Graduate topics course in Winter Term 2020:

"Elements of random matrix theory"

(Instructor: Andu Nica)

Course outline

The study of random matrices is a vigorous area of current research which offers interesting ideas to reflect on, and interesting problems to work on for a pure mathematician. Here is an outline of a few such ideas that we would like to discuss in this course.

In order to talk about random matrices, one needs to have a space of matrices, endowed with a probability measure. For someone with background in analysis, a natural first example of this kind is found by looking at a compact matrix group (e.g. the group $\mathcal{U}(n)$ of unitary $n \times n$ matrices) endowed with its Haar measure. We will start by studying how integration is performed in this kind of framework, via a method which goes under the name of "Weingarten calculus".

Our next job will be to take on some elements of spectral theory for complex Hermitian random matrices. I will streamline this part of the course around the discussion of statistical properties for the eigenvalues of a sum of two Hermitian random matrices. To put the things into perspective: if we are given two non-random Hermitian $n \times n$ matrices A and B, a well-known linear algebra result called "Horn's conjecture" provides precise conditions on how the eigenvalues of $A + B$ are constrained via inequalities involving the eigenvalues of A and those of B. In order to "randomize" our setting, what we do (for the same A and B as above) is to consider the matrix $A + UBU^*$, where U is a random unitary matrix sampled according to the Haar measure of the group $\mathcal{U}(n)$. We will study the distribution of eigenvalues of the random matrix $A + UBU^*$, and we will find that for large values of n (or rather: when we make $n \to \infty$, in a sense which will be made precise in the course) this is governed by a certain binary operation "$\boxplus$" with probability distributions, called free additive convolution.

We will then pay some special attention to the most basic instance of a random Hermitian matrix, the so-called "Gaussian ensemble", or GUE for short. We will see how a classical analysis gadget called "Hermite polynomials" can be used in the study of an $n \times n$ GUE matrix. Also, in a context where we once again make $n \to \infty$, we will discuss the behaviour of eigenvalues that are "in the bulk" or "at the edge" of the spectrum of the GUE.

Background and Prerequisites.

The course will run, of course, in the "rigorous" style specific to a Pure Math course. Here are some highlights on what is required on the lines of background.

• A solid background in analysis: abstract measure and integration, basic complex analysis, some elements of abstract harmonic analysis – in particular some familiarity with the notion of Haar measure on a compact group.

• A minimal background in probability, including e.g. a few basic facts about distributions, and about independence for random variables on a probability space $(\Omega, \mathcal{F}, P)$.

• (Optional). If you happen to have some previous exposure to the framework of a

$*$-probability space (as studied e.g. in the PMath 990 course of Fall Term 2017), this could help to make you feel more comfortable with the setting used in some of the segments of the course.

References.

[1] G.W. Anderson, A. Guionnet, O. Zeitouni. *An introduction to random matrices*, Cambridge University Press, 2010.
(Available at https://cims.nyu.edu/∼zeitouni/cupbook.pdf)

[2] J.A. Mingo, R. Speicher. *Random matrices and free probability*, Springer Verlag, 2017.
(Available at https://rolandspeicher.files.wordpress.com/2019/02/mingo-speicher.pdf)

[3] T. Tao. *Topics in random matrix theory*, American Mathematical Society, 2012.
(Available at https://terrytao.files.wordpress.com/2011/02/matrix-book.pdf)