ is linearly dependent, it is immediate that both sides are $\vec{0}$. Otherwise, for that fixed $\{\vec{b}, \vec{c}\}$, both sides are linear functions of $\vec{a}$, so need only be seen to agree for $\vec{a}$ in the basis $\{ \vec{b} \times \vec{c} \,,\, \vec{b} \,,\, \vec{c} \}$. [**Ex.** As an amusement if nothing else, using only bilinearity, alternativity, (2) and (3), show that indeed, if $\{ \vec{b} \,,\, \vec{c} \}$ is linearly independent, then so is $\{ \vec{b} \times \vec{c} \,,\, \vec{b} \,,\, \vec{c} \}$!] When $\vec{a} = \vec{b} \times \vec{c}$, both sides are $\vec{0}$, using (2) twice on the right. When $\vec{a} = \vec{b}$, identity (5) reduces to (4); and essentially does so when $\vec{a} = \vec{c}$ as well.

Either adding two instances of (5), or just (3) on its own, yields

$$\vec{a} \times (\vec{b} \times \vec{c}) \;+\; \vec{c} \times (\vec{a} \times \vec{b}) \;=\; < \vec{b}, \vec{c} > \vec{a} \;-\; < \vec{b}, \vec{a} > \vec{c} \quad .$$

But (5) shows that the right-hand side is $\vec{b} \times (\vec{a} \times \vec{c})$ which equals $-\vec{b} \times (\vec{c} \times \vec{a})$, so we've now proved the Jacobi identity also.

Let us finish this appendix by recording four easy, useful, and closely related analytical identities. Take $\vec{f} \;=\; \vec{f}(\vec{q}, \vec{p})$, and also $\vec{g}$, to be functions $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$. If $\vec{f}$ factors through the *right* projection $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ (more prosaically, 'is a function of $\vec{p}$ only, not $\vec{q}$ '), then

$$\vec{\nabla}_* < \vec{q}, \vec{f} > \;=\; \vec{f} \quad \text{and} \quad \vec{\nabla}_* < \vec{p}, \vec{f} > \;=\; \vec{0} \qquad\qquad (6) .$$

Similarly, if $\vec{f}$ factors through the *left* projection (more prosaically, 'is a function of $\vec{q}$ only, not $\vec{p}$ '), then

$$\vec{\nabla}_{**} < \vec{p}, \vec{f} > \ = \ \vec{f} \quad \text{and} \quad \vec{\nabla}_{**} < \vec{q}, \vec{f} > \ = \ \vec{0} \qquad (7) \ .$$

These follow immediately from the definitions of $\vec{\nabla}_*$ and $\vec{\nabla}_{**}$, and the evident product rules for them. The second identity in the displays should be generalized to

$$\vec{\nabla}_* < \vec{g}, \vec{f} > \ = \ \vec{0} \quad \text{or} \quad \vec{\nabla}_{**} < \vec{g}, \vec{f} > \ = \ \vec{0} \quad \text{respectively} \ ,$$

when $\vec{f}$ and $\vec{g}$ are functions of $\vec{p}$ only, resp. of $\vec{q}$ only.
Exercise. By applying (1) and (5), show that

$$< \vec{a} \times \vec{b} \ , \ \vec{u} \times \vec{v} > \ = \ < \vec{a} \ , \ \vec{u} >< \vec{b} \ , \ \vec{v} > - < \vec{a} \ , \ \vec{v} >< \vec{b} \ , \ \vec{u} > \ .$$

Then deduce that

$$\vec{\nabla}_{**} < \vec{p} \times \vec{a} \ , \ \vec{p} \times \vec{b} > \ = \ \vec{a} \times (\vec{p} \times \vec{b}) \ + \ \vec{b} \times (\vec{p} \times \vec{a}) \ .$$

for $\vec{a}$ and $\vec{b}$ which are functions of $\vec{q}$ only.

Proof of Conservation$_0$.

