On irregularities of distribution in shifts and dilations of integer sequences, II

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Dedicated to Professor A. Schinzel on the occasion of his sixtieth birthday

1 Introduction

Let N be a positive integer. In 1964, Roth [3], see also [4] and [5], proved that no matter how we partition $\{1,\ldots,N\}$ into two sets there will always be an arithmetic progression which contains a preponderance of terms from one of the two sets. In particular, let $\varepsilon_1,\ldots,\varepsilon_N$ be elements of $\{1,-1\}$ and put $\varepsilon_i=0$ for $i<1$ and $i>N$. He proved that there exist positive numbers c_0 and c_1 such that if N exceeds c_0 then

$$\max_{a,q,t\in\mathbb{Z}^+}\left|\sum_{j=1}^{t}\varepsilon_{a+j\,q}\right|>c_1N^{1/4}\ . \tag{1}$$

In 1981, Beck [1] proved that (1) is best possible, apart from a logarithmic factor, by proving that there exists a positive number c_2 such that for each integer N larger than one there is a sequence $(\varepsilon_1,\ldots,\varepsilon_N)$ of plus and minus ones for which

$$\max_{a,q,t\in\mathbb{Z}^+}\left|\sum_{j=1}^{t}\varepsilon_{a+j\,q}\right|<c_2N^{1/4}(\log N)^{5/2}\ . \tag{2}$$

Recently Matousek and Spencer [2] proved that the factor $(\log N)^{5/2}$ can be removed in (2) and so (1) is best possible up to a constant factor.

In 1987 Sárközy and Stewart [7] generalized Roth's result by studying shifts and dilations of arbitrary sequences of positive integers in place of arithmetical

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progressions. In particular, let $\varepsilon_1, \ldots, \varepsilon_N$ be complex numbers and put $\varepsilon_i = 0$ for $i < 1$ and $i > N$. Let $(b_1, b_2, \ldots)$ be any sequence of positive integers. Their objective was to estimate

$$\max_{a \in \mathbb{Z}, q, t \in \mathbb{Z}^+} \left| \sum_{j=1}^{t} \varepsilon_{a+b_j q} \right| . \tag{3}$$

They were able to obtain non-trivial lower bound for (3) when $(b_1, b_2, \ldots)$ is an increasing sequence which does not grow more quickly than the sequence of squares. For the sequence of squares they proved the following result. Let δ be a positive real number and suppose that $\varepsilon_1 \ldots, \varepsilon_N$ are of absolute value one. There is a number $c_3(\delta)$, which is effectively computable in terms of δ, such that if N exceeds $c_3(\delta)$ then

$$\max_{a \in \mathbb{Z}, q, t \in \mathbb{Z}^+} \left| \sum_{j=1}^{t} \varepsilon_{a+j^2 q} \right| > N^{1/6} \exp(-(1+\delta)(\log 2 \log N)/3 \log\log N) . \tag{4}$$

Our goal in this paper is to strengthen and to extend the range of applicability of the estimates of [7]. For instance, we shall prove that the exponential factor on the right hand side of inequality (4) may be replaced by a constant. Further we are able to treat shifts and dilations of the sequence of rth powers for any positive integer r.

Theorem 1. *Let N, t and Q be positive integers with*

$$2t \leq Q . \tag{5}$$

Let $b_1, \ldots, b_t$ be positive integers with $b_t = \max_i b_i$ and let $\varepsilon_1, \ldots, \varepsilon_N$ be complex numbers. Put $\varepsilon_i = 0$ for $i < 1$ and $i > N$. Then

$$\sum_{q=1}^{Q} \sum_{a=-Qb_t+1}^{N} \left| \sum_{j=1}^{t} \varepsilon_{a+b_j q} \right|^2 \geq \frac{tQ}{4} \sum_{n=1}^{N} |\varepsilon_n|^2 . \tag{6}$$

