PM 945 Topics in Algebra:
Clifford Algebras, Spinors, and Calibrations
FALL 2020

- **Instructor:** Spiro Karigiannis
- **Telephone:** 519-888-4567 ext 32810
- **Office Hours:** See note below.

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Course Lectures: Mon/Wed 9:30am–10:50am in B1-271

NOTE: This course is being offered in-person on campus, *not online*. If the university allows me to hold in-person office hours, I will. This may have to be in the form of a weekly in-person "tutorial" that will function essentially as office hours in a large classroom so we can all maintain physical distance. Regardless, "office hours" will also be handled using **Piazza**. You can expect a response from me within two hours most of the time, if your question is posted between 8am and 10pm. Questions posted in the middle of the night will be answered the following morning. One-on-one virtual appointments may also be possible.

Course description: This is a graduate course in *algebra* covering various specialized topics that have applications in geometry. There is *no geometry*. We only discuss structures at the level of vector spaces. *It is all just algebra.*

Prerequisites: Linear algebra, group theory, and ring theory. No geometry background is assumed or needed. This course could be taken by any graduate student in pure mathematics.

Main topics: Tensor products and exterior algebra; inner products, symplectic structures, and complex structures; normed real division algebras; cross products and calibrations; Clifford algebras and the related Spin groups; Jordan algebras. There are two main themes running through the second half of the course:

- Quaternions and octonions can be used to understand some specific structures in a concrete way, such as specific Spin groups, orthogonal groups, and Jordan algebras, in particular low dimensions.
- There is a way to generalize the Cauchy-Schwartz inequality from inner product spaces to spaces with calibrations using normed division algebras.

Textbook: There is no required textbook for this course, because *there is no textbook that adequately covers all the material that I plan to cover.* Some supplementary material may be distributed, and some useful additional references can be found below, but the best way to make sure that you learn everything is to come to all of the lectures. One of the reasons that I am giving this course is because I plan to eventually write a book that covers all of this material from the point of view that I like to think about it.

Some useful additional references (only some parts of each of these are relevant):

- Chevalley, *The algebraic theory of spinors*, Columbia University Press, 1954.
- Conway and Smith, *On quaternions and octonions: their geometry, arithmetic, and symmetry*, A K Peters, 2003.
- Harvey, *Spinors and calibrations* (Perspectives in Mathematics), Academic Press, 1990.
- Lawson and Michelson, *Spin geometry* (Princeton Mathematical Series), Princeton University Press, 1989.
- Porteous, *Clifford algebras and the classical groups* (Cambridge Studies in Advanced Mathematics), Cambridge University Press, 1995.

Marking scheme

Course marks will be determined as follows.

- Assignments: 78% (six assignments, about one every two weeks, 13% each)
- Paper/presentation: 22% (approx 12–15 pages, typewritten; 16% for paper and 6% for presentation)

Please note that you are encouraged to work together with your classmates on the assignment problems, but you must write up and turn in your own solutions to the problems.

Detailed outline of course topics. (Tentative and subject to change.)

[1] Review of dual spaces and the tensor product of vector spaces. The exterior algebra of a vector space. Bilinear forms on real vector spaces and sesquilinear forms on complex vector spaces. Inner products on vector spaces and their induced inner products on the tensor powers and exterior powers. Orientations and volume forms. The Hodge star operator. Relation between the exterior and interior product. Poincaré duality at the level of vector spaces. Self-duality in 4 dimensions. The orthogonal group.

[2] Symplectic vector spaces. The canonical symplectic basis theorem for non-degenerate skew-symmetric bilinear forms. Lagrangian, isotropic, and coisotropic subspaces of symplectic vector spaces. The symplectic reduction theorem at the level of vector spaces. The symplectic group.

[3] Inner product spaces with signature $(n, 1)$: Lorentzian vector spaces. Null cones. Timelike and spacelike vectors and subspaces. The Cauchy-Schwartz inequality in Lorentzian spaces. The Lorentz group.

[4] Complex structures on real vector spaces. The associated Kähler form. The Hodge star operator and the relation of the Kähler form to the volume form. The classical Wirtinger inequality at the level of linear algebra. Characterization of complex subspaces as those calibrated by powers of the Kähler form. Decomposition of the complexified exterior algebra into irreducible representations of the complex linear group. The Lefschetz decomposition.