Abbreviating $H_{\rm free}$ to just H, we have, with $\mu = \frac{1}{2m}$,

$$H(\vec{q}, \vec{p}) = \mu(p_1^2 + p_2^2 + p_3^2) \quad \text{so} \quad \vec{\nabla}_* H = \vec{0} \ \text{ and } \ \vec{\nabla}_{**} H = 2\mu\vec{p} \ .$$

Thus, for any function E,

$$\{ \ H \ , \ E \ \} \ = \ < \vec{\nabla}_{**} H, \vec{\nabla}_* E > \ = \ 2\mu < \vec{p}, \vec{\nabla}_* E > \ .$$

Now for $E = \phi_0(0, B_{\vec{v}})$, we have $E(\vec{q}, \vec{p}) =< \vec{p}, \vec{v} >$, giving $\vec{\nabla}_* E = \vec{0}$ by (6), and so $\{ \ H \ , \ E \ \} = 0$, as required.
And for $E = \phi_0(B_{\vec{v}}, 0)$, we have $E(\vec{q}, \vec{p}) =< \vec{q} \times \vec{p}, \vec{v} >=< \vec{q}, \vec{p} \times \vec{v} >$, giving $\vec{\nabla}_* E = \vec{p} \times \vec{v}$ by (6), and so

$$\{ \ H \ , \ E \ \} \ = \ 2\mu < \vec{p}, \vec{p} \times \vec{v} > \ = \ 0 \ ,$$

by (2), as required.

Proof of $\mathbf{Lie}_0$.

The hardest identity needed, which arises from the formula for $[(B,0),(C,0)]_{{}_0}$ in the definition of ℓ_0, together with the formula defining $\phi_0\ (B_{\vec{v}},0)$, reduces to

$$\{\ <\vec{q}\times\vec{p},\vec{u}>\ ,\ <\vec{q}\times\vec{p},\vec{v}>\ \}\ =\ -<\vec{q}\times\vec{p},\vec{u}\times\vec{v}>\ ,$$

by using the 'isomorphism' between ℓ and the cross-product :
$[B_{\vec{u}}\ ,\ B_{\vec{v}}]\ =\ -B_{\vec{u}\times\vec{v}}$. One is thinking of $\vec{u}$ and $\vec{v}$ as fixed, and both sides as functions of $(\vec{q},\vec{p})$. (See the final case in the proof of $\mathbf{Lie}_+$ further down for details concerning a much harder example of reducing as claimed just above.)

The left-hand side, using (1), (6) and (7), is

$$<\vec{\nabla}_{**}<\vec{q}\times\vec{p},\vec{u}>,\vec{\nabla}_{*}<\vec{q}\times\vec{p},\vec{v}>>-<\vec{\nabla}_{*}<\vec{q}\times\vec{p},\vec{u}>,\vec{\nabla}_{**}<\vec{q}\times\vec{p},\vec{v}>>$$

$$=\ <\vec{\nabla}_{**}<-\vec{p},\vec{q}\times\vec{u}>,\vec{\nabla}_{*}<\vec{q},\vec{p}\times\vec{v}>>-<\vec{\nabla}_{*}<\vec{q},\vec{p}\times\vec{u}>,\vec{\nabla}_{**}<-\vec{p},\vec{q}\times\vec{v}>>$$

$$=\ <\vec{p}\times\vec{u},\vec{q}\times\vec{v}>-<\vec{q}\times\vec{u},\vec{p}\times\vec{v}>\ .$$

The right-hand side, using Jacobi and (1), is

$$-<\vec{q},\vec{p}\times(\vec{u}\times\vec{v})>\ =\ <\vec{q},\vec{v}\times(\vec{p}\times\vec{u})+\vec{u}\times(\vec{v}\times\vec{p})>$$

$$=\ <\vec{q}\times\vec{v},\vec{p}\times\vec{u}>+<\vec{q}\times\vec{u},\vec{v}\times\vec{p}>\ =\ <\vec{p}\times\vec{u},\vec{q}\times\vec{v}>-<\vec{q}\times\vec{u},\vec{p}\times\vec{v}>\ ,$$

agreeing with the previous, as required.

