Submitted to Rev. Mat. Iberoam., 1–30

On the representation of k-free integers by binary forms

C.L. Stewart and Stanley Yao Xiao

Abstract. Let F be a binary form with integer coefficients, non-zero discriminant and degree d with d at least 3 and let r denote the largest degree of an irreducible factor of F over the rationals. Let k be an integer with $k \geq 2$ and suppose that there is no prime p such that p^k divides $F(a,b)$ for all pairs of integers (a,b). Let $R_{F,k}(Z)$ denote the number of k-free integers of absolute value at most Z which are represented by F. We prove that there is a positive number $C_{F,k}$ such that $R_{F,k}(Z)$ is asymptotic to $C_{F,k}Z^{\frac{2}{d}}$ provided that k exceeds $\frac{7r}{18}$ or (k,r) is $(2,6)$ or $(3,8)$.

1. Introduction

Let F be a binary form with integer coefficients, non-zero discriminant $\Delta(F)$ and degree d with $d \geq 3$. For any positive number Z let $\mathcal{R}_F(Z)$ denote the set of non-zero integers h with $|h| \leq Z$ for which there exist integers x and y such that $F(x,y) = h$. Denote the cardinality of a set $\mathcal{S}$ by $|\mathcal{S}|$ and let $R_F(Z) = |\mathcal{R}_F(Z)|$. In [37] Stewart and Xiao proved that there exists a positive number C_F such that

$$R_F(Z) \sim C_F Z^{\frac{2}{d}}. \tag{1.1}$$

Such a result had been obtained earlier by Hooley in [16], [21], [23] and [24] when F is an irreducible binary cubic form, when F is a quartic form of the shape

$$F(x,y) = ax^4 + bx^2y^2 + cy^4.$$

and when F is the product of linear forms with integer coefficients. In addition, a number of authors including Bennett, Dummigan, and Wooley [1], Browning [5], Greaves [11], Heath-Brown [13], Hooley [19], [20], [22], Skinner and Wooley [34] and Wooley [40] obtained asymptotic estimates for $R_F(Z)$ when F is a binomial form.

Mathematics Subject Classification (2010): Primary 11D45; Secondary 11N36, 11E76.
Keywords: binary forms, determinant method, k-free integers.

Let k be an integer with $k \geq 2$. An integer is said to be k-free if it is not divisible by the k-th power of a prime number. For any positive number Z let $\mathcal{R}_{F,k}(Z)$ denote the set of k-free integers h with $|h| \leq Z$ for which there exist integers x and y such that $F(x,y) = h$ and put $R_{F,k}(Z) = |\mathcal{R}_{F,k}(Z)|$. Extending work of Hooley [16], [18], Gouvêa and Mazur [8] in 1991 proved that if there is no prime p such that p^2 divides $F(a,b)$ for all pairs of integers (a,b), if all the irreducible factors of F over $\mathbb{Q}$ have degree at most 3 and if ε is a positive real number then there are positive numbers C_1 and C_2, which depend on ε and F, such that if Z exceeds C_1 then

$$R_{F,2}(Z) > C_2 Z^{\frac{2}{d}-\epsilon}. \tag{1.2}$$

This was subsequently extended by Stewart and Top in [36]. Let r be the largest degree of an irreducible factor of F over $\mathbb{Q}$. Let k be an integer with $k \geq 2$ and suppose that there is no prime p such that p^k divides $F(a,b)$ for all integer pairs (a,b). They showed, by utilizing work of Greaves [10] and Erdős and Mahler [6], that if k is at least $(r-1)/2$ or $k = 2$ and $r = 6$ then there are positive numbers C_3 and C_4, which depend on k and F, such that if Z exceeds C_3 then

$$R_{F,k}(Z) > C_4 Z^{\frac{2}{d}}. \tag{1.3}$$

The estimates (1.2) and (1.3) were used by Gouvêa and Mazur [8] and Stewart and Top [36] in order to estimate, for any elliptic curve defined over $\mathbb{Q}$, the number of twists of the curve for which the rank of the Mordell-Weil group is at least 2.

For any real number x let $\lceil x \rceil$ denote the least integer u such that $x \leq u$. In 2016 [41] Xiao extended the range for which (1.3) holds by generalizing the determinant method of Heath-Brown [14] and Salberger [31], [32] to the setting of weighted projective space. He proved that if

$$k > \min\left\{\frac{7r}{18}, \left\lceil \frac{r}{2} \right\rceil - 2\right\}, \tag{1.4}$$

and (k,r) is not (3,8) then (1.3) holds. In addition, the related problem of estimating $B_{F,k}(Z)$, the number of pairs of integers (x,y) with $\max(|x|,|y|) \leq Z$ for which $F(x,y)$ is k-free, has been studied by Browning [4], Filaseta, [7], Granville [9], Greaves [10], Helfgott [15], Hooley [25], [26], Murty and Pasten [28], Poonen [30] and Xiao [41]. Recently Bhargava [2] and Bhargava, Shankar and Wang [3] have extended these estimates to the case of discriminant forms.

By building on the method used to prove (1.1) we shall give the first asymptotic estimates for $R_{F,k}(Z)$. We shall do so under the assumption that k satisfies (1.4).

Theorem 1.1. *Let F be a binary form with integer coefficients, non-zero discriminant and degree d with $d \geq 3$ and let r denote the largest degree of an irreducible factor of F over the rationals. Let k be an integer with $k \geq 2$ and suppose that there is no prime p such that p^k divides $F(a,b)$ for all pairs of integers (a,b). Suppose* *that (1.4) holds. Then there exists a positive number $C_{F,k}$ such that*

$$R_{F,k}(Z) = C_{F,k} Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/g_{k,r}(Z)\right) \tag{1.5}$$

where

$$g_{k,r}(Z) = \begin{cases} \log Z \log\log Z & \text{if } (k,r) \neq (2,6) \text{ or } (3,8) \\ (\log Z)^{\frac{(d-2)(0.7043)}{d}} & \text{if } (k,r) = (2,6) \\ \left(\frac{\log\log Z}{\log\log\log Z}\right)^{1-\frac{2}{d}} & \text{if } (k,r) = (3,8). \end{cases} \tag{1.6}$$

Throughout this article we make use of the standard notation "O", "o" and "$\sim$", for instance as in Section 1.6 of [12], with the convention that the implicit constant denoted by the symbol "O" may be determined in terms of the subscripts attached to it.

For a positive number Z we put

$$\mathcal{N}_F(Z) = \{(x,y) \in \mathbb{Z}^2 : 1 \leq |F(x,y)| \leq Z\}$$

and

$$N_F(Z) = |\mathcal{N}_F(Z)|.$$

We also put

$$A_F = \mu(\{(x,y) \in \mathbb{R}^2 : |F(x,y)| \leq 1\}) \tag{1.7}$$

where $\mu(\cdot)$ denotes the area of a set in $\mathbb{R}^2$. In 1933 Mahler [27] proved that if F is a binary form with integer coefficients and degree d with $d \geq 3$ which is irreducible over $\mathbb{Q}$ then

$$N_F(Z) = A_F Z^{\frac{2}{d}} + O_F\left(Z^{\frac{1}{d-1}}\right).$$

The assumption that F is irreducible may be replaced with the weaker requirement that F has non-zero discriminant; see [39].

Let k be an integer with $k \geq 2$. For a positive number Z we put

$$\mathcal{N}_{F,k}(Z) = \{(x,y) \in \mathbb{Z}^2 : F(x,y) \text{ is } k\text{-free and } 1 \leq |F(x,y)| \leq Z\}$$

and

$$N_{F,k}(Z) = |\mathcal{N}_{F,k}(Z)|.$$

For each positive integer m we put

$$\rho_F(m) = |\{(i,j) \in \{0,\cdots,m-1\}^2 : F(i,j) \equiv 0 \pmod{m}\}|$$

and

$$\lambda_{F,k} = \prod_p \left(1 - \frac{\rho_F(p^k)}{p^{2k}}\right),$$

where the product is taken over the primes p. Observe that the product converges since $k \geq 2$ and $\rho_F(p^k)$ is at most $p^{2k-2} + dp^k$ provided that p does not divide the discriminant $\Delta(F)$, see [35]. Further $\lambda_{F,k} = 0$ whenever there is a prime p such that p^k divides $F(a,b)$ for all (a,b) in $\mathbb{Z}^2$. Next we put

$$c_{F,k} = \lambda_{F,k} A_F. \tag{1.8}$$

In order to prove Theorem 1.1 we shall first establish the following result which is an analogue of Mahler's theorem for the case of k-free values assumed by a binary form.

Theorem 1.2. *Let F be a binary form with integer coefficients, non-zero discriminant and degree d with $d \geq 3$ and let r denote the largest degree of an irreducible factor of F over $\mathbb{Q}$. Let k be an integer with $k \geq 2$ and suppose that (1.4) holds. Then, with $c_{F,k}$ defined by (1.8), we have*

$$N_{F,k}(Z) = c_{F,k} Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}} / g_{k,r}(Z)\right) \tag{1.9}$$

with $g_{k,r}(Z)$ given by (1.6).

Let A be an element of $\mathrm{GL}_2(\mathbb{Q})$ with

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}.$$

Put $F_A(x,y) = F(a_1x + a_2y, a_3x + a_4y)$. We say that A fixes F if $F_A = F$. The set of A in $\mathrm{GL}_2(\mathbb{Q})$ which fix F is the *automorphism group* of F and we shall denote it by $\mathrm{Aut}\, F$. Let G_1 and G_2 be subgroups of $\mathrm{GL}_2(\mathbb{Q})$. We say that they are equivalent under conjugation if there is an element T in $\mathrm{GL}_2(\mathbb{Q})$ such that $G_1 = TG_2T^{-1}$. There are 10 equivalence classes of finite subgroups of $\mathrm{GL}_2(\mathbb{Q})$ under $\mathrm{GL}_2(\mathbb{Q})$-conjugation to which $\mathrm{Aut}\, F$ might belong, see [29] and [37], and we give a representative of each equivalence class together with its generators in Table 1 below.