[5] Real normed division algebras: real numbers, complex numbers, quaternions, and octonions. Their classification and the Hurwitz Theorem. Their construction via the Cayley-Dickson process. Geometric interpretation of the real normed division algebras, and their relation to the orthogonal groups. The Lie group $SO(3)$ in terms of the quaternions, and the exceptional Lie group G_2 in terms of octonions. Decomposition of the exterior algebra in seven dimensions into irreducible representations of G_2. Decomposition of the exterior algebra in eight dimensions into irreducible representations of $\mathrm{Spin}(7)$. Relations with self-duality in four dimensions.

[6] Cross products on inner product spaces and their classification. Special subspaces and calibrations. Relation to the Cauchy-Schwarz inequality. Description of instantons and branes at the level of linear algebra. Comparison of general calibrations with the Kähler calibrations.

[7] The classical Clifford algebras over $\mathbb{R}$ and $\mathbb{C}$. Their construction and properties. Bott periodicity of Clifford algebras. The Spin and Pin groups, and their relation to the orthogonal groups. Vector spaces of spinors and representations of Clifford algebras on spinor spaces. Relation of quaternions and octonions to Spin groups. Triality. The exceptional Lie group F_4 in terms of spin groups and Clifford algebras. Relation between calibrations and spinors at the level of linear algebra. (Calibrations are essentially "squares" of spinors.)

[8] The projective spaces over the normed real division algebras and their properties. Proofs that $\mathbb{RP}^1 \cong U(1)$ and that $\mathbb{RP}^3 \cong SO(3)$. The classical Hopf fibrations of projective spaces over spheres with spherical fibres.

[9] Jordan algebras and their basic properties. Jordan algebras as the "symmetric" analogue of Lie algebras. Examples. The exceptional Jordan algebra in 27 dimensions and its relation to F_4 and to the octonions.

Academic offences

Academic Integrity: In order to maintain a culture of academic integrity, members of the University of Waterloo community are expected to promote honesty, trust, fairness, respect and responsibility. Please see `http://www.uwaterloo.ca/academicintegrity/` for more information.

Grievance: A student who believes that a decision affecting some aspect of his/her university life has been unfair or unreasonable may have grounds for initiating a grievance. Read Policy 70 - Student Petitions and Grievances, Section 4, `http://www.adm.uwaterloo.ca/infosec/Policies/policy70.htm`. When in doubt please be certain to contact the department's administrative assistant who will provide further assistance.

Discipline: A student is expected to know what constitutes academic integrity, to avoid committing academic offenses, and to take responsibility for his/her actions. A student who is unsure whether an action constitutes an offense, or who needs help in learning how to avoid offenses (e.g., plagiarism, cheating) or about rules for group work/collaboration should seek guidance from the course professor, academic advisor, or the Undergraduate Associate Dean. For information on categories of offenses and types of penalties, students should refer to Policy 71, Student Discipline, `http://www.adm.uwaterloo.ca/infosec/Policies/policy71.htm`. For typical penalties check Guidelines for the Assessment of Penalties, `http://www.adm.uwaterloo.ca/infosec/guidelines/penaltyguidelines.htm`.

Avoiding Academic Offenses: Most students are unaware of the line between acceptable and unacceptable academic behaviour, especially when discussing assignments with classmates and using the work of other students. For information on commonly misunderstood academic offenses and how to avoid them, students should refer to the Faculty of Mathematics Cheating and Student Academic Discipline Policy, `http://www.math.uwaterloo.ca/navigation/Current/cheating_policy.shtml`

Appeals: A student may appeal the finding and/or penalty in a decision made under Policy 70 - Student Petitions and Grievances (other than regarding a petition) or Policy 71 - Student Discipline if a ground for an appeal can be established. Read Policy 72 - Student Appeals, `http://www.adm.uwaterloo.ca/infosec/Policies/policy72.htm`

Note for students with disabilities

The AccessAbility Services (AS) Office, located in Needles Hall, Room 1132, collaborates with all academic departments to arrange appropriate accommodations for students with disabilities without compromising the academic integrity of the curriculum. If you require academic accommodations to lessen the impact of your disability, please register with the AS Office at the beginning of each academic term.