The other two identities are quicker :

$$\{\ <\vec{p},\vec{u}>\ ,\ <\vec{p},\vec{v}>\ \}\ =\ 0\quad\text{since}\quad\vec{\nabla}_{*}<\vec{p},\vec{u}>\ =\ \vec{0}\ =\ \vec{\nabla}_{*}<\vec{p},\vec{v}>\ .$$

$$\{\ <\vec{q}\times\vec{p},\vec{u}>\ ,\ <\vec{p},\vec{v}>\ \}\ =\ <\vec{\nabla}_{**}<\vec{q}\times\vec{p},\vec{u}>,\vec{0}>-<\vec{\nabla}_{*}<\vec{q}\times\vec{p},\vec{u}>,\vec{\nabla}_{**}<\vec{p},\vec{v}>>$$

$$=\ -<\vec{\nabla}_{*}<\vec{q},\vec{p}\times\vec{u}>,\vec{v}>\ =\ -<\vec{p}\times\vec{u},\vec{v}>\ =\ -<\vec{p},\vec{u}\times\vec{v}>\ .$$

(Again, the negative sign is correct because of the connection between ℓ and the cross-product.)

Proposition. *Recall $\vec{\Lambda}$, from the definition of ϕ_+ . We have, for all $\vec{v}$,*

$$\vec{\nabla}_* < \vec{\Lambda}, \vec{v} > \;=\; -\,\vec{p}\times(\vec{p}\times\vec{v}) \;-\; \frac{\nu}{2\mu|\vec{q}|^3}\,\vec{q}\times(\vec{v}\times\vec{q})\ ,$$

and

$$\vec{\nabla}_{**} < \vec{\Lambda}, \vec{v} > \;=\; \vec{q}\times(\vec{p}\times\vec{v}) \;+\; \vec{v}\times(\vec{p}\times\vec{q})\ .$$

The right-hand sides can be made even less readable using (5)!

Proof. Using the product rule for the 1st equality, the left-hand side of the top one is

$$\vec{\nabla}_* < \vec{p}\times(\vec{q}\times\vec{p}), \vec{v} > -\frac{\nu}{2\mu}\vec{\nabla}_*(\frac{1}{|\vec{q}|} < \vec{q}, \vec{v} >)$$

$$= \;\vec{\nabla}_* < -\vec{q}\times\vec{p}, \vec{p}\times\vec{v} > -\frac{\nu}{2\mu}(\frac{1}{|q|}\vec{v} + \frac{< \vec{q}, \vec{v} >}{|\vec{q}|^3}\vec{q}\ )$$

$$= \;-\,\vec{\nabla}_* < \vec{q}, \vec{p}\times(\vec{p}\times\vec{v}) > -\frac{\nu}{2\mu|q|^3}(|q|^2\vec{v} - < \vec{q}, \vec{v} > \vec{q}\ )\ ,$$

which agrees with its right-hand side by general identities (6) and (4) in the recent appendix.

The left-hand side of the 2nd equality is

$$\vec{\nabla}_{**} < \vec{p}\times(\vec{q}\times\vec{p}), \vec{v} > -\frac{\nu}{2\mu}\vec{\nabla}_{**} < \frac{\vec{q}}{|q|}, \vec{v} > \;=\; \vec{\nabla}_{**} < -\vec{q}\times\vec{p}\ ,\ \vec{p}\times\vec{v} > -\vec{0}$$

$$= \;\vec{\nabla}_{**} < \vec{p}\times\vec{q}\ ,\ \vec{p}\times\vec{v} >\ ,$$

agreeing with its right-hand side, by the exercise at the end of the appendix.

Proof of Conservation$_+$.
Abbreviating H_{closed} to just H, we have

$$H(\vec{q}, \vec{p}) = \mu(p_1^2 + p_2^2 + p_3^2) - \nu(q_1^2 + q_2^2 + q_3^2)^{-1/2}$$

so $$\vec{\nabla}_* H = \frac{\nu\vec{q}}{|\vec{q}|^3} \qquad \text{and} \qquad \vec{\nabla}_{**} H = 2\mu\vec{p}\ .$$

Thus, for any function E,

$$\{ H \ , \ E \} \ = \ 2\mu < \vec{p}, \vec{\nabla}_* E > - \ \frac{\nu}{|\vec{q}|^3} < \vec{q}, \vec{\nabla}_{**} E > \ .$$