Table 1

Group	Generators	Group	Generators
$\mathbf{C}_1$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\mathbf{D}_1$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
$\mathbf{C}_2$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\mathbf{D}_2$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$
$\mathbf{C}_3$	$\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$	$\mathbf{D}_3$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$
$\mathbf{C}_4$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\mathbf{D}_4$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$
$\mathbf{C}_6$	$\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$	$\mathbf{D}_6$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}$

Let Λ be the sublattice of $\mathbb{Z}^2$ consisting of (u,v) in $\mathbb{Z}^2$ for which $A\binom{u}{v}$ is in $\mathbb{Z}^2$ for all A in $\operatorname{Aut} F$.

When $\operatorname{Aut} F$ is conjugate to $\mathbf{D}_3$ it has three subgroups G_1, G_2 and G_3 of order 2 with generators A_1, A_2 and A_3 respectively, and one, G_4 say, of order 3 with generator A_4. Let $\Lambda_i = \Lambda(A_i)$ be the sublattice of $\mathbb{Z}^2$ consisting of (u,v) in $\mathbb{Z}^2$ for which $A_i\binom{u}{v}$ is in $\mathbb{Z}^2$ for $i = 1,2,3,4$.

When $\operatorname{Aut} F$ is conjugate to $\mathbf{D}_4$ there are three subgroups G_1, G_2 and G_3 of order 2 of $\operatorname{Aut} F/\{\pm I\}$ where I denotes the 2×2 identity matrix. Let Λ_i be the sublattice of $\mathbb{Z}^2$ consisting of (u,v) in $\mathbb{Z}^2$ for which $A\binom{u}{v}$ is in $\mathbb{Z}^2$ for A in a generator of G_i for $i = 1,2,3$.

Finally when $\operatorname{Aut} F$ is conjugate to $\mathbf{D}_6$ there are three subgroups G_1, G_2 and G_3 of order 2 and one, G_4 say, of order 3 in $\operatorname{Aut} F/\{\pm I\}$. Let A_i be in a generator of G_i for $i = 1,2,3,4$. Let Λ_i be the sublattice of $\mathbb{Z}^2$ consisting of (u,v) in $\mathbb{Z}^2$ for which $A_i\binom{u}{v}$ is in $\mathbb{Z}^2$ for $i = 1,2,3,4$.

Let L be a sublattice of $\mathbb{Z}^2$. We define $c_{F,k,L}$ in the following manner. For any basis $\{\omega_1, \omega_2\}$ of L with $\omega_1 = (a_1, a_3)$ and $\omega_2 = (a_2, a_4)$ we define $F_{\omega_1,\omega_2}(x,y) = F(a_1x + a_2y, a_3x + a_4y)$. Notice that if $\{\omega_1', \omega_2'\}$ is another basis for L then it is related to $\{\omega_1, \omega_2\}$ by a unimodular transformation. As a consequence,

$$c_{F_{\omega_1,\omega_2},k} = c_{F_{\omega_1',\omega_2'},k}$$

and so we may define $c_{F,k,L}$ by putting

$$c_{F,k,L} = c_{F_{\omega_1,\omega_2},k}.$$

Observe that if $L = \mathbb{Z}^2$ then $c_{F,k,L} = c_{F,k}$. For brevity, we shall write

$$c(L) = c_{F,k,L}. \tag{1.10}$$

We are now able to determine the positive number $C_{F,k}$ in (1.5) of Theorem 1.1 explicitly in terms of $\operatorname{Aut} F$ and the lattices described above.

Theorem 1.3. *The positive number $C_{F,k}$ in the statement of Theorem 1.1 is given by the following table:*

Rep(F)	$C_{F,k}$	Rep(F)	$C_{F,k}$
$\mathbf{C}_1$	$c_{F,k}$	$\mathbf{D}_1$	$c_{F,k} - \frac{c(\Lambda)}{2}$
$\mathbf{C}_2$	$\frac{c_{F,k}}{2}$	$\mathbf{D}_2$	$\frac{1}{2}\left(c_{F,k} - \frac{c(\Lambda)}{2}\right)$
$\mathbf{C}_3$	$c_{F,k} - \frac{2c(\Lambda)}{3}$	$\mathbf{D}_3$	$c_{F,k} - \frac{c(\Lambda_1)}{2} - \frac{c(\Lambda_2)}{2} - \frac{c(\Lambda_3)}{2} - \frac{2c(\Lambda_4)}{3} + \frac{4c(\Lambda)}{3}$
$\mathbf{C}_4$	$\frac{1}{2}\left(c_{F,k} - \frac{c(\Lambda)}{2}\right)$	$\mathbf{D}_4$	$\frac{1}{2}\left(c_{F,k} - \frac{c(\Lambda_1)}{2} - \frac{c(\Lambda_2)}{2} - \frac{c(\Lambda_3)}{2} + \frac{3c(\Lambda)}{4}\right)$
$\mathbf{C}_6$	$\frac{1}{2}\left(c_{F,k} - \frac{2c(\Lambda)}{3}\right)$	$\mathbf{D}_6$	$\frac{1}{2}\left(c_{F,k} - \frac{c(\Lambda_1)}{2} - \frac{c(\Lambda_2)}{2} - \frac{c(\Lambda_3)}{2} - \frac{2c(\Lambda_4)}{3} + \frac{4c(\Lambda)}{3}\right)$

Here Rep(F) *denotes a representative of the equivalence class of* $\operatorname{Aut} F$ *under* $\mathrm{GL}_2(\mathbb{Q})$*-conjugation,* Λ *and* Λ_i*'s are defined above,* $c_{F,k}$ *is as in (1.8), and* $c(\Lambda)$ *and* $c(\Lambda_i)$ *as in (1.10).*

Recall (1.1). We remark that while $C_{F,k}$ is equal to $\lambda_{F,k}C_F$ when $\operatorname{Aut} F$ is conjugate to either $\mathbf{C}_1$ or $\mathbf{C}_2$, in general $C_{F,k}$ is different from $\lambda_{F,k}C_F$. For instance if $G(x,y) = 8x^3 + y^3$ then, by Lemma 3.3 of [37], $\operatorname{Aut} G = \left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1/2\\ 2 & 0\end{pmatrix}\right)$ so $\operatorname{Aut} G$ is conjugate to $\mathbf{D}_1$ and, by Corollary 1.3 of [37],

$$C_G = \frac{3}{4}A_G. \tag{1.11}$$

Furthermore Λ, the sublattice of $\mathbb{Z}^2$ consisting of (u,v) in $\mathbb{Z}^2$ for which $A\binom{u}{v}$ is in $\mathbb{Z}^2$ for all A in $\operatorname{Aut} G$, is generated by $\omega_1 = (1,0)$ and $\omega_2 = (0,2)$. Observe that

$$G_{\omega_1,\omega_2}(x,y) = 8(x^3 + y^3) \tag{1.12}$$

and so when k is 2 or 3 we have $\lambda_{G_{\omega_1,\omega_2},k} = 0$ since $\rho_{G_{\omega_1,\omega_2}}(2^k) = 2^{2k}$. Thus, when k is 2 or 3, $c(\Lambda) = c_{G_{\omega_1,\omega_2},k} = 0$ and, by Theorem 1.3,

$$C_{G,k} = c_{G,k} \tag{1.13}$$

hence, by (1.8) and (1.11),

$$C_{G,k} = \frac{4}{3}\lambda_{G,k}C_G. \tag{1.14}$$

We conjecture that the estimates for $R_{F,k}(Z)$ in Theorem 1.1 and for $N_{F,k}(Z)$ in Theorem 1.2 apply without hypothesis (1.4).

Conjecture 1.4. *Let F be a binary form with integer coefficients, non-zero discriminant and degree d with d at least 3. Let k be an integer larger than 1. Then either $c_{F,k} = 0$ or*

$$N_{F,k}(Z) \sim c_{F,k}Z^{\frac{2}{d}} \tag{1.15}$$

where $c_{F,k}$ is defined by (1.8). If there is no prime p such that p^k divides $F(a, b)$ for all pairs of integers (a, b) then

$$R_{F,k}(Z) \sim C_{F,k}Z^{\frac{2}{d}} \tag{1.16}$$

where $C_{F,k}$ is the positive number given by Theorem 1.3.

Let F be a binary form with integer coefficients, non-zero discriminant and degree d with $d \geq 3$. Granville [9] established an asymptotic estimate for $B_{F,2}(Z)$, the number of pairs of integers (x, y) with absolute value at most Z for which $F(x, y)$ is squarefree subject to the *abc* conjecture, see eg. [38]. Let k be an integer with $k > 1$. The same analysis allows one to give an asymptotic estimate for $B_{F,k}(Z)$, the number of pairs of integers (x, y) with absolute value at most Z for which $F(x, y)$ is k-free. We may use such an estimate in conjunction with the arguments used to prove Theorem 1.1 and Theorem 1.2 in order to prove Conjecture 1.4, see the final paragraph of Section 6. In particular, Conjecture 1.4 follows from the *abc* conjecture.

2. Preliminary lemmas

Let $F(x, y)$ be a binary form with integer coefficients, non-zero discriminant and degree d with $d \geq 3$. Suppose that F factors over $\mathbb{C}$ as

$$F(x, y) = \prod_{i=1}^{d}(\gamma_i x + \beta_i y) \tag{2.1}$$

and put

$$\mathcal{H}(F) = \prod_{i=1}^{d}\sqrt{|\gamma_i|^2 + |\beta_i|^2}.$$

Then $\mathcal{H}(F)$ does not depend on the factorization in (2.1).

A special case of Theorem 3 of Thunder [39] is the following explicit version of a result of Mahler [27].

Lemma 2.1. *Let F be a binary form with integer coefficients, non-zero discriminant and degree $d \geq 3$. Let Z be a real number with $Z \geq 1$. Then, with A_F defined by (1.7), we have*

$$\left| N_F(Z) - A_F Z^{\frac{2}{d}} \right| = O\left(Z^{\frac{1}{d-1}} \mathcal{H}(F)^{d-2} \right),$$

where the implied constant is absolute.

