ON THE *abc* CONJECTURE, II

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Abstract

Let x, y, and z be coprime positive integers with $x + y = z$. In this paper we give upper bounds for z in terms of the greatest square-free factor of xyz.

1. Introduction

Let x, y, and z be positive integers, and define $G = G(x, y, z)$ by

$$G = G(x, y, z) = \prod_{\substack{p \mid xyz \\ p \text{ a prime}}} p.$$

Thus G is the greatest square-free factor of xyz. In 1985, D. Masser [6] proposed a refinement of a conjecture that had been recently formulated by J. Oesterlé. Masser conjectured that for each positive real number ε there is a positive number $c(\varepsilon)$, which depends on ε only, such that, for all positive integers x, y, and z with

$$x + y = z \qquad \text{and} \qquad (x, y, z) = 1, \tag{1}$$

we have

$$z < c(\varepsilon) G^{1+\varepsilon}. \tag{2}$$

The conjecture is now known as the *abc* conjecture. It captures in a succinct way the idea that the additive and the multiplicative structure of the integers should be independent, and, accordingly, it has profound consequences (cf. [1], [3], [4], [5], [11], [13]).

In 1986, C. Stewart and R. Tijdeman [11] obtained an upper bound for z as a function of G. They proved that there exists an effectively computable positive constant c_1 such that, for all positive integers x, y, and z satisfying (1),

$$z < \exp\left(c_1 G^{15}\right). \tag{3}$$

The proof depends on a p-adic estimate for linear forms in the logarithms of algebraic

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numbers due to A. van der Poorten [8]. In 1991 Stewart and K. Yu [12] strengthened (3). They proved, by combining a p-adic estimate for linear forms in the logarithms of algebraic numbers due to Yu [15] with an earlier Archimedean estimate due to M. Waldschmidt [14], that there exists an effectively computable positive constant c_2 such that, for all positive integers x, y, and z, with $z > 2$, satisfying (1),

$$z < \exp\left(G^{2/3+c_2/\log\log G}\right). \tag{4}$$

Our purpose in this paper is to present two further improvements on (4).

THEOREM 1
There exists an effectively computable positive number c such that, for all positive integers x, y, and z with $x + y = z$ and $(x, y, z) = 1$,

$$z < \exp\left(c\, G^{1/3}(\log G)^3\right). \tag{5}$$

The key new ingredient in our proof of Theorem 1 is an estimate of Yu [17] for p-adic linear forms in the logarithms of algebraic numbers which has a better dependence on the number of terms in the linear form than previous p-adic estimates; for the Archimedean case, see the earlier work of E. Matveev [7]. We employ this estimate in order to control the p-adic order of x, y, and z at the small primes p dividing x, y, and z. Next we combine the contributions from the small primes in order to reduce the number of terms in our linear forms. We conclude with a further application of estimates for linear forms in the logarithms of algebraic numbers in a fashion similar to [12]. Here we appeal to a p-adic estimate due to Yu [16] and its earlier Archimedean counterpart due to A. Baker and G. Wüstholz [2].

An examination of our proof reveals that the impediment to a further refinement of Theorem 1 is not the dependence on the number of terms in the estimates for linear forms in logarithms but instead is the dependence on the parameter p in the p-adic estimates. This fact is highlighted by our next result, which shows that if the greatest prime factor of one of x, y, and z is small relative to G, then the estimate for z from Theorem 1 can be improved. In particular, let p_x, p_y, and p_z denote the greatest prime factors of x, y, and z, respectively, with the convention that the greatest prime factor of 1 is 1. Put

$$p' = \min\left\{p_x, p_y, p_z\right\}.$$

Denote the ith iterate of the logarithmic function by $\log_i$, so that $\log_1 t = \log t$ and $\log_i t = \log(\log_{i-1} t)$ for $i = 2, 3, \ldots$.

THEOREM 2
There exists an effectively computable positive number c such that, for all positive integers x, y, and z with $x + y = z$, $(x, y, z) = 1$, and $z > 2$,

$$z < \exp\left(p' G^{c \log_3 G^* / \log_2 G}\right), \tag{6}$$

where $G^* = \max(G, 16)$.

Thus, for each $\varepsilon > 0$ there exists a number $c_3(\varepsilon)$, which is effectively computable in terms of ε, such that for all positive integers x, y, and z with $x + y = z$ and $(x, y, z) = 1$,

$$z < \exp\left(c_3(\varepsilon) p' G^{\varepsilon}\right).$$

Observe that

$$p' \le \left(p_x p_y p_z\right)^{1/3} \le G^{1/3},$$

and so we immediately obtain

$$z < \exp\left(c_3(\varepsilon) G^{1/3+\varepsilon}\right),$$

a slightly weaker version of Theorem 1. On the other hand, if p' is appreciably smaller than $G^{1/3}$, (6) gives a sharper upper bound than (5).