Now for $E = \phi_0(B_{\vec{v}}, 0)$, we have $\vec{\nabla}_* E = \vec{p} \times \vec{v}$ (as in the proof of **Conservation**$_0$), and $\vec{\nabla}_{**} E = -\vec{q} \times \vec{v}$ (symmetrically). So

$$\{ H \ , \ E \} \ = \ 2\mu < \vec{p}, \vec{p} \times \vec{v} > + \ \frac{\nu}{|\vec{q}|^3} < \vec{q}, \vec{q} \times \vec{v} > \ = \ 0 + 0 \ = \ 0 \ ,$$

using (2) twice, as required.
And for $E = \phi_0(0, B_{\vec{v}})$, we must prove

$$2\mu < \vec{p} \ , \ \vec{\nabla}_* < \vec{\Lambda}, \vec{v} >> \ = \ \frac{\nu}{|\vec{q}|^3} < \vec{q} \ , \ \vec{\nabla}_{**} < \vec{\Lambda}, \vec{v} >> \ .$$

This suffices, because H itself is conserved, and the set of conserved functions is closed under ordinary products and under composing with real valued functions, so the factor $1/\sqrt{-\mu^{-1}H}$ can be ignored here, and we need only check that $\{ \ H(\vec{q}, \vec{p}) \ , \ < \vec{\Lambda}(\vec{q}, \vec{p}) \ , \ \vec{v} > \ \} \ = \ 0$. Using the proposition and that $< \vec{q} \ , \ \vec{q} \times \vec{a} >= 0$, the right-hand side of the display is

$$\frac{\nu}{|\vec{q}|^3} < \vec{q} \ , \ \vec{q} \times (\vec{p} \times \vec{v}) \ + \ \vec{v} \times (\vec{p} \times \vec{q}) > \ = \ \frac{\nu}{|\vec{q}|^3} < \vec{q} \ , \ \vec{v} \times (\vec{p} \times \vec{q}) > \ .$$

Again using the proposition and that $< \vec{p} \ , \ \vec{p} \times \vec{a} >= 0$, the left-hand side is

$$2\mu < \vec{p} \ , \ -\vec{p} \times (\vec{p} \times \vec{v}) \ - \ \frac{\nu}{2\mu|\vec{q}|^3} \vec{q} \times (\vec{v} \times \vec{q}) > \ = \ - \frac{\nu}{|\vec{q}|^3} < \vec{p} \ , \ \vec{q} \times (\vec{v} \times \vec{q}) > \ .$$

But the answers agree, as required, since, applying (1),

$$< \vec{q} \ , \ \vec{v} \times (\vec{p} \times \vec{q}) > \ = \ - < \vec{p} \times \vec{q} \ , \ \vec{v} \times \vec{q}) > \ = \ - < \vec{p} \ , \ \vec{q} \times (\vec{v} \times \vec{q}) > \ .$$

Proof of Lie$_+$.

The easiest of the three identities here is the one called "hardest" in the proof of **Lie**$_0$; and its proof is the same as there, since ϕ_0 and ϕ_+ agree on $(B_{\vec{v}}, 0)$, and since $[\ ,\ ]_0$ and $[\ ,\ ]_+$ agree on pairs $(B, 0), (C, 0)$.

The next one, for pairs $(B_{\vec{u}}, 0), (0, B_{\vec{v}})$, easily reduces, as in the proof of $\mathbf{Lie}_0$, to showing [for functions of $(\vec{q}, \vec{p})$—the $(\vec{u}, \vec{v})$ is fixed]

$$\{ \ < \vec{q} \times \vec{p}, \vec{u} > \ , \ < \vec{\Lambda}, \vec{v} > \ \} \ = \ - < \vec{\Lambda}, \vec{u} \times \vec{v} > \ .$$

Using the proposition and the appendix identities, the left-hand side may be calculated as

$$< \vec{\nabla}_{**} < \vec{q} \times \vec{p}, \vec{u} >, \vec{\nabla}_{*} < \vec{\Lambda}, \vec{v} >> - < \vec{\nabla}_{*} < \vec{q} \times \vec{p}, \vec{u} >, \vec{\nabla}_{**} < \vec{\Lambda}, \vec{v} >>$$

$$< -\vec{q} \times \vec{u} \, , \, -\vec{p} \times (\vec{p} \times \vec{v}) - \frac{\nu}{2\mu |\vec{q}|^3} \vec{q} \times (\vec{v} \times \vec{q}) > - < \vec{p} \times \vec{u} \, , \, \vec{q} \times (\vec{p} \times \vec{v}) + \vec{v} \times (\vec{p} \times \vec{q}) > \ .$$