We may write

$$F(x,y) = a \prod_{i=1}^{d} (x - \alpha_i y) \tag{2.2}$$

where a is a positive integer and $\alpha_1, \ldots, \alpha_d$ are the roots of $F(x,1)$ provided that y is not a factor of $F(x,y)$. In the latter case, since the discriminant of F is non-zero, we have

$$F(x,y) = ay \prod_{i=1}^{d-1} (x - \alpha_i y). \tag{2.3}$$

Put

$$E_F = \frac{2 \max\limits_{1 \leq j \leq k} (1, |\alpha_j|)}{\min(1, \min\limits_{i \neq j} |\alpha_i - \alpha_j|)} \tag{2.4}$$

where $k = d$ if (2.2) holds and $k = d-1$ if (2.3) holds.

Lemma 2.2. *Let F be a binary form with integer coefficients, non-zero discriminant and degree d with $d \geq 3$. Let Z be a real number with $Z \geq 1$. For any positive real number β larger than E_F the number of pairs of integers (x,y) with*

$$0 < |F(x,y)| \leq Z$$

for which

$$\max(|x|,|y|) > Z^{\frac{1}{d}} \beta$$

is

$$O_F\left(Z^{\frac{1}{d}} \log(1+Z) + \frac{Z^{\frac{2}{d}}}{\beta^{d-2}} \right).$$

Proof. We shall follow Heath-Brown's proof of Theorem 8 in [14]. Accordingly put

$$S(Z;C) = |\{(x,y) \in \mathbb{Z}^2 : 0 < |F(x,y)| \leq Z, C < \max(|x|,|y|) \leq 2C, \gcd(x,y) = 1\}|.$$

Suppose that

$$C \geq Z^{\frac{1}{d}} E_F. \tag{2.5}$$

Heath-Brown observes that by Roth's theorem $S(Z;C)=0$ unless $C \ll_F Z^2$.

Suppose that we are in the case when (2.2) holds and that (x,y) is a pair of integers with $\gcd(x,y)=1$,

$$0<|F(x,y)|\le Z$$

and

$$C<\max\{|x|,|y|\}\le 2C \tag{2.6}$$

Further suppose that i_0 is an index for which

$$|x-\alpha_{i_0}y|\le|x-\alpha_j y|$$

for $j=1,\dots,d$. Note that then

$$|x-\alpha_{i_0}y|\le Z^{1/d}. \tag{2.7}$$

We have two cases to consider. The first case is when $\max(|x|,|y|)=|y|$. In this case we have, for $j\ne i_0$,

$$|x-\alpha_j y|=|(x-\alpha_{i_0}y)+(\alpha_{i_0}-\alpha_j)y|\ge|\alpha_{i_0}-\alpha_j||y|-|x-\alpha_{i_0}y| \tag{2.8}$$

and, by (2.5), (2.6) and (2.7),

$$\frac{1}{2}|\alpha_{i_0}-\alpha_j||y|-|x-\alpha_{i_0}y|\ge\frac{1}{2}|\alpha_{i_0}-\alpha_j|Z^{1/d}E_F-Z^{1/d}\ge 0. \tag{2.9}$$

Thus, by (2.8) and (2.9),

$$|x-\alpha_j y|\ge\frac{1}{2}|\alpha_{i_0}-\alpha_j||y|\ge\frac{1}{2}|\alpha_{i_0}-\alpha_j|C. \tag{2.10}$$

The second case is when $\max(|x|,|y|)=|x|$. Then

$$|\alpha_{i_0}(x-\alpha_j y)|=|(\alpha_{i_0}-\alpha_j)x+\alpha_j(x-\alpha_{i_0}y)|\ge|\alpha_{i_0}-\alpha_j||x|-|\alpha_j||x-\alpha_{i_0}y|, \tag{2.11}$$

and, by (2.4), (2.10) and (2.11),

$$\frac{1}{2}|\alpha_{i_0}-\alpha_j||x|-|\alpha_j||x-\alpha_{i_0}y|\ge\frac{1}{2}|\alpha_{i_0}-\alpha_j|Z^{1/d}E_F-|\alpha_j|Z^{1/d}\ge 0. \tag{2.12}$$

Thus, by (2.11) and (2.12),

$$|x-\alpha_j y|\ge\frac{1}{2|\alpha_{i_0}|}|\alpha_{i_0}-\alpha_j|C. \tag{2.13}$$

It now follows from (2.6), (2.10) and (2.13) that

$$C\ll_F|x-\alpha_j y|\ll_F C. \tag{2.14}$$

We obtain (2.14) in a similar fashion when (2.3) holds.

Thus, by (2.14),

$$|x - \alpha_{i_0} y| \ll_F Z/C^{d-1}. \tag{2.15}$$

The number of coprime integer pairs (x, y) satisfying (2.6) and (2.15) for some index i_0 is an upper bound for $S(Z;C)$ and therefore, by Lemma 1, part (vii) of [14],

$$S(Z;C) \ll_F 1 + \frac{Z}{C^{d-2}}. \tag{2.16}$$

Put

$$S^{(1)}(Z;C) = |\{(x,y) \in \mathbb{Z}^2 : 0 < |F(x,y)| \le Z, C < \max(|x|,|y|), \gcd(x,y) = 1\}|.$$

Therefore on replacing C by $2^j C$ in (2.16) for $j = 1, 2, \cdots$ and summing we find that

$$S^{(1)}(Z;C) \ll_F \log(1 + Z) + \frac{Z}{C^{d-2}}.$$

Next put

$$S^{(2)}(Z;C) = |\{(x,y) \in \mathbb{Z}^2 : 0 < |F(x,y)| \le Z, C < \max(|x|,|y|)\}|.$$

Then

$$S^{(2)}(Z;C) \ll_F \sum_{h \le Z^{1/d}} S^{(1)}\left(\frac{Z}{h^d}, \frac{C}{h}\right)$$

and since $C > Z^{\frac{1}{d}} E_F$ we see that

$$\frac{C}{h} > \left(\frac{Z}{h^d}\right)^{\frac{1}{d}} E_F,$$

hence

$$\begin{aligned} S^{(2)}(Z;C) &\ll_F \sum_{h \le Z^{1/d}} \left(\log(1+Z) + \frac{Z}{h^2 C^{d-2}}\right) \\ &\ll_F Z^{\frac{1}{d}} \log(1+Z) + \frac{Z}{C^{d-2}}. \end{aligned}$$

Our result now follows on taking $C = Z^{\frac{1}{d}}\beta$ since $\beta > E_F$. □

For any positive real numbers Z and β put

$$N_F(Z,\beta) = |\{(x,y) \in \mathbb{Z}^2 : |F(x,y)| \le Z, \max(|x|,|y|) \le Z^{\frac{1}{d}}\beta\}|.$$

Lemma 2.3. *Let F be a binary form of degree $d \ge 3$ with integer coefficients and non-zero discriminant. Let Z be a real number with $Z \ge 1$. Let E_F be as in (2.4) and suppose that β is a real number with $\beta > E_F$. Then*

$$N_F(Z,\beta) = A_F Z^{\frac{2}{d}} + O_F\left(Z^{\frac{1}{d-1}} + Z^{\frac{1}{d}}\beta + Z^{\frac{2}{d}}\beta^{-(d-2)}\right).$$

Proof. This follows from Lemma 2.1 and Lemma 2.2 on noting that the number of pairs of integers (x, y) with $\max(|x|, |y|\} \le Z^{\frac{1}{d}}\beta$ for which $F(x, y) = 0$ is at most $O_F\left(Z^{\frac{1}{d}}\beta\right)$. □

In order to facilitate the determination of the main terms in Theorems 1.1 and 1.2 we introduce the quantity $\tilde{N}_F(Z, \beta)$, which is the number of pairs of integers (x, y) such that $|F(x+\theta_1, y+\theta_2)| \le Z$ and $\max(|x+\theta_1|, |y+\theta_2|) \le Z^{\frac{1}{d}}\beta$ whenever $0 \le \theta_1 < 1$ and $0 \le \theta_2 < 1$.

Lemma 2.4. *Let F be a binary form with integer coefficients, non-zero discriminant and degree $d \ge 3$. Let Z be a real number, let E_F be as in (2.4) and suppose that β is a real number with $Z^{1/d^2} > \beta > E_F$. Then*

$$\tilde{N}_F(Z, \beta) = A_F Z^{\frac{2}{d}} + O_F\left(Z^{\frac{1}{d-1}} + Z^{\frac{2}{d}}\beta^{-(d-2)} + Z^{\frac{1}{d}}\beta^{d-1}\right).$$

Proof. Plainly

$$\tilde{N}_F(Z, \beta) \le N_F(Z, \beta). \tag{2.17}$$

Note that for integers x, y with $(x, y) \ne (0, 0)$ there is a number κ with $\kappa \ge 1$, which depends on F, such that for (θ_1, θ_2) in $\mathbb{R}^2$ with $|\theta_i| \le 1$ for $i = 1, 2$ we have

$$|F(x+\theta_1, y+\theta_2)| \le |F(x, y)| + \kappa \max(|x|, |y|)^{d-1}.$$

Put

$$Z_1 = Z - \kappa\left(Z^{\frac{1}{d}}\beta\right)^{d-1} \tag{2.18}$$

and observe that we may assume that Z exceeds a positive number depending on F and, in particular, that $Z_1 \ge 1$. Thus if $\max(|x|, |y|) \le Z^{\frac{1}{d}}\beta$ and

$$|F(x, y)| \le Z_1$$

then, for $(\theta_1, \theta_2) \in \mathbb{R}^2$ with $|\theta_i| \le 1$ for $i = 1, 2$, we have

$$|F(x+\theta_1, y+\theta_2)| \le Z. \tag{2.19}$$

Furthermore, since $\beta < Z^{1/d^2}$,

$$Z^{\frac{1}{d}} - Z_1^{\frac{1}{d}} = Z^{\frac{1}{d}} - Z^{\frac{1}{d}}\left(1 - \frac{\kappa\beta^{d-1}}{Z^{\frac{1}{d}}}\right)^{\frac{1}{d}} \tag{2.20}$$

and so

$$Z^{\frac{1}{d}} - Z_1^{\frac{1}{d}} = \frac{\kappa\beta^{d-1}}{d} + O_F\left(Z^{\frac{-1}{d}}\beta^{2(d-1)}\right). \tag{2.21}$$

Since $\kappa \ge 1$ and $\beta \ge 2$, if $\max(|x|, |y|) \le Z_1^{\frac{1}{d}}\beta$ then

$$\max(|x|+1, |y|+1) \le Z^{\frac{1}{d}}\beta \tag{2.22}$$

and hence, by (2.19) and (2.22),

$$N_F(Z_1,\beta) \le \tilde{N}_F(Z,\beta) \tag{2.23}$$

for Z sufficiently large in terms of F. Note that

$$Z_1^{\frac{2}{d}} = \left(Z - \kappa\left(Z^{\frac{1}{d}}\beta\right)^{d-1}\right)^{\frac{2}{d}} = Z^{\frac{2}{d}} + O_F\left(Z^{\frac{1}{d}}\beta^{d-1}\right). \tag{2.24}$$

The result now follows from Lemma 2.3, (2.17), (2.23) and (2.24).