Condition (5) is necessary and cannot be weakened by a factor of 2 as the following example shows. Let p be a prime number and put $Q = t = p - 1$, and $b_j = j$ for $j = 1, \ldots, t$. Let N be a positive integer and put $\varepsilon_n = \left(\dfrac{n}{p}\right)$ for $n = 1, \ldots, N$, where $\left(\dfrac{n}{p}\right)$ denotes the Legendre symbol of n modulo p. Notice that

$$\left| \sum_{j=1}^{t} \varepsilon_{a+b_j q} \right| = \left| \sum_{j=1}^{p-1} \left(\frac{a+jq}{p} \right) \right| \leq 1$$

for $1 \le q \le t = p-1$ and for $0 \le a \le N - t^2$. For $-Qb_t + 1 = -t^2 + 1 \le a < 0$ and for $N - t^2 < a \le N$ we certainly have

$$\left| \sum_{j=1}^{t} \varepsilon_{a+b_j q} \right| \le t$$

and thus the left hand side of inequality (6) is at most $tN + 2t^5$. On the other hand the right hand side of inequality (6) is at least $\frac{tQ}{4} N \left(\frac{p-1}{p}\right)$, hence at least $\frac{Nt^2}{8}$. A comparison of these two estimates yields a contradiction for t in the range $16 < t < (N/32)^{1/3}$. Furthermore, the lower bound in (6) is, in general, best possible apart from a constant factor as may be seen from the Rudin-Shapiro construction from harmonic analysis, see [6, Theorem 4] for details.

Choosing $Q = 2t$ in Theorem 1 we obtain the next result.

Corollary 1. *Let N and t be positive integers. Let $b_1, \ldots, b_t$ be positive integers with $b_t = \max_i b_i$ and let $\varepsilon_1, \ldots, \varepsilon_N$ be complex numbers. Put $\varepsilon_i = 0$ for $i < 1$ and $i > N$. If*

$$2tb_t \le N \ , \tag{7}$$

then

$$\max_{\substack{1 \le q \le 2t \\ -N < a \le N}} \left| \sum_{j=1}^{t} \varepsilon_{a+b_j q} \right| \ge \frac{t^{1/2}}{\sqrt{8}} \left(\frac{1}{N} \sum_{n=1}^{N} |\varepsilon_n|^2 \right)^{1/2} .$$

It follows from the result of Matousek and Spencer [2] that condition (7) can not be weakened by more than a constant factor.

For any real number x, let $[x]$ denote the greatest integer less than or equal to x. Let r be a positive integer and put

$$t = [(N/2)^{1/(r+1)}] \ .$$

We now apply Corollary 1 with $b_j = j^r$ for $j = 1, \ldots, t$. Observe that $t \ge (N^{1/r+1})/2$ provided that $N \ge 5^{r+1}$ and so we obtain the result below.

Corollary 2. *Let N and r be positive integers and assume N is at least 5^{r+1}. Let $\varepsilon_1, \ldots, \varepsilon_N$ be complex numbers of absolute value 1. Put $\varepsilon_i = 0$ for $i < 1$ and $i > N$. Then*

$$\max_{\substack{1 \le q \le 2N^{1/(r+1)} \\ -N < a \le N}} \left| \sum_{1 \le j \le (\frac{N}{2})^{1/(r+1)}} \varepsilon_{a+j^r q} \right| \ge \frac{1}{4} \ N^{1/2(r+1)} \ . \tag{8}$$

Certainly from (8) we have

$$\max_{a \varepsilon \mathbb{Z}, q, t \varepsilon \mathbb{Z}^+} \left| \sum_{j=1}^{t} \varepsilon_{a+j^r q} \right| \ge \frac{1}{4} N^{1/2(r+1)} \tag{9}$$

and we suspect that the lower bound on the right hand side of inequality (9) is best possible, up to a constant factor for each positive integer r. This is certainly the case when $r = 1$ by the result of Matousek and Spencer [2]. Finally, we remark that on taking the ε_i's to be plus or minus one we induce a partition of $\{1, \ldots, N\}$. Therefore, by Corollary 2, if N exceeds 5^{r+1} then no matter how we split $\{1, \ldots, N\}$ into two disjoint sets there will always be a shift and dilation of the sequence of r-th powers of positive integers which contains $\frac{1}{4}\, N^{1/2(r+1)}$ more terms from one set than from the other.