The right-hand side is, by the definition of $\vec{\Lambda}$,

$$- < \vec{p} \times (\vec{q} \times \vec{p}) \ , \ \vec{u} \times \vec{v} > \ + \ \frac{\nu}{2\mu |\vec{q}|} < \vec{q} \ , \ \vec{u} \times \vec{v} > \ .$$

To check their equality, firstly the terms containing a fraction with ν in the numerator agree by applying identity (4) to $\vec{q} \times (\vec{v} \times \vec{q})$, producing two summands, one of which disappears because $< \vec{q} \times \vec{u} \ , \ \vec{q} > \ = 0$; and the other summand agrees with the $\frac{\nu}{2\mu|\vec{q}|} < \vec{q} \ , \ \vec{u} \times \vec{v} >$ from the right-hand side. To finish, we may write all the four remaining terms in the form $< \vec{u} \ , \ *** >$. Then one must show that the $***$ agree. Specifically this becomes

$$-\vec{q} \times (\vec{p} \times (\vec{p} \times \vec{v})) \ + \ \vec{p} \times (\vec{q} \times (\vec{p} \times \vec{v})) \ + \ \vec{p} \times (\vec{v} \times (\vec{p} \times \vec{q})) \ = \ -\vec{v} \times (\vec{p} \times (\vec{q} \times \vec{p}) \ .$$

It can be dealt with by a semi-massive application of identity (5) from the appendix (the massive one being in the final part of this proof!). The above identity becomes, keeping the same order,

$$- < \vec{q} \ , \ \vec{p} \times \vec{v} > \vec{p} \ + \ < \vec{q} \ , \ \vec{p} > \vec{p} \times \vec{v} \ + \ < \vec{p} \ , \ \vec{p} \times \vec{v} > \vec{q}$$

$$- \ < \vec{p} \ , \ \vec{q} > \vec{p} \times \vec{v} \ + \ < \vec{p} \ , \ \vec{p} \times \vec{q} > \vec{v} \ - \ < \vec{p} \ , \ \vec{v} > \vec{p} \times \vec{q}$$

$$= \ - < \vec{v} \ , \ \vec{q} \times \vec{p} > \vec{p} \ + \ < \vec{v} \ , \ \vec{p} > \vec{q} \times \vec{p} \ .$$

The 3rd and 5th summands are $\vec{0}$ because $< \vec{p} \ , \ \vec{p} \times \vec{a} >= 0$; and the rest easily pair off into three canceling pairs.

(One is comforted, from the viewpoint of physical units, that all the summands in those last two unpleasant displays, are 'products' of two p's, one q, and one v ; that doesn't differ from one summand to the next.)

For the final identity needed, we shall include all the small initial details. (So this can serve as a model for filling in the analogous, but simpler, details in the earlier two parts of this proof and in all three parts of the proof of $\mathbf{Lie}_0$.)

We are asked to prove, for all $\vec{u}$ and $\vec{v}$, that

$$\{\ \phi_+(0, B_{\vec{u}})\ ,\ \phi_+(0, B_{\vec{v}})\ \}\ =\ \phi_+[(0,\ B_{\vec{u}}), (0,\ B_{\vec{v}})]_+\quad ;$$

that is,

$$\{\ \frac{1}{\sqrt{-\mu^{-1}H}} < \vec{\Lambda}, \vec{u} >\ ,\ \frac{1}{\sqrt{-\mu^{-1}H}} < \vec{\Lambda}, \vec{v} >\ \}\ =\ \phi_+([B_{\vec{u}}, B_{\vec{v}}], 0)$$

$$=\ \phi_+(-B_{\vec{u}\times\vec{v}}, 0)\ =\ - < \vec{q}\times\vec{p}, \vec{u}\times\vec{v} > \quad .$$

But for any smooth real-valued functions β and γ of a real variable, and any H-conserved quantities F and G, we have

$$\{\ \beta\circ H\ ,\ F\ \}\ =\ \beta'\circ\{\ H\ ,\ F\ \}\ =\ \beta'\circ 0\ =\ 0\quad .$$