□

We now put, for a real number Z, an integer k with $k \ge 2$ and positive numbers γ and β,

$$N_{F,k}(Z,\gamma,\beta) = |\{(x,y)\in\mathbb{Z}^2 : |F(x,y)| \le Z, \max(|x|,|y|) \le Z^{\frac{1}{d}}\beta \text{ and}$$
$$F(x,y) \text{ is not divisible by } p^k \text{ for any prime } p \text{ with } p \le \gamma\}|,$$

and

$$N_{F,k}(Z,\gamma) = |\{(x,y)\in\mathbb{Z}^2 : 0 < |F(x,y)| \le Z \text{ and } F(x,y) \text{ is not divisible by}$$
$$p^k \text{ for any prime } p \text{ with } p \le \gamma\}|.$$

Lemma 2.5. *Let F be a binary form with integer coefficients, non-zero discriminant and degree d with $d \ge 3$. Let Z be a real number with $Z \ge 4$ and let k be an integer with $k \ge 2$. Then*

$$N_{F,k}\left(Z, \frac{1}{2kd}\log Z\right) = c_{F,k}Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/(\log Z \log\log Z)\right)$$

with $c_{F,k}$ given by (1.8).

Proof. We have

$$N_{F,k}\left(Z, \frac{1}{2kd}\log Z\right) = N_{F,k}\left(Z, \frac{1}{2kd}\log Z, (\log Z)^6\right) + O_{F,k}\left(Z^{\frac{1}{d}}(\log Z)^6\right)$$
$$+ O_{F,k}\left(|\{(x,y)\in\mathbb{Z}^2 : 0 < |F(x,y)| \le Z \text{ and } \max(|x|,|y|) > Z^{\frac{1}{d}}(\log Z)^6\}|\right).$$

By Lemma 2.2, since $d \ge 3$,

$$N_{F,k}\left(Z, \frac{1}{2kd}\log Z\right) = N_{F,k}\left(Z, \frac{1}{2kd}\log Z, (\log Z)^6\right) + O_{F,k}\left(Z^{\frac{2}{d}}/(\log Z)^6\right). \tag{2.25}$$

Next we put

$$V = V(d,k,Z) = \prod_{p \le \log Z/(2kd)} p^k,$$

where the product is taken over primes p. By the Prime Number Theorem,

$$V = O\left(Z^{\frac{1}{2d}+\frac{1}{d^2}}\right). \tag{2.26}$$

For each pair of integers (a, b) we define $B(a, b)$ by

$$B(a, b) = \{(t, u) \in \mathbb{R}^2 : aV \leq t < (a + 1)V, bV \leq u < (b + 1)V\}.$$

Observe that $B(a, b)$ is a square in $\mathbb{R}^2$. We say that $B(a, b)$ is admissible if

$$|F(t, u)| \leq Z \text{ and } \max(|t|, |u|) \leq Z^{\frac{1}{d}}(\log Z)^6 \tag{2.27}$$

whenever (t, u) is in $B(a, b)$. Let B_1 denote the number of admissible squares $B(a, b)$. We have

$$B_1 = \tilde{N}_F\left(\frac{Z}{V^d}, (\log Z)^6\right)$$

and so by Lemma 2.4 and (2.26), since $d \geq 3$,

$$B_1 = A_F\frac{Z^{\frac{2}{d}}}{V^2} + O_F\left(\left(\frac{Z}{V^d}\right)^{\frac{1}{d-1}} + \frac{Z^{\frac{2}{d}}}{V^2(\log Z)^6} + \frac{Z^{\frac{1}{d}}}{V}(\log Z)^{6(d-1)}\right). \tag{2.28}$$

Therefore the number of pairs of integers (x, y) which are in one of the admissible squares is B_1V^2 and so is

$$A_FZ^{\frac{2}{d}} + O_F\left(Z^{\frac{1}{d-1}}V^{\frac{d-2}{d-1}} + \frac{Z^{\frac{2}{d}}}{(\log Z)^6} + Z^{\frac{1}{d}}V(\log Z)^{6(d-1)}\right). \tag{2.29}$$

By Lemma 2.3

$$N_F(Z, (\log Z)^6) = A_FZ^{\frac{2}{d}} + O_F\left(Z^{\frac{1}{d-1}} + Z^{\frac{1}{d}}(\log Z)^6 + Z^{\frac{2}{d}}(\log Z)^{-6(d-2)}\right) \tag{2.30}$$

and so the number of pairs of integers (x, y) for which $|F(x, y)| \leq Z$ and $\max(|x|, |y|) \leq Z^{\frac{1}{d}}(\log Z)^6$ which are not in an admissible square is

$$O_F\left(Z^{\frac{1}{d-1}}V^{\frac{d-2}{d-1}} + Z^{\frac{2}{d}}/(\log Z)^6 + Z^{\frac{1}{d}}V(\log Z)^{6(d-1)}\right). \tag{2.31}$$

We may now apply the Chinese Remainder Theorem to conclude that within each admissible square the number of integer pairs (x, y) for which $F(x, y)$ is not divisible by p^k for any prime p with $p \leq \frac{1}{2kd}\log Z$ is precisely

$$\prod_{p\leq\log Z/(2kd)}\left(1 - \frac{\rho_F(p^k)}{p^{2k}}\right)V^2.$$

Thus the number of integer pairs (x, y) in some admissible square and for which $F(x, y)$ is not divisible by p^k for any prime p with $p \leq \frac{1}{2kd}\log Z$ is

$$B_1\prod_{p\leq\log Z/(2kd)}\left(1 - \frac{\rho_F(p^k)}{p^{2k}}\right)V^2. \tag{2.32}$$

Therefore, by (2.28), (2.31) and (2.32),

$$N_{F,k}\left(Z, \frac{1}{2kd}\log Z\right) = A_F \prod_{p \le \log Z/(2kd)} \left(1 - \frac{\rho_F(p^k)}{p^{2k}}\right) Z^{\frac{2}{d}} + \tag{2.33}$$

$$O_{F,k}\left(Z^{\frac{1}{d-1}} V^{\frac{d-2}{d-1}} + Z^{\frac{1}{d}} V (\log Z)^{6(d-1)} + Z^{\frac{2}{d}}/(\log Z)^6\right).$$

By(2.26) and (2.33),

(2.34)

$$N_{F,k}\left(Z, \frac{1}{2kd}\log Z\right) = A_F \prod_{p \le \log Z/(2kd)} \left(1 - \frac{\rho_F(p^k)}{p^{2k}}\right) Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/(\log Z)^6\right).$$

Note that the number of integer pairs (a,b) with $0 \le a < p^k$ and $0 \le b < p^k$ for which p divides both a and b is p^{2k-2}. Further the number of pairs (a,b) for which p does not divide both a and b and for which $F(a,b) \equiv 0 \pmod{p^k}$ is at most dp^k provided that p does not divide $\Delta(F)$, see [35]. Thus for primes p which do not divide $\Delta(F)$, we have

$$\rho_F(p^k) \le p^{2k-2} + dp^k \le (d+1)p^{2k-2}, \tag{2.35}$$

since $k \ge 2$. Put

$$P = \prod_{p \le \log Z/(2kd)} \left(1 - \frac{\rho_F(p^k)}{p^{2k}}\right), P_1 = \prod_{p} \left(1 - \frac{\rho_F(p^k)}{p^{2k}}\right)$$

and

$$t = \sum_{p > \log Z/(2kd)} \log\left(1 - \frac{\rho_F(p^k)}{p^{2k}}\right).$$

Then

$$P_1 - P = P(e^t - 1) = -P\left(-t - \frac{t^2}{2!} - \frac{t^3}{3!} - \cdots\right).$$

Since t is negative,

$$0 \le P - P_1 \le -Pt. \tag{2.36}$$

Further,

$$-t = O_{F,k}\left(\sum_{p > \log Z/(2kd)} \frac{\rho_F(p^k)}{p^{2k}}\right)$$

and by (2.35),

$$-t = O_{F,k}\left(\sum_{p > \log Z/(2kd)} \frac{1}{p^2}\right). \tag{2.37}$$

We have

$$\sum_{p>\log Z/(2kd)} \frac{1}{p^2} = \sum_{j=0}^{\infty} \sum_{2^j \frac{\log Z}{2kd} < p < 2^{j+1} \frac{\log Z}{2kd}} \frac{1}{p^2}$$

and so, by the Prime Number Theorem,

(2.38)

$$\sum_{p>\log Z/(2kd)} \frac{1}{p^2} = O_k\left(\sum_{j=0}^{\infty}\left(\frac{2^{j+1}\log Z}{(j+1)\log 2 + \log\log Z}\right)\frac{1}{2^{2j}(\log Z)^2}\right)$$
$$= O_k\left(\frac{1}{\log Z \log\log Z}\right).$$

Therefore, by (2.36), (2.37) and (2.38),

$$P = P_1 + O_{F,k}\left(\frac{1}{\log Z \log\log Z}\right). \tag{2.39}$$

It now follows from (2.34) and (2.39) that

$$N_{F,k}\left(Z, \frac{1}{2kd}\log Z\right) = c_{F,k}Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/(\log Z \log\log Z)\right), \tag{2.40}$$

as required. □

We say that an integer h is *essentially represented* by F if whenever $(x_1, y_1), (x_2, y_2)$ are in $\mathbb{Z}^2$ and

$$F(x_1, y_1) = F(x_2, y_2) = h$$

then there exists A in $\operatorname{Aut} F$ such that

$$A\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

We remark that if there is only one integer pair (x, y) for which $F(x, y) = h$ then h is essentially represented since I is in $\operatorname{Aut} F$.

