Pure Mathematics Topology and Real Analysis Qualifying Examination
University of Waterloo
September 13, 2022

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.
2. Check for questions on both sides of each page.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
4. Do not write on the Crowdmark QR code at the top of each page.
5. Use a dark pencil or pen for your work.
6. All questions are equally weighted.

1. Let X be a compact metric space with the metric d.

   (a) Consider $f : X \to X$ satisfying $d(f(x), f(y)) < d(x, y)$ for any $x \neq y$. Prove that f has a unique fixed point.

   (b) Consider $f : X \to X$ satisfying $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that f is surjective. Does it still remain true if X is non-compact?

Extra page for answers. Please specify the question number here and the use of this page on the question page.

2. Let $\lambda > 0$. A function $f : [0,1] \to \mathbb{R}$ is called Hölder continuous with exponent λ (or λ-Hölder) if

$$C_\lambda(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\lambda}$$

is finite.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of λ-Hölder functions $f_n : [0,1] \to \mathbb{R}$ such that

$$|f_n(x)| \leq 1 \qquad \forall x \in [0,1],\ \forall n \in \mathbb{N} \qquad \text{and} \qquad C_\lambda(f_n) \leq 1 \qquad \forall n \in \mathbb{N}.$$

Prove that $(f_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

3. Let $C^1[0,1]$ be the space of real valued functions $f \in C[0,1]$ whose derivative f' exists on $(0,1)$ and extends to a continuous function $f' \in C[0,1]$. We equip $C^1[0,1]$ with the norm

$$||f|| = ||f||_\infty + ||f'||_\infty$$

where $||g||_\infty = \sup_{x \in [0,1]} |g(x)|$. Show that $C^1[0,1]$ with the above norm is a complete metric space.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

4. Let $f \in C[0,1]$ have the property that

$$\int_0^1 f(x)\Big(\log\left(1+\sin\left(\frac{\pi x}{2}\right)\right)\Big)^n dx = 0 \qquad n = 0, 1, 2, \ldots$$

Is it true that $f = 0$? If yes, give a proof. Otherwise, find a counterexample.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

Extra page for answers. Please specify the question number here and the use of this page on the question page.