Winter Term 2023, PMath 990 (Graduate Topics Course)
Course Information

Topic of the course: Limit theorems for non-commutative random variables

Instructor: Andu Nica (Professor, Pure Mathematics Department), anica@uwaterloo.ca

Room and time: MWF 12:30-1:20 pm, in ML 349

Prerequisites: The required background for the course is a good knowledge of basic analysis in abstract framework, including measure and integration, and including some rudimentary facts about bounded linear operators on a Hilbert space. There is, of course, some basic level of mathematical maturity which will be assumed in connection to the handling of algebraic objects such as groups or commonly encountered combinatorial structures. The lectures will be largely self-contained; for the results that have to be accepted without proof, I will provide specific references as the course moves on. Some textbooks that I may invoke at one point or another are listed below.

References: For Part I of the course, the standard reference is the monograph of Billingsley [1]. Some good sources for parts II and III are the books [2], [3], also the draft of manuscript that was recently posted by Banica [4].

[1] P. Billingsley. *Convergence of probability measures*, 2nd edition, Wiley, 1999.

[2] J.A. Mingo, R. Speicher. *Free probability and random matrices*, Springer, 2017.

[3] A. Nica, R. Speicher. *Lectures on the combinatorics of free probability*, Cambridge University Press, 2006.

[4] T. Banica. *Methods of free probability*, available at https://banica.u-cergy.fr/pdf/fp.pdf

Course-grade. This will be computed based on homework assignments (60%) and on an end-of-term project (40%).

Some more details concerning the content of the course.
In order to talk about limit laws, it stands to reason that we should first clarify:
→ what is the framework where these laws appear, and
→ in what sense are we considering the notion of limit.
This will be done in the Parts II and I of the course (respectively).

Part I. In what sense are we considering our limits.
Here we will go over some basic facts about the *weak convergence* of probability measures on the Borel sigma-algebra of the real line. In order to do so, we will organize these probability measures into a metric space $\text{Prob}(\mathbb{R})$ and we will discuss some properties of this metric space, e.g. the description of its compact sets, as given by the so-called "Helly's selection theorem". Particular attention will be given to measures $\mu \in \text{Prob}(\mathbb{R})$ which have finite moments of all orders; for such measures μ we will see some growth conditions on the sequence of moments which ensure that μ is uniquely determined by this sequence, or ensure that μ has compact support.

Part II. What is our framework: non-commutative probability spaces.
We now start to look at real random variables and their distributions. We will be interested in such random variables that have finite moments of all orders. Taking advantage of the

fact that random variables can be added and multiplied together, we will introduce an axiomatic approach for looking at them. In this approach the primary object of study is a pair $(\mathcal{A}, \mathbb{E})$ where $\mathcal{A}$ is an abstract algebra (taken, for convenience, over the field of scalars $\mathbb{C}$) and $\mathbb{E} : \mathcal{A} \to \mathbb{C}$ is a linear functional, satisfying a suitable list of axioms. We think of the elements of $\mathcal{A}$ as *non-commutative random variables*, and for $a \in \mathcal{A}$ we think of $\mathbb{E}(a)$ as some kind of *expectation* of the (non-commutative) random variable a. The qualifier "non-commutative" is used as a reminder that, in a departure from the framework of classical probability, we do not assume the algebra $\mathcal{A}$ to be commutative. This allows for a score of interesting examples, for instance we can allow $\mathcal{A}$ to be the group algebra $\mathbb{C}[G]$ of a discrete (not necessarily commutative) group G, and where $\mathbb{E} : \mathbb{C}[G] \to \mathbb{C}$ is the so-called "trace functional". In order to ensure that we can talk about distributions in the framework of $(\mathcal{A}, \mathbb{E})$, we will have a look at a Hilbert space $L^2(\mathcal{A}, \mathbb{E})$ which is naturally associated to it, and at how the elements $a \in \mathcal{A}$ may act on this Hilbert space.

Part III. Limit laws.
At this point, we can start in earnest on the topic announced in the title of the course: we want to look at significant situations where we consider a natural sequence $(a_n)_{n=1}^{\infty}$ of selfadjoint elements in an $(\mathcal{A}, \mathbb{E})$, and we find that there is a limit (in the sense of weak convergence in $\text{Prob}(\mathbb{R})$) for the distributions of the a_n's. The prototypical result of this kind occurs when the a_n's are obtained as renormalized averages $a_n = \frac{1}{\sqrt{n}}(x_1 + \cdots + x_n)$, with the x_n's being independent. But wait – what do we mean when we say that the x_n's are *independent*? This question leads into very interesting developments! Thinking in terms of the $\mathbb{C}[G]$ example, we can naturally create a candidate of independent sequence of x_n's when we let G be either a *direct product* or a *free product* of countably many copies of the 2-element group $\{-1, 1\}$. The corresponding limit theorems both go under the name of Central Limit Theorem (CLT): in the first case we have the *classical DeMoivre-Laplace CLT*, where the limit is the Gaussian distribution, while in the second case we have an instance of the *CLT of free probability*, where the limit distribution is the so-called "semicircle law of Wigner". These CLT results will be covered in great detail in the course. We will in fact see a mechanism which gives both the classical and the free CLT at the same time, based on a symmetry principle called *exchangeability* for the sequence of x_n's that are being averaged.

The realm of limit theorems that can be considered definitely goes beyond the above mentioned CLT results. For example you may have heard about the Poisson limit theorem, where the limit is (no wonder) the Poisson distribution. We will recover this limit theorem, and also get its counterpart in free probability. For the latter counterpart, the limit could be called "free Poisson distribution", but it was actually discovered in the 1960s by people working in random matrix theory, and is known as "the Marchenko-Pastur distribution".

I would venture to say that, generally speaking, the universe of non-commutative probability is generous in allowing for limit theorems to hold whenever one considers "natural examples" of sequences of random variables. This may cover, for instance, sequences of generators in groups, statistical quantities measured on random combinatorial objects (such as trees or set-partitions), or eigenvalues of random matrices. But there is a big leap between guessing that such a limit theorem holds and actually *proving* it, even more so if we want to determine precisely what is the limit distribution. The course will discuss some of the tools which were developed in order to deal with this issue in classical probability (convolution of probability distributions, characteristic functions, cumulants), and will also discuss the counterparts of these notions in free probability (*free* convolution of probability distributions, R-transforms, *free* cumulants).

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https://uwaterloo.ca/secretariat/policies-procedures-guidelines/policy-70
When in doubt please be certain to contact the department's administrative assistant who will provide further assistance.

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https://uwaterloo.ca/secretariat/policies-procedures-guidelines/policy-71

Appeals: A decision made or penalty imposed under Policy 70, Student Petitions and Grievances (other than a petition) or Policy 71, Student Discipline may be appealed if there is a ground. A student who believes he/she has a ground for an appeal should refer to Policy 72, Student Appeals,
https://uwaterloo.ca/secretariat/policies-procedures-guidelines/policy-72

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