And so, applying the Leibniz rule a few times,

$$\{\ (\beta\circ H)\cdot F\ ,\ (\gamma\circ H)\cdot G\ \}$$

$$=\ (\beta\circ H)\cdot(\gamma\circ H)\{\ F\ ,\ G\} + F\cdot(\gamma\circ H)\{\ \beta\circ H\ ,\ G\} + G\cdot(\beta\circ H)\{\ F\ ,\ \gamma\circ H\} + F\cdot G\{\ \beta\circ H\ ,\ \gamma\circ H\}$$

$$=\ (\beta\circ H)\cdot(\gamma\circ H)\{\ F\ ,\ G\} + 0 + 0 + 0\ =\ (\beta\circ H)\cdot(\gamma\circ H)\{\ F\ ,\ G\}\ .$$

Applying this with $\beta(t) = \gamma(t) = \frac{1}{\sqrt{-\mu^{-1}t}}$, our required identity simplifies to

$$\{\ < \vec{\Lambda}, \vec{u} >\ ,\ < \vec{\Lambda}, \vec{v} >\ \}\ =\ \mu^{-1}H < \vec{q}\times\vec{p}, \vec{u}\times\vec{v} > \quad .$$

(The right-hand side does have two canceling negative signs.) This completes the initial details referred to above.

By the proposition, the left-hand side is

$$< \vec{\nabla}_{**} < \vec{\Lambda}, \vec{u} >, \vec{\nabla}_{*} < \vec{\Lambda}, \vec{v} >> - < \vec{\nabla}_{*} < \vec{\Lambda}, \vec{u} >, \vec{\nabla}_{**} < \vec{\Lambda}, \vec{v} >>$$

$$= \ < \vec{q} \times (\vec{p} \times \vec{u}) \ + \ \vec{u} \times (\vec{p} \times \vec{q}) \ , \ \ - \vec{p} \times (\vec{p} \times \vec{v}) \ - \ \frac{\nu}{2\mu|\vec{q}|^3} \ \vec{q} \times (\vec{v} \times \vec{q}) >$$

$$- \ < \vec{q} \times (\vec{p} \times \vec{v}) \ + \ \vec{v} \times (\vec{p} \times \vec{q}) \ , \ \ - \vec{p} \times (\vec{p} \times \vec{u}) \ - \ \frac{\nu}{2\mu|\vec{q}|^3} \ \vec{q} \times (\vec{u} \times \vec{q}) >$$

$$= \ < \vec{q}\times(\vec{p}\times\vec{v}) + \vec{v}\times(\vec{p}\times\vec{q}) \ , \ \vec{p}\times(\vec{p}\times\vec{u}) > \ - \ < \vec{q}\times(\vec{p}\times\vec{u}) + \vec{u}\times(\vec{p}\times\vec{q}) \ , \ \vec{p}\times(\vec{p}\times\vec{v}) >$$

$$+ \frac{\nu}{\mu|\vec{q}|} \ \frac{1}{2|\vec{q}|^2}(< \vec{q}\times(\vec{p}\times\vec{v}) + \vec{v}\times(\vec{p}\times\vec{q}) \ , \ \ \vec{q}\times(\vec{u}\times\vec{q}) > \ - \ < \vec{q}\times(\vec{p}\times\vec{u}) + \vec{u}\times(\vec{p}\times\vec{q}) \ , \ \ \vec{q}\times(\vec{v}\times\vec{q}) >).$$

The right-hand side is, by the definition of $\vec{\Lambda}$,

$$|\vec{p}|^2 < \vec{q} \times \vec{p} \ , \ \vec{u} \times \vec{v} > \ \ - \ \frac{\nu}{\mu|\vec{q}|} < \vec{q} \times \vec{p} \ , \ \vec{u} \times \vec{v} > \ \ .$$

Unsurprisingly, we can split the required equality of the last two displays into two separate equalities—the summands with, and those without, the factor $\frac{\nu}{\mu|\vec{q}|}$. So it remains to prove the following two equalities.