For any positive number Z let $R_F^{(2)}(Z)$ denote the number of integers h with $0 < |h| \le Z$ which are represented by F but which are not essentially represented by F. For each binary form F with integer coefficients, non-zero discriminant and degree d with $d \ge 3$ we define β_F in the following way. If F has a linear factor in $\mathbb{R}[x, y]$ we put

(2.41)

$$\beta_F = \begin{cases} \dfrac{12}{19} & \text{if } d = 3 \text{ and } F \text{ is irreducible over } \mathbb{Q} \\[2ex] \dfrac{4}{7} & \text{if } d = 3 \text{ and } F \text{ has exactly one linear factor over } \mathbb{Q} \\[2ex] \dfrac{5}{9} & \text{if } d = 3 \text{ and } F \text{ has three linear factors over } \mathbb{Q} \\[2ex] \dfrac{3}{(d-2)\sqrt{d}+3} & \text{if } 4 \le d \le 8 \\[2ex] \dfrac{1}{d-1} & \text{if } d \ge 9. \end{cases}$$

If F does not have a linear factor over $\mathbb{R}$ then d is even and we put

$$\beta_F = \begin{cases} \dfrac{3}{d\sqrt{d}} & \text{if } d = 4, 6, 8 \\[2ex] \dfrac{1}{d} & \text{if } d \ge 10. \end{cases} \tag{2.42}$$

In [37], Stewart and Xiao, building on work of Heath-Brown [14], Salberger [31], [32] and Colliot-Thélène [14], proved the following result.

Lemma 2.6. *Let F be a binary form with integer coefficients, non-zero discriminant and degree d with $d \ge 3$. Then for each $\varepsilon > 0$,*

$$R_F^{(2)}(Z) = O_{F,\varepsilon}\left(Z^{\beta_F+\varepsilon}\right)$$

where β_F is given by (2.41) and (2.42).

The proof of Lemma 2.6 is based on the p-adic determinant method of Heath-Brown as elaborated in [14].

Recall that if F is a binary form we denote by Λ the sublattice of $\mathbb{Z}^2$ consisting of integer pairs (u, v) for which $A\binom{u}{v}$ is in $\mathbb{Z}^2$ for all A in $\operatorname{Aut} F$. Further, if $\operatorname{Aut} F$ is conjugate to $\mathbf{D}_3, \mathbf{D}_4$ and $\mathbf{D}_6$ we define Λ_i for $i = 1, 2, 3, 4$ as in our discussion following Table 1 in the introduction.

Lemma 2.7. *Let F be a binary form with integer coefficients, non-zero discriminant and degree $d \ge 3$. If A is an element of order 3 in $\operatorname{Aut} F$ then*

$$\Lambda(A^2) = \Lambda(A).$$

If $\operatorname{Aut} F$ is equivalent under conjugation in $\mathrm{GL}_2(\mathbb{Q})$ to $\mathbf{D}_3, \mathbf{D}_4$ or $\mathbf{D}_6$ then

$$\Lambda_i \cap \Lambda_j = \Lambda \textit{ for } i \ne j.$$

Lemma 2.7 is Lemma 3.2 of [37] .

3. Outline of the proof of Theorem 1.2

Let N_1 denote the number of integer pairs (x, y) for which

(i) $0 < |F(x, y)| \le Z$,

and

(ii) $p^k \nmid F(x, y)$ for $1 \le p \le \frac{1}{2kd} \log Z$.

By Lemma 2.5

$$N_1 = c_{F,k} Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/(\log Z \log\log Z)\right). \tag{3.1}$$

Our objective is to show that the number N of integer pairs for which (i) holds and

(iii) $p^k \nmid F(x, y)$ for p a prime,

satisfies a similar estimate to (3.1). To that end let N_2 denote the number of integer pairs (x, y) for which (i) holds and p divides both x and y for some prime $p > \frac{1}{2kd} \log Z$.

Let $F(1, 0) = u$ and $F(0, 1) = v$. Notice that we may suppose that $uv \neq 0$ since if $uv = 0$ we may replace F by F_A where A is a unimodular 2×2 matrix and $F_A(1, 0)F_A(0, 1) \neq 0$. Next observe that if x and y are integers and p is a prime which divides $F(x, y)$ and y but does not divide x then p divides u. Similarly if p divides $F(x, y)$ and x but does not divide y then p divides v. We shall suppose that Z is sufficiently large that

$$|uv| < \frac{1}{2kd} \log Z. \tag{3.2}$$

Thus if p is larger than $\frac{1}{2kd} \log Z$, divides $F(x, y)$ and does not divide both x and y then p does not divide either x or y. Let N_3 denote the number of pairs of integers (x, y) for which (i) holds and for some prime p with

$$\frac{1}{2kd} \log Z < p \le (\log Z)^9$$

we have $p^k | F(x, y)$ and $p \nmid \gcd(x, y)$. Let N_4 denote the number of integer pairs (x, y) for which (i) holds and for some prime p with

$$(\log Z)^9 < p \le \frac{Z^{\frac{2}{d}}}{(\log Z)^9},$$

$p^k | F(x, y)$ and $p \nmid \gcd(x, y)$. Finally let N_5 denote the number of integer pairs (x, y) for which (i) holds and for some prime p with

$$\frac{Z^{\frac{2}{d}}}{(\log Z)^9} < p,$$

$p^k|F(x,y)$ and $p \nmid \gcd(x,y)$. Then

$$N = N_1 + O(N_2 + N_3 + N_4 + N_5). \tag{3.3}$$

In order to establish Theorem 1.2 it suffices, by (3.1) and (3.3), to prove that

$$N_i = O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right)$$

for $i = 2, 3, 4$ and 5 where

$$u(z) = \log Z \log\log Z \tag{3.4}$$

when k and r satisfy (1.4) with (k,r) not $(2,6)$ or $(3,8)$,

$$u(z) = (\log Z)^{\frac{(d-2)\delta}{d}} \tag{3.5}$$

with $\delta = 0.7043$ when (k,r) is $(2,6)$ and

$$u(z) = (\log\log Z/\log\log\log Z)^{1-\frac{2}{d}} \tag{3.6}$$

when (k,r) is $(3,8)$.

We may suppose that F factors over $\mathbb{Q}$ as

$$F(x,y) = \prod_{i=1}^{t} F_i(x,y) \tag{3.7}$$

with F_i in $\mathbb{Z}[x,y]$ and irreducible over $\mathbb{Q}$ for $i = 1, \cdots, t$. Let r_i be the degree of F_i for $i = 1, \cdots, t$ and put

$$r = \max(r_1, \cdots, r_t). \tag{3.8}$$

4. An estimate for N_2 and for N_3

Notice that if p divides a and b and $0 < |F(a,b)| \le Z$ then $|F(a,b)| = p^d|F(a/p, b/p)|$, so $p \le Z^{\frac{1}{d}}$. As a consequence

$$N_2 = O\left(\sum_{\frac{1}{2kd}\log Z < p \le Z^{\frac{1}{d}}} \left|\left\{(x,y) \in \mathbb{Z}^2 : 0 < |F(x,y)| \le \frac{Z}{p^d}\right\}\right|\right). \tag{4.1}$$

Further, by Lemma 2.1, for each prime p with $p \le Z^{\frac{1}{d}}$,

$$\left|\left\{(x,y) \in \mathbb{Z}^2 : 0 < |F(x,y)| \le \frac{Z}{p^d}\right\}\right| = A_F \frac{Z^{\frac{2}{d}}}{p^2} + O_F\left(\left(\frac{Z}{p^d}\right)^{\frac{1}{d-1}}\right). \tag{4.2}$$

Thus by (4.1) and (4.2),

$$N_2 = O_F\left(\left(\sum_{\frac{1}{2kd}\log Z<p}\frac{1}{p^2}\right)Z^{\frac{2}{d}}\right). \tag{4.3}$$

It now follows from (2.38) and (4.3) that

$$N_2 = O_{F,k}\left(Z^{\frac{2}{d}}/(\log Z\log\log Z)\right). \tag{4.4}$$

The integer pairs (a, b) with $F(a, b) \equiv 0 \pmod{p^k}$ and for which p does not divide both a and b lie in at most d sublattices L_θ of $\mathbb{Z}^2$, provided that p does not divide the discriminant $\Delta(F)$ of F, see [10]. Further, if p does not divide uv then each sublattice L_θ is defined by a congruence of the form

$$a \equiv \theta b \pmod{p^k}$$

for some integer θ with $0 \le \theta < p^k$. Let (a_1, a_3) and (a_2, a_4) be a basis for L_θ chosen so that $\max(|a_1|, |a_2|, |a_3|, |a_4|)$ is minimized. Then

$$\max(|a_1|, |a_2|, |a_3|, |a_4|) \le p^k.$$

Put

$$F_{L_\theta}(x, y) = F(a_1x + a_2y, a_3x + a_4y)$$

and notice that

$$|\mathcal{N}_F(Z) \cap L_\theta| = N_{F_{L_\theta}}(Z).$$

Observe that

$$\mathcal{H}(F_{L_\theta}) \le 4^d p^{kd}\mathcal{H}(F).$$

Therefore by Lemma 2.1

$$N_{F_{L_\theta}}(Z) = A_{F_{L_\theta}}Z^{\frac{2}{d}} + O_F\left(p^{kd(d-2)}Z^{\frac{1}{d-1}}\right)$$

and, since the lattice L_θ has determinant p^k,

$$N_{F_{L_\theta}}(Z) = \frac{A_F Z^{\frac{2}{d}}}{p^k} + O_F\left(p^{kd(d-2)}Z^{\frac{1}{d-1}}\right). \tag{4.5}$$

