

PMath 990, Winter Term 2027

Free and finite free probability

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Course description. Free probability is a sibling of “usual” probability theory, where instead of the customary notion of independence for real random variables one studies the notion of *free independence* for selfadjoint elements in a (necessarily non-commutative) $\ast$ -algebra over $\mathbb{C}$. This area of research was founded in the 1980’s and has, since then, accumulated an extensive body of work. Its name comes from the fact that examples of freely independent families of elements arise naturally in the group algebras of free groups. Another important class of examples arises in the context of certain models of random $d \times d$ matrices over $\mathbb{C}$, where one finds families of random Hermitian matrices which are “asymptotically” freely independent in the $d \rightarrow \infty$ limit.

The course will not aim to present a large chunk of material, the emphasis will rather be on giving the students a good understanding of some free probability basics. In order to fix the ideas, we will concentrate for a good part of the term on notions of *free convolution* which reflect the operations of adding and multiplying freely independent elements. Despite its apparent simple-mindedness, this topic can serve as a great motivation for introducing some relevant tools of the trade used in free probabilistic arguments.

The instructor will aim to make the lectures clear and as self-contained as possible, so that in-class notes can be used as reference towards absorbing the course material. When needed, we will draw on the notes from a course on free probability taught by Roland Speicher at the Saarland University in Germany, <https://www.uni-saarland.de/lehrstuhl/speicher/roland-speicher.html> (go to Lecture Notes, then to Free Probability Theory).

In the last part of the term we will devote some time to ideas from a recent direction of development called *finite free probability*. This can be seen as a way of studying random $d \times d$ matrices by focusing on their characteristic polynomials, and by introducing some operations of *polynomial convolution*, which relate to free convolution in the $d \rightarrow \infty$ limit. For this part of the course, our main references will be two relevant papers written by the creators of this direction of research, <https://arXiv.org/pdf/1504.00350> and <https://arxiv.org/pdf/2108.07054>.