$$< \vec{q}\times(\vec{p}\times\vec{v}) + \vec{v}\times(\vec{p}\times\vec{q}) \ , \ \vec{p}\times(\vec{p}\times\vec{u}) > \ - \ < \vec{q}\times(\vec{p}\times\vec{u}) + \vec{u}\times(\vec{p}\times\vec{q}) \ , \ \vec{p}\times(\vec{p}\times\vec{v}) >$$

$$= \ |\vec{p}|^2 < \vec{q} \times \vec{p} \ , \ \vec{u} \times \vec{v} > \qquad (*) \ ;$$

and

$$< \vec{q}\times(\vec{p}\times\vec{v}) + \vec{v}\times(\vec{p}\times\vec{q}) \ , \ \ \vec{q}\times(\vec{u}\times\vec{q}) > \ - \ < \vec{q}\times(\vec{p}\times\vec{u}) + \vec{u}\times(\vec{p}\times\vec{q}) \ , \ \ \vec{q}\times(\vec{v}\times\vec{q}) >$$

$$= \ - 2|\vec{q}|^2 < \vec{q} \times \vec{p} \ , \ \vec{u} \times \vec{v} > \qquad (**) \ .$$

(It is again edifying that each of these has ‘physical units’ which are the same for all summands, roughly p^3quv and q^3puv.) For the right-hand sides of both of $(*)$ and $(**)$, we use the exercise at the end of the appendix to replace

$$< \vec{q} \times \vec{p} \ , \ \vec{u} \times \vec{v} > \quad \text{by} \quad < \vec{q} \ , \ \vec{u} >< \vec{p} \ , \ \vec{v} > - < \vec{q} \ , \ \vec{v} >< \vec{p} \ , \ \vec{u} > \ \ .$$

We shall describe how the left-hand sides produce this last expression, multiplied respectively by $|\vec{p}|^2$ and by $-2|\vec{q}|^2$, as required.

Proving $(*)$ is done by a brute-force massive application of (5) from the appendix to the left-hand side. (See the immediately previous case in this proof for a more detailed example of this type of calculation.) This produces 16 summands. Five of these are $|\vec{p}|^2 < \vec{q}\ ,\ \vec{u} >< \vec{p}\ ,\ \vec{v} >$, with three having positive signs and two negative. Another five are $|\vec{p}|^2 < \vec{q}\ ,\ \vec{v} >< \vec{p}\ ,\ \vec{u} >$, this time three with negative signs. So those ten summands give the correct answer, as long as the other six summands cancel in pairs. They do. If you don't believe me, just try it! And even if you do, try it anyway.

The proof of $(**)$ is very similar. Massive application of (5) to the left-hand side produces 16 summands once again. This time, there are four of the form $|\vec{q}|^2 < \vec{q}\ ,\ \vec{u} >< \vec{p}\ ,\ \vec{v} >$, with three being negative and one positive. Another four are $|\vec{q}|^2 < \vec{q}\ ,\ \vec{v} >< \vec{p}\ ,\ \vec{u} >$, this time three being positive. So those eight summands give the correct answer, since the remaining eight summands cancel in pairs.

Thus $\mathbf{Lie}_+$ is proved.

These calculations are not at all difficult, just tedious. However, our 'explanation' of the Runge-Lenz phenomenon is hardly a triumph of the conceptual/geometric over the calculational/algebraic! The treatment in the first chapter of [**Guill-Stern**] appears rather more elegant. However, some of the calculation is left to the reader. I have been assiduously thorough (within reason) I think! And the elegance of [**Guill-Stern**]'s treatment is in the opposite direction really, to our straightforward verification that the formulas produce conserved quantities which generate a Lie algebra of a particular type, namely $o(4)$ [and $e(3)$ for the free particle]. Roughly speaking, they *assume* the existence of a Runge-Lenz-type conserved vector, and deduce from that the need for the hamiltonian to take the form of H_{Kepler}. That is, they basically *deduce* the inverse-square law for the central force! That's cool.

So it seems that Noether's theorem, or its transmogrification into hamiltonian form, is just as often used to look for the symmetry which might have produced some 'accidentally' discovered conservation law, as it is to employ one's geometric intuition to find symmetries, and then transform them into conservation laws. In fact, as mentioned in the previous paragraph, the thrust of the calculations just above is mainly to work out the nature of the symmetry, that is, the isomorphism type of the Lie algebra generated by the conserved quantities. A previous exercise shows that, in the case of non-closed 'orbits' for the Kepler problem, the Lie algebra generated by combining angular momentum conservation with a Runge-Lenz-type conserved vector is isomorphic to $o(3,1)$, the Lie algebra of the generalized Lorentz group from special relativity. Chapter 2 onwards in [**Guill-Stern**] illustrates profusely how important this type of calculation can be for understanding quantization theory.

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