Thus

$$\begin{aligned} N_3 &= O_{F,k}\left(Z^{\frac{2}{d}}\sum_{\frac{1}{2kd}\log Z<p\le(\log Z)^9}\frac{1}{p^k}\right)\\ &= O_{F,k}\left(Z^{\frac{2}{d}}\sum_{\frac{1}{2kd}\log Z<p}\frac{1}{p^2}\right),\end{aligned}$$

and so, by (2.38),

$$N_3 = O_{F,k}\left(Z^{\frac{2}{d}}/(\log Z\log\log Z)\right). \tag{4.6}$$

5. An estimate for N_4

In order to estimate N_4 we note that

$$N_4 = O\left(N_4^{(1)} + N_4^{(2)}\right) \tag{5.1}$$

where $N_4^{(1)}$ is the number of integer pairs (x, y) for which

$$\max(|x|, |y|) \le Z^{\frac{1}{d}} (\log Z)^{7/2} \tag{5.2}$$

and for which p^k divides $F(x, y)$ for some p with

$$(\log Z)^9 < p \le Z^{\frac{2}{d}} / (\log Z)^9 \tag{5.3}$$

which does not divide both x and y. Further $N_4^{(2)}$ is the number of integer pairs (x, y) for which $0 < |F(x, y)| \le Z$ and

$$\max(|x|, |y|) > Z^{\frac{1}{d}} (\log Z)^{7/2}.$$

By Lemma 2.2 we have, since d is at least 3,

$$N_4^{(2)} = O_F\left(Z^{\frac{2}{d}} / (\log Z)^{\frac{7}{2}}\right). \tag{5.4}$$

It remains only to estimate $N_4^{(1)}$ and we shall do so by a modification of an argument of Greaves [10] based on the geometry of numbers.

Recall (3.7). For $i = 1, \ldots, t$ we let $N_{4,i}^{(1)}$ be the number of integer pairs (x, y) for which $F(x, y) \neq 0$, (5.2) holds and p^k divides $F_i(x, y)$ for some prime satisfying (5.3) which does not divide both x and y. Notice that if p divides $F_i(x, y)$ and $F_j(x, y)$ with $i \neq j$ then p divides $\Delta(F)$. We may suppose that Z is sufficiently large that $(\log Z)^9$ exceeds $|\Delta(F)|$. Then

$$N_4^{(1)} = O(N_{4,1}^{(1)} + \cdots + N_{4,t}^{(1)}). \tag{5.5}$$

Suppose that (a, b) is an integer pair for which p^k divides $F_i(a, b)$ for some prime p satisfying (5.3) which does not divide both a and b. Then, since p does not divide uv by (3.2) and (5.3), (a, b) belongs to one of at most r_i lattices L_θ defined by a congruence

$$a \equiv \theta b \pmod{p^k}.$$

Following Greaves [10] we let $M = M(\theta, p^k)$ denote the minimal positive value of $\max(|a|, |b|)$ as we range over (a, b) in L_θ. For any real number X let $N_{\theta,k}(X)$ denote the number of pairs (a, b) in L_θ for which $|a| \le X$ and $|b| \le X$. Then, by Lemma 1 of [10],

$$N_{\theta,k}(X) \le \frac{4X^2}{p^k} + O\left(\frac{X}{M}\right) + O(1). \tag{5.6}$$

(Note that in the statement of Lemma 1 of [10] a term $O(1)$ has been omitted.) It then follows from (5.6) with $X = Z^{\frac{1}{d}}(\log Z)^{\frac{7}{2}}$ that

$$N_{4,i}^{(1)} \leq \sum_{(\log Z)^9 < p \leq Z^{\frac{2}{d}}(\log Z)^{-9}} \sum_{\theta} N_{\theta,k}(X)$$
$$\leq \sum_{(\log Z)^9 < p \leq Z^{\frac{2}{d}}(\log Z)^{-9}} \sum_{\theta} \left(\frac{4Z^{\frac{2}{d}}(\log Z)^7}{p^k} + O\left(\frac{Z^{\frac{1}{d}}(\log Z)^{\frac{7}{2}}}{M(\theta, p^k)} \right) + O(1) \right).$$

For each prime p we have at most d terms θ in the inner sum. Thus

(5.7)
$$N_{4,i}^{(1)} = O_F\left(Z^{\frac{2}{d}}(\log Z)^7 \sum_{(\log Z)^9 < p \leq Z^{\frac{2}{d}}(\log Z)^{-9}} \frac{1}{p^k} \right)$$
$$+ O_F\left(Z^{\frac{1}{d}}(\log Z)^{\frac{7}{2}} \sum_{(\log Z)^9 < p \leq Z^{\frac{2}{d}}(\log Z)^{-9}} \sum_{\theta} \frac{1}{M(\theta, p^k)} \right) + O_F\left(Z^{\frac{2}{d}}(\log Z)^{-10} \right).$$

Certainly

(5.8)
$$\sum_{(\log Z)^9 < p \leq Z^{\frac{2}{d}}(\log Z)^{-9}} \frac{1}{p^k} = O\left(\sum_{(\log Z)^9 < p} \frac{1}{p^k} \right)$$
$$= O\left(\frac{1}{(\log Z)^{9(k-1)}} \right)$$
$$= O\left(\frac{1}{(\log Z)^9} \right).$$

Further, since $M(\theta, p^k)$ is at least 1,

(5.9)
$$\sum_{(\log Z)^9 < p \leq Z^{\frac{1}{2d}}(\log Z)^2} \sum_{\theta} \frac{1}{M(\theta, p^k)} = O_F\left(Z^{\frac{1}{2d}}(\log Z)^2 \right).$$

It remains to estimate S where

$$S = \sum_{Z^{\frac{1}{2d}}(\log Z)^2 < p < Z^{\frac{2}{d}}(\log Z)^{-9}} \sum_{\theta} \frac{1}{M(\theta, p^k)}.$$

Notice that if $r_i = 1$ then $F_i(x, y) = O_F(Z^{\frac{1}{d}}(\log Z)^{\frac{7}{2}})$ and so if p^k divides $F_i(x, y)$ then, since $k \geq 2$, $p = O_F(Z^{\frac{1}{2d}}(\log Z)^{\frac{7}{4}})$.Thus if $r_i = 1$ then

(5.10)
$$S = O_F(1).$$

We shall now estimate S under the assumption that $r_i > 1$. We put $S = S_1 + S_2$ where S_1 is the sum over pairs p, θ with

$$M(\theta, p^k) \geq \frac{Z^{\frac{1}{d}}}{(\log Z)^5}$$

and S_2 is the sum over the other pairs (p, θ). Certainly

$$S_1 = O_F\left(\sum_{p \leq Z^{\frac{2}{d}}(\log Z)^{-9}} \frac{(\log Z)^5}{Z^{\frac{1}{d}}}\right) = O_F\left(\frac{Z^{\frac{1}{d}}}{(\log Z)^5}\right). \tag{5.11}$$

On the other hand S_2 consists of the sum over pairs p, θ with

$$1 \leq M \leq \frac{Z^{\frac{1}{d}}}{(\log Z)^5},$$

and $p > Z^{\frac{1}{2d}}(\log Z)^2$. To each pair p, θ we may associate a pair of integers (r, s) for which $\max(|r|, |s|) = M(\theta, p^k)$. Note that since $r_i > 1$ we have $F_i(r, s) \neq 0$. Further there are at most $O_F(1)$ pairs (p, θ) with $p > Z^{\frac{1}{2d}}(\log Z)^2$ which can be associated with a given pair (r, s) since $F_i(r, s) = O_F(Z^{\frac{r_i}{d}})$. Thus

$$S_2 = O\left(\sum_{1 \leq s \leq Z^{\frac{1}{d}}(\log Z)^{-5}} \frac{1}{s} \sum_{0 \leq r \leq s} \sum_{\substack{p^k | F(r,s) \\ F(r,s) \neq 0 \\ p > Z^{\frac{1}{2d}}(\log Z)^2}} 1\right) = O_{F,k}\left(Z^{\frac{1}{d}}(\log Z)^{-5}\right). \tag{5.12}$$

Therefore, by (5.5), (5.7), (5.8), (5.9), (5.10), (5.11) and (5.12),

$$N_4^{(1)} = O_{F,k}\left(Z^{\frac{2}{d}}/(\log Z)^{\frac{3}{2}}\right). \tag{5.13}$$

Further, by (5.1), (5.4) and (5.13),

$$N_4 = O_{F,k}\left(Z^{\frac{2}{d}}/(\log Z)^{\frac{3}{2}}\right). \tag{5.14}$$

6. An estimate for N_5

For any real number T let $B^*_{F,k}(T)$ denote the number of pairs of integers (x, y) with $\max(|x|, |y|) \leq T$ and for which $F(x, y)$ is divisible by p^k with p a prime larger than $T^2/(\log T)^{12}$. We shall suppose that $T^2/(\log T)^{12}$ exceeds $|\Delta(F)|$. Then

$$B^*_{F,k}(T) = O(B^*_{F_1,k}(T) + ... + B^*_{F_t,k}(T)). \tag{6.1}$$

If $r \leq 2k+1$ then Greaves used Selberg's sieve to prove that

$$B^*_{F_i,k}(T) = O_{F,k}\left(T^{2-\frac{1}{20}}\right) \tag{6.2}$$

for $i = 1, \cdots, t$. This follows from the proof of Lemma 4 of [10] on taking $x = T$ and $\eta = (\log T)^{-16}$; Greaves required the constraint $\eta \geq (\log T)^{-2}$ but it may be replaced with the weaker constraint $\eta \geq (\log T)^{-16}$. Xiao dealt with the case when

$$\frac{7}{18} < \frac{k}{r} < \frac{1}{2}$$

in [41] by means of the determinant method applied to weighted projective spaces. It follows from [41] that in this case

$$B^*_{F_i,k}(T) = O_{F,k}\left(T^2/(\log T)^4\right) \tag{6.3}$$

for $i = 1, \cdots, t$. Therefore for $\frac{k}{r} > \frac{7}{18}$

$$B^*_{F,k}(T) = O_{F,k}\left(T^2/(\log T)^4\right). \tag{6.4}$$

By a result of Helfgott, see the proof of Theorem 5.2 of [15], when (k, r) is $(2, 6)$

$$B^*_{F,2}(T) = O_{F,2}\left(T^2/(\log T)^{\delta}\right) \tag{6.5}$$

where

$$\delta = 0.7043.$$

Hooley in 2009 established an asymptotic estimate for the number of integer pairs (x, y) in a box for which $F(x, y)$ is cubefree when F is a binary form of degree 8 with integer coefficients which is irreducible over the rationals, see [25] and Theorem 2 of [26]. Xiao [42] extended this work to decomposable forms F and an examination of his proof yields an explicit error term from which we find that

$$B^*_{F,3}(T) = O_F\left(T^2/(\log\log T/\log\log\log T)\right) \tag{6.6}$$

when (k, r) is $(3, 8)$.

