ON SHIFTED PRODUCTS WHICH ARE POWERS

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§1. *Introduction.* Fermat gave the first example of a set of four positive integers $\{a_1, a_2, a_3, a_4\}$ with the property that $a_i a_j + 1$ is a square for $1 \leqslant i < j \leqslant 4$. His example was $\{1, 3, 8, 120\}$. Baker and Davenport [**1**] proved that the example could not be extended to a set of 5 positive integers such that the product of any two of them plus one is a square. Kangasabapathy and Ponnudurai [**6**], Sansone [**9**] and Grinstead [**4**] gave alternative proofs. The construction of such sets originated with Diophantus who studied the problem when the a_i are rational numbers. It is conjectured that there do not exist five positive integers whose pairwise products are all one less than the square of an integer. Recently Dujella [**3**] proved that there do not exist nine such integers. In this note we address the following related problem. Let V denote the set of pure powers, that is, the set of positive integers of the form x^k with x and k positive integers and $k > 1$. How large can a set of positive integers A be if $aa' + 1$ is in V whenever a and a' are distinct integers from A? We expect that there is an absolute bound for $|A|$, the cardinality of A. While we have not been able to establish this result, we have been able to prove that such sets cannot be very dense.

THEOREM 1. *Let N be a positive integer and let A be a subset of $\{1, \ldots, N\}$ with the property that $aa' + 1$ is in V whenever a and a' are distinct integers from A. There exists a positive real number N_0 such that, if N exceeds N_0, then*

$$|A| < 340(\log N)^2/\log\log N.$$

We shall deduce our result from the theorem below. For each integer k, with k at least 2, define V_k by

$$V_k = \{x^l \mid x \in \mathbb{Z}^+ \text{ and } 2 \leqslant l \leqslant k\}.$$

THEOREM 2. *Let k be an integer with $k \geqslant 2$. Let N be a positive integer and let A be a subset of $\{1, \ldots, N\}$ with the property that $aa' + 1$ is in V_k whenever a and a' are distinct integers from A. There exists a positive real number N_1, such that, if N exceeds N_1, then*

$$|A| < 160 \frac{k^2}{(\log k)^2} \log\log N.$$

Notice that Theorem 1 follows from Theorem 2 on observing that, if x^k is a positive integer from $\{2, \ldots, N\}$, then k is at most $(\log N)/\log 2$.

The proof of Theorem 2 depends upon a gap principle, the result of Dujella and two results from extremal graph theory.

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§2. *Preliminary lemmas.*

LEMMA 1. *Let k be an integer with $k \geqslant 2$, and let a, b, x and y be positive integers with $a < b$ and $x < y$. If $ax+1$, $ay+1$, $bx+1$ and $by+1$ are kth powers, then*

$$yb > (xa)^{k-1}.$$

Proof. This follows from the proof of Theorem 1 of [**5**].

LEMMA 2 (Turán's Theorem). *Let n and r be positive integers with $r \geqslant 2$, and let G be a graph with n vertices. If the number of edges in G exceeds*

$$\sum_{0 \leqslant i < j < r-1} \left[\frac{n+i}{r-1}\right]\left[\frac{n+j}{r-1}\right],$$

then G contains a complete graph of order r.

Proof. This is Theorem 1.1, Chapter VI of [**2**]; see also [**10**].

LEMMA 3. *Let G be a graph with n ($\geqslant 1$) vertices and m edges, and suppose that*

$$m > \tfrac{1}{2}(n^{3/2} + n - n^{1/2}).$$

Then G contains a cycle of length 4.

Proof. This is a special case of Theorem 2.3, Chapter VI of [**2**], and is due to Kövári, Sós and Turán [**7**].

§3. *Proof of Theorem* 2. We suppose that

$$|A| \geqslant 160(k/\log k)^2 \log\log N,$$

and show that this leads to a contradiction. For N sufficiently large, there is an integer m, with $1 \leqslant m \leqslant 1 + (\log(\log N/\log 2))/\log 2$, such that A has more than $110(k/\log k)^2$ elements from $\{2^{2^m}, 2^{2^m}+1, \ldots, 2^{2^{m+1}}-1\}$. Let us denote the set of these elements by A_m, and put $n = |A_m|$. Then

$$n > 110(k/\log k)^2. \tag{1}$$

Form the complete graph G whose vertices are the elements of A_m. Next, colour the edges between two vertices a and a' by the smallest integer l larger than one for which $aa'+1$ is a perfect lth power. Note that each edge is coloured by a prime number.

For $i = 2, 3, \ldots, k$, let b_i denote the number of edges of G which are coloured with the integer i. It now follows readily from the method of Lagrange multipliers that

$$\sum_{0 \leqslant i < j < 8} \left[\frac{n+i}{8}\right]\left[\frac{n+j}{8}\right] \leqslant \binom{8}{2}\left(\frac{n}{8}\right)^2 = \frac{7}{16}n^2,$$

and so, by Lemma 2, if b_2 exceeds $7n^2/16$, then there is a complete graph on 9 vertices coloured with the integer 2. But Dujella [**3**] has proved that there do not exist 9 such positive integers. Accordingly,

$$b_3 + \cdots + b_k \geqslant \binom{n}{2} - \frac{7}{16}n^2 = \frac{n^2}{16} - \frac{n}{2}.$$

By Corollary 2 of Rosser and Schoenfeld [**8**], the number of primes up to k is at most $5k/4\log k$. Thus there exists a prime p, with $3 \leqslant p \leqslant k$, such that

$$b_p \geqslant \frac{4\log k}{5k}\left(\frac{n^2}{16} - \frac{n}{2}\right). \tag{2}$$

Let G_p be the subgraph of G whose vertices are those of G and whose edges are the edges of G coloured with the prime p. By (1),

$$\frac{4\log k}{5k}\left(\frac{n^2}{16} - \frac{n}{2}\right) = \frac{\log k}{k}\frac{n^2}{20}\left(1 - \frac{8}{n}\right)$$

$$> n^{3/2}\frac{\sqrt{110}}{20}\left(1 - \frac{8}{110}\left(\frac{\log 3}{3}\right)^2\right) > 0{\cdot}519n^{3/2}. \tag{3}$$

Further,

$$\tfrac{1}{2}(n^{3/2} + n - n^{1/2}) < \tfrac{1}{2}n^{3/2}\left(1 + \frac{1}{\sqrt{n}}\right) < \tfrac{1}{2}n^{3/2}\left(1 + \frac{1}{\sqrt{110}}\frac{\log 3}{3}\right) < 0{\cdot}518n^{3/2}. \tag{4}$$

Therefore, by (2), (3) and (4),

$$b_p > \tfrac{1}{2}(n^{3/2} + n - n^{1/2}),$$

whence, by Lemma 3, there is a cycle of length 4 in G_p. In particular, there exist integers a, b, x, y which are vertices of G_p with a and b both connected by edges to x and y. Without loss of generality, we may assume that $a < b$ and $x < y$. Then $ax+1, bx+1, ay+1$ and $by+1$ are pth powers and so, by Lemma 1,

$$yb > (xa)^{p-1} \geqslant (xa)^2 \tag{5}$$

But a, b, x and y are in $\{2^{2^m}, \ldots, 2^{2^{m+1}} - 1\}$, and hence

$$yb < (2^{2^{m+1}} - 1)^2 < 2^{2^{m+2}} \leqslant (xa)^2,$$

which contradicts (5). The results now follows.

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