2 The proof of Theorem 1

For any real number α we denote $e^{2\pi i\alpha}$ by $e(\alpha)$. Put

$$S(\alpha) = \sum_{n=1}^{N} \varepsilon_n e(n\alpha) \ .$$

We now introduce the Féjer kernel. For each positive integer Q denote $F_Q(\alpha)$ by

$$F_Q(\alpha) = \left| \sum_{q=0}^{Q} e(q\alpha) \right|^2 = Q + 1 + \sum_{q=1}^{Q} (Q + 1 - q)(e(q\alpha) + e(-q\alpha)) \ ,$$

and observe that

$$F_Q(0) = (Q + 1)^2 \ . \tag{10}$$

Next, put

$$G(\alpha) = \sum_{j=1}^{t} \sum_{k=1}^{t} F_Q((b_j - b_k)\alpha) \ .$$

Observe that

$$G(\alpha) = t^2(Q + 1) + 2 \sum_{q=1}^{Q} (Q + 1 - q) \sum_{j=1}^{t} \sum_{k=1}^{t} e((b_j - b_k)q\alpha)$$

hence

$$G(\alpha) = t^2(Q + 1) + 2 \sum_{q=1}^{Q} (Q + 1 - q) \left| \sum_{j=1}^{t} e(b_j q\alpha) \right|^2 \ . \tag{11}$$

Our proof proceeds by a comparison of estimates for

$$J = \int_0^1 |\ S(\alpha)|^2\, G(\alpha) d\alpha \ .$$

We establish a lower bound first. Since $F_Q(\alpha)$ is always non-negative,

$$J \geq \int_0^1 |S(\alpha)|^2 \sum_{j=1}^{t} F_Q((b_j - b_j)\alpha) d\alpha \ ,$$

and so, by (10),

$$J \geq t(Q+1)^2 \ \int_0^1 |S(\alpha)|^2 \, d\alpha \ .$$

Therefore, by Parseval's formula,

$$J \geq t(Q+1)^2 \ \sum_{n=1}^{N} \ |\varepsilon_n|^2 \ . \tag{12}$$

On the other hand, by (11),

$$J = t^2(Q+1)\int_0^1 |S(\alpha)|^2 d\alpha + 2\sum_{q=1}^{Q}(Q+1-q)\int_0^1 \left| S(\alpha)\sum_{j=1}^{t} e(-b_j q\alpha)\right|^2 d\alpha. \tag{13}$$

Since

$$S\ (\alpha) \ \sum_{j=1}^{t} \ e(-b_j q\alpha) = \sum_{n=1}^{N} \ \sum_{j=1}^{t} \ \varepsilon_n \ e((n - b_j q)\alpha) \ ,$$

we have

$$S\ (\alpha) \ \sum_{j=1}^{t} \ e(-b_j q\alpha) = \sum_{a=-\infty}^{\infty} \ \delta_{a,q} \ e(a\alpha)$$

where

$$\delta_{a,q} = \sum_{j=1}^{t} \ \varepsilon_{a+b_j q} \quad \text{for } a \in \mathbb{Z} \ .$$

Notice that $\delta_{a,q} = 0$ when a exceeds N or when a is at most $-Qb_t$. From (13) and Parseval's formula we obtain

$$J \leq t^2(Q+1)\sum_{n=1}^{N} \ |\varepsilon_n|^2 + 2(Q+1)\sum_{q=1}^{Q} \ \sum_{a=-Qb_t+1}^{N} \ |\sum_{j=1}^{t} \ \varepsilon_{a+b_j q}|^2 \ . \tag{14}$$

The result now follows from (5), (12) and (14).

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