Define $g(T)$ by

$$g(T) = \begin{cases} (\log T)^4 & \text{if } \frac{k}{r} > \frac{7}{18} \\ (\log T)^{\delta} & \text{if } (k,r) = (2,6) \\ \log\log T/\log\log\log T & \text{if } (k,r) = (3,8). \end{cases} \tag{6.7}$$

Then by (6.4), (6.5) and (6.6),

$$B^*_{F,k}(T) = O_{F,k}\left(T^2/g(T)\right) \tag{6.8}$$

for (k,r) satisfying (1.4).

Put

$$f(T)=g\left(T^{\frac{1}{d}}\right)^{\frac{1}{d}}. \tag{6.9}$$

Let $N_5^{(1)}$ be the number of integer pairs (x,y) for which $F(x,y)\neq 0$ and $p^k|F(x,y)$ for some prime p with

$$p>Z^{\frac{2}{d}}/(\log Z)^9 \tag{6.10}$$

which does not divide both x and y and for which

$$\max(|x|,|y|)\leq Z^{\frac{1}{d}}f(Z). \tag{6.11}$$

Further, write $N_5^{(2)}$ for the number of integer pairs (x,y) for which $0<|F(x,y)|\leq Z$ and

$$\max(|x|,|y|)>Z^{\frac{1}{d}}f(Z). \tag{6.12}$$

Notice that $N_5=O\left(N_5^{(1)}+N_5^{(2)}\right)$. By Lemma 2.2,

$$N_5^{(2)}=O_{F,k}\left(Z^{\frac{2}{d}}/f(Z)^{d-2}\right)=O_{F,k}\left(Z^{\frac{2}{d}}/g(Z^{\frac{1}{d}})^{\frac{d-2}{d}}\right). \tag{6.13}$$

Furthermore, on taking $T=Z^{\frac{1}{d}}f(Z)$, we see from (6.8) and (6.9) that

$$N_5^{(1)}=O_{F,k}\left(Z^{\frac{2}{d}}f(Z)^2/g(Z^{\frac{1}{d}}f(Z))\right)$$

Since $g(T)$ is eventually increasing and tends to infinity with T it follows from (6.9) that $f(Z)$ is at least 1 for Z sufficiently large. We then have $g(Z^{\frac{1}{d}})\leq g(Z^{\frac{1}{d}}f(Z))$ and so

$$N_5^{(1)}=O_{F,k}\left(Z^{\frac{2}{d}}f(Z)^2/g(Z^{\frac{1}{d}})\right)$$

But $f(Z)^2/g(Z^{\frac{1}{d}})=f(Z)^{-d+2}$, by (6.9), and thus

$$N_5^{(1)}=O_{F,k}\left(Z^{\frac{2}{d}}/g(Z^{\frac{1}{d}})^{\frac{d-2}{d}}\right).$$

Therefore

$$N_5=O_{F,k}\left(Z^{\frac{2}{d}}/g(Z^{\frac{1}{d}})^{\frac{d-2}{d}}\right). \tag{6.14}$$

Theorem 1.2 now follows from (3.1), (3.3), (4.4), (4.6), (5.14), (6.7) and (6.14).

If F is a binary form with integer coefficients, nonzero discriminant and degree at least 3 and k is an integer larger than 1 then there exists a positive monotone increasing function $g_1(t)$ on the positive real numbers with $0 \le g_1(t) \le \log(t+2)$ for all positive real numbers t and

$$\lim_{t\to\infty} g_1(t) = \infty$$

such that

$$B^*_{F,k}(T) = O_{F,k}\left(T^2/g_1(T)\right), \tag{6.15}$$

subject to the *abc* conjecture. Granville [9] showed this when $k = 2$ and his argument extends readily to the general case. Arguing as above we deduce that Conjecture 1.4 holds for $N_{F,k}(Z)$. With this estimate for $N_{F,k}(Z)$ we are then able to establish Conjecture 1.4 for $R_{F,k}(Z)$ as in the next section.

7. The proof of Theorems 1.1 and 1.3

If $\operatorname{Aut} F = \mathbf{C}_1$ then every integer pair (x, y) for which $F(x, y)$ is essentially represented with $0 < |F(x, y)| \le Z$ gives rise to a distinct integer h with $0 < |h| \le Z$. It follows from Theorem 1.2 and Lemma 2.6 that

$$R_{F,k}(Z) = c_{F,k} Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right) \tag{7.1}$$

where $u(z)$ is defined as in (3.4) when k and r satisfy (1.4) with (k, r) not $(2, 6)$ or $(3, 8)$, as in (3.5) when (k, r) is $(2, 6)$ and satisfies (3.6) when (k, r) is $(3, 8)$. Similarly if $\operatorname{Aut} F = \mathbf{C}_2$ then (7.1) holds with $\frac{1}{2} c_{F,k}$ in place of $c_{F,k}$.

Suppose now that $\operatorname{Aut} F$ is conjugate to $\mathbf{C}_3$. Then for A in $\operatorname{Aut} F$ with $A \ne I$ we have, by Lemma 2.7, $\Lambda(A) = \Lambda(A^2)$. Thus whenever (x, y) is in Λ, $F(x, y) = h$ and h is essentially represented there are exactly two other pairs $(x_1, y_1), (x_2, y_2)$ for which $F(x_i, y_i) = h$ for $i = 1, 2$. When (x, y) is in $\mathbb{Z}^2$ but not in Λ and $F(x, y)$ is essentially represented then $F(x, y)$ has exactly one representative.

Let $\{\omega_1, \omega_2\}$ be a basis for Λ with $\omega_1 = (a_1, a_3)$ and $\omega_2 = (a_2, a_4)$ and such that $\max(|a_1|, |a_2|, |a_3|, |a_4|)$ is minimized. Recall that

$$F_{\omega_1,\omega_2}(x, y) = F(a_1 x + a_2 y, a_3 x + a_4 y).$$

Since

$$|\mathcal{N}_{F,k}(Z) \cap \Lambda| = N_{F_{\omega_1,\omega_2},k}(Z),$$

by Theorem 1.2 we have

$$|\mathcal{N}_{F,k}(Z) \cap \Lambda| = c(\Lambda) Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right). \tag{7.2}$$

Note that since ω_1 and ω_2 are chosen so that $\max(|a_1|, |a_2|, |a_3|, |a_4|)$ is minimized the implicit constant in the error term may be determined in terms of F and

k. By (7.2) and Lemma 2.6 the number of integer pairs (x, y) in Λ for which $0 < |F(x, y)| \le Z$ and $F(x, y)$ is k-free and essentially represented is

$$c(\Lambda)Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right).$$

Each pair (x, y) is associated with two other pairs which represent the same integer. These pairs yield

$$\frac{c(\Lambda)}{3}Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right) \tag{7.3}$$

integers h with $0 < |h| \le Z$. It now follows from Theorem 1.2 and Lemma 2.6 that there are

$$(c_{F,k} - c(\Lambda))Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right) \tag{7.4}$$

integer pairs (x, y) not in Λ for which $F(x, y)$ is k-free and essentially represented and each pair gives rise to an integer h with $0 < |h| \le Z$ which is uniquely represented by F. It follows from (7.3) and (7.4) that when $\operatorname{Aut} F$ is equivalent to $\mathbf{C}_3$ we have

$$R_{F,k}(Z) = \left(c_{F,k} - \frac{2}{3}c(\Lambda)\right)Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right).$$

A similar analysis applies to the case when $\operatorname{Aut} F$ is equivalent to $\mathbf{D}_1, \mathbf{D}_2, \mathbf{C}_4$ or $\mathbf{C}_6$. These groups are cyclic with the exception of $\mathbf{D}_2$ but $\mathbf{D}_2/\{\pm I\}$ is cyclic and that is sufficient for our purposes.

We are left with the possibility that $\operatorname{Aut} F$ is conjugate to $\mathbf{D}_3, \mathbf{D}_4$ or $\mathbf{D}_6$. We first consider the case when $\operatorname{Aut} F$ is equivalent to $\mathbf{D}_4$. In this case (7.2) holds as before and since each h which is essentially represented by F and for which $h = F(x, y)$ with (x, y) in Λ is represented by 8 integer pairs the number of k-free integers h with $0 < |h| \le Z$ for which there exists an integer pair (x, y) in Λ with $F(x, y) = h$ is

$$\frac{c(\Lambda)}{8}Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right). \tag{7.5}$$

By Lemma 2.7 $\Lambda_i \cap \Lambda_j = \Lambda$ for $1 \le i < j \le 3$ and so the number of integer pairs (x, y) in Λ_1, Λ_2 or Λ_3 but not in Λ for which $F(x, y)$ is essentially represented and k-free with $0 < |F(x, y)| \le Z$ is, by Theorem 1.2,

$$\left(c(\Lambda_1) + c(\Lambda_2) + c(\Lambda_3) - 3c(\Lambda)\right)Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right).$$

Each such integer $F(x, y)$ has precisely four representatives and so the terms in $\Lambda_1, \Lambda_2, \Lambda_3$ but not in Λ contribute

$$\frac{1}{4}\left(c(\Lambda_1) + c(\Lambda_2) + c(\Lambda_3) - 3c(\Lambda)\right)Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right) \tag{7.6}$$

terms to $\mathcal{R}_{F,k}(Z)$. Finally the terms (x,y) in $\mathcal{N}_{F,k}(Z)$ but not in Λ_1, Λ_2 or Λ_3 for which $F(x,y)$ is essentially represented have cardinality

$$(c_{F,k} - c(\Lambda_1) - c(\Lambda_2) - c(\Lambda_3) + 2c(\Lambda))\, Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right).$$

Each integer represented by such a term has 2 representations and therefore these terms contribute

$$\frac{1}{2}\,(c_{F,k} - c(\Lambda_1) - c(\Lambda_2) - c(\Lambda_3) + 2c(\Lambda))\, Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right) \tag{7.7}$$

terms to $\mathcal{R}_{F,k}(Z)$. It now follows from (7.5), (7.6), (7.7) and Lemma 2.6 that

$$R_{F,k}(Z) = \frac{1}{2}\left(c_{F,k} - \frac{c(\Lambda_1)}{2} - \frac{c(\Lambda_2)}{2} - \frac{c(\Lambda_3)}{2} + \frac{3c(\Lambda)}{4}\right) Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right),$$

as required.

Next suppose that $\operatorname{Aut} F$ is conjugate to $\mathbf{D}_3$. As before the pairs (x,y) in $\mathcal{N}_{F,k}(Z) \cap \Lambda$ for which $F(x,y)$ is essentially represented yield

$$\frac{c(\Lambda)}{6} Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right) \tag{7.8}$$

terms in $\mathcal{R}_{F,k}(Z)$. Since $\Lambda_i \cap \Lambda_j = \Lambda$ for $1 \le i < j \le 3$ by Lemma 2.7 the pairs (x,y) in $\mathcal{N}_{F,k}(Z) \cap \Lambda_i$ for $i = 1,2,3$ which are not in Λ and which are essentially represented contribute

$$\left(\frac{c(\Lambda_1)}{2} + \frac{c(\Lambda_2)}{2} + \frac{c(\Lambda_3)}{2} - \frac{3c(\Lambda)}{2}\right) Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right) \tag{7.9}$$

terms to $\mathcal{R}_{F,k}(Z)$. The pairs (x,y) in $\mathcal{N}_{F,k}(Z) \cap \Lambda_4$ which are not in Λ and for which $F(x,y)$ is essentially represented contribute

$$\left(\frac{c(\Lambda_4)}{3} - \frac{c(\Lambda)}{3}\right) Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right) \tag{7.10}$$

terms to $\mathcal{R}_{F,k}(Z)$. Finally the pairs (x,y) in $\mathcal{N}_{F,k}(Z)$ which do not lie in Λ_i for $i = 1,2,3,4$ contribute, by Lemma 2.7,

$$(c_{F,k} - c(\Lambda_1) - c(\Lambda_2) - c(\Lambda_3) - c(\Lambda_4) + 3c(\Lambda))\, Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right) \tag{7.11}$$

terms to $\mathcal{R}_{F,k}(Z)$. It then follows from (7.8), (7.9), (7.10), (7.11) and Lemma 2.6 that

$$R_{F,k}(Z) = \left(c_{F,k} - \frac{c(\Lambda_1)}{2} - \frac{c(\Lambda_2)}{2} - \frac{c(\Lambda_3)}{2} - \frac{2c(\Lambda_4)}{3} + \frac{4c(\Lambda)}{3}\right) Z^{\frac{2}{d}} + O_{F,k}\left(Z^{\frac{2}{d}}/u(z)\right),$$

as required.

When $\operatorname{Aut} F$ is equivalent to $\mathbf{D}_6$ the analysis is the same as for $\mathbf{D}_3$ taking into account the fact that $\operatorname{Aut} F$ contains $-I$ and so the weighting factor is one half of what it is when $\operatorname{Aut} F$ is equivalent to $\mathbf{D}_3$.

Finally we note that, since there is no prime p such that p^k divides $F(a,b)$ for all pairs of integers (a,b), $c_{F,k}$ is a positive number. Since an integer which is essentially represented by F has at most $|\operatorname{Aut} F|$ representations we have

$$C_{F,k} \geq c_{F,k}/|\operatorname{Aut} F|$$

and, since the order of the automorphism group of F is at most 12, the order of $\mathbf{D}_6$, we deduce that $C_{F,k}$ is positive. This completes the proof of Theorems 1.1 and 1.3.

References

[1] M. A. Bennett, N. P. Dummigan, T. D. Wooley: The representation of integers by binary additive forms. *Comp. Math.* **111** (1998), 15-33.

[2] M. Bhargava: The geometric sieve and the density of squarefree values of invariant polynomials. *arXiv:1402.0031 [math.NT]*

[3] M. Bhargava, A. Shankar, X. Wang: Squarefree values of polynomial discriminants. *arXiv:1611.09806 [math.NT]*

[4] T. D. Browning: Power-free values of polynomials. *Archiv der Mathematik* **96** (2011) no. [2], 139-150.

[5] T. D. Browning: Equal sums of two kth powers. *J. Number Theory* **96** (2002), 293-318.

[6] P. Erdős, K. Mahler: On the number of integers which can be represented by a binary form. *J. London Math. Soc.* **13** (1938), 134-139.

[7] M. Filaseta: Powerfree values of binary forms. *J. Number Theory* **49** (1994), 250-268.

[8] F. Q. Gouvêa, B. Mazur: The square-free sieve and the rank of elliptic curves. *J. Amer. Math. Soc.* **4** (1991) no.[1], 793-805.

[9] A. Granville: *ABC* allows us to count squarefrees. *Int. Math. Res. Notices* **9** (1998), 991-1009.

[10] G. Greaves: Power-free values of binary forms. *Quart. J. Math.* **43** (1992) no. [2], 45-65.

[11] G. Greaves: Representation of a number by the sum of two fourth powers. *Mat. Zametki* **55** (1994), 47-58.

[12] G.H. Hardy and E.M. Wright: *An introduction to the theory of numbers.* 5th ed., Oxford University Press, 1979.

[13] D. R. Heath-Brown: The density of rational points on cubic surfaces. *Acta. Arith.* **79** (1997), 17-39.

[14] D. R. Heath-Brown: The density of rational points on curves and surfaces. *Ann. Math.* **155** (2002) no.[2], 553-598.

[15] H. A. Helfgott: On the square-free sieve. *Acta Arith.* **115** (2004), 349-402.

[16] C. Hooley: On the power free values of polynomials. *Mathematika* **14** (1967), 21-26.

[17] C. Hooley: On binary cubic forms. *J. reine angew. Math.* **226** (1967), 30-87.

[18] C. Hooley: *Applications of sieve methods to the theory of numbers.* Cambridge University Press, Cambridge, 1976.

[19] C. Hooley: On the numbers that are representable as the sum of two cubes. *J. reine angew. Math.* **314** (1980), 146-173.

[20] C. Hooley: On another sieve method and the numbers that are a sum of two h^{th} powers. *Proc. London Math. Soc.* **43** (1981), 73-109.

[21] C. Hooley: On binary quartic forms. *J. reine angew. Math.* **366** (1986), 32-52.

[22] C. Hooley: On another sieve method and the numbers that are a sum of two h^{th} powers. II. *J. reine angew. Math.* **475** (1996), 55-75.

[23] C. Hooley: On binary cubic forms: II. *J. reine angew. Math.* **521** (2000), 185-240.

[24] C. Hooley: On totally reducible binary forms: II. *Hardy-Ramanujan Journal* **25** (2002), 22-49.

[25] C. Hooley: On the power-free values of polynomials in two variables. In *Analytic Number Theory.*, 235-266. Cambridge University Press, Cambridge, 2009.

[26] C. Hooley: On the power-free values of polynomials in two variables: II. *J. Number Theory* **129** (2009), 1443-1455.

[27] K. Mahler: Zur Approximation algebraischer Zahlen. III. (Über die mittlere Anzahl der Darstellungen grosser Zahlen durch binäre Formen). *Acta Math.* **62** (1933), 91-166.

[28] R. Murty, H. Pasten: Counting square free values of polynomials with error term. *Int. J. Number Theory* **10** (2014) no.[7], 1743-1760.

[29] M. Newman: *Integral Matrices.* Pure and Appl. Math. (S. Eilenberg and P.A.Smith, eds.), vol.45, Academic Press, New York, 1972.

[30] B. Poonen: Squarefree values of multivariate polynomials. *Duke Math. J.* **118** (2003) no.[2], 353-373.

[31] P. Salberger: On the density of rational and integral points on algebraic varieties. *J. reine angew. Math.* **606** (2007), 123-147.

[32] P. Salberger: Counting rational points on projective varieties. Preprint 2009.

[33] P. Salberger: Uniform bounds for rational points on cubic hypersurfaces. In *Arithmetic and Geometry.*, 401-421. London Math. Soc. Lecture Note Ser.,**420**, Cambridge University Press, Cambridge 2015

[34] C. Skinner, T. D. Wooley: Sums of two k-th powers. *J. reine angew. Math.* **462** (1995), 57-68.

[35] C. L. Stewart: On the number of solutions to polynomial congruences and Thue equations. *J. Amer. Math. Soc.* **4** (1991) no.[4], 793-835.

[36] C. L. Stewart, J. Top: On ranks of twists of elliptic curves and power-free values of binary forms. *J. Amer. Math. Soc.* **8** (1995) no.[4], 943-972.

[37] C. L. Stewart, S. Y. Xiao: On the representation of integers by binary forms. *Math. Ann.*, to appear.

[38] C. L. Stewart, K. Yu: On the *abc* conjecture, II. *Duke Math. J* **108** (2001), 169-181.

[39] J. L. Thunder: Decomposable form inequalities. *Ann. Math.* **153** (2001) no.[3], 767-804.

[40] T. D. Wooley: Sums of two cubes. *Int. Math. Res. Notices* **4** (1995), 181-185.

[41] S. Y. Xiao: Power-free values of binary forms and the global determinant method. *Int. Math. Res. Notices* **16** (2016), 5078-5135.

[42] S. Y. Xiao: Square-free values of decomposable forms. *Canadian J. Math.* **70** (2018), 1390-1415.

The research of the first author was supported in part by the Canada Research Chairs Program and by Grant A3528 from the Natural Sciences and Engineering Research Council of Canada.

Received ??

Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario, Canada
N2L 3G1
E-mail: `cstewart@uwaterloo.ca`

Department of Mathematics
University of Toronto
Toronto, Ontario, Canada
M5S 2E4
E-mail: `syxiao@math.toronto.edu`