For any integer n with $n > 1$, let $P(n)$ denote the greatest prime factor of n. As an illustration of the above remark, we deduce from Theorem 2 that there exists an effectively computable positive number c_4 such that if x and y are coprime positive integers with $x < y$ and $y \ge 16$, then

$$P = P(xy(x+y)) > c_4 \frac{\log_2 y \log_3 y}{\log_4^* y}, \tag{7}$$

where $\log_4^* y = \max\{\log_4 y, 1\}$, a result that improves upon the lower bound of $c_5 \log_2 y$ obtained by T. Shorey, van der Poorten, Tijdeman, and A. Schinzel [10]. Suppose $y \ge 16$, and suppose (1) holds. Then by (6) there exists an effectively computable positive number c_6 such that

$$\log\log y < \log P + c_6 \frac{\log G \log_3 G^*}{\log_2 G}, \tag{8}$$

where G denotes the greatest square-free factor of $xy(x+y)$. Plainly,

$$G = \prod_{p \mid xy(x+y)} p \le \exp\left(\sum_{p \le P} \log p\right),$$

and so, by the prime number theorem, we have

$$\log G < c_7 P, \tag{9}$$

where c_7 is an effectively computable positive number (cf. J. Rosser and L. Schoenfeld [9, Theorem 9]). Estimate (7) follows directly from (8) and (9).

2. Preliminary lemmas

For any algebraic number α, let $h_0(\alpha)$ denote its absolute logarithmic Weil height, so that

$$h_0(\alpha) = d^{-1}\left(\log|a_d| + \sum_{j=1}^{d}\log\max\left(1, \left|\alpha^{(j)}\right|\right)\right),$$

where the minimal polynomial for α over $\mathbb{Z}$ is

$$a_d x^d + \cdots + a_1 x + a_0 = a_d\left(x-\alpha^{(1)}\right)\cdots\left(x-\alpha^{(d)}\right).$$

Let p be a prime number, and put

$$q = \begin{cases} 2 & \text{if } p > 2, \\ 3 & \text{if } p = 2 \end{cases} \qquad \text{and} \qquad \alpha_0 = \begin{cases} -1 & \text{if } p \equiv 3 \pmod 4, \\ i = \sqrt{-1} & \text{if } p \equiv 1 \pmod 4, \\ e^{2\pi i/3} & \text{if } p = 2. \end{cases}$$

Put $K = \mathbb{Q}(\alpha_0)$. Let $\mathfrak{p}$ be a prime ideal of $\mathscr{O}_K$, the ring of algebraic integers in K. Suppose that $\mathfrak{p}$ lies above p with ramification index $e_\mathfrak{p}$ and residue class degree $f_\mathfrak{p}$. Note that

$$e_\mathfrak{p} = 1 \qquad \text{and} \qquad f_\mathfrak{p} = \begin{cases} 1 & \text{if } p > 2, \\ 2 & \text{if } p = 2 \end{cases} \tag{10}$$

(see [15, appendix] for the case $p \equiv 1 \bmod 4$ and the case $p = 2$). For nonzero α in K, we write $\mathrm{ord}_\mathfrak{p}\alpha$ for the exponent to which $\mathfrak{p}$ divides the fractional ideal generated by α in K.

Let $\alpha_1, \ldots, \alpha_n$ be nonzero elements of K, and put

$$h_j = \max\left(h_0(\alpha_j), \log p\right),$$

for $j = 1, \ldots, n$. Let $b_1, \ldots, b_n$ be rational integers with absolute values at most $B\ (\geq 3)$. Next put

$$\Theta = \alpha_1^{b_1}\cdots\alpha_n^{b_n} - 1.$$

LEMMA 1

Suppose that $\left[K\left(\alpha_0^{1/q}, \ldots, \alpha_n^{1/q}\right) : K\right] = q^{n+1}$, $\mathrm{ord}_\mathfrak{p}\alpha_j = 0$ *for* $j = 1, \ldots, n$, *and* $\Theta \neq 0$. *Then there exists an effectively computable positive number* c_8 *such that*

$$\mathrm{ord}_\mathfrak{p}\Theta < \frac{p}{(\log p)^2} c_8^n\, h_1 \cdots h_n \log B.$$

Proof

This follows from [17, Theorem 1] on applying (10). □

LEMMA 2

Suppose that $\mathrm{ord}_{\mathfrak{p}}\alpha_j = 0$ *for* $j = 1, \ldots, n$, *and suppose that* $\Theta \neq 0$. *Then there exists an effectively computable positive number* c_9 *such that*

$$\mathrm{ord}_{\mathfrak{p}}\Theta < \frac{p}{\log p}(c_9 n)^{2n}\left(\frac{h_1}{\log p}\right)\cdots\left(\frac{h_n}{\log p}\right)\log B.$$

Proof

This follows from [16, Theorem 1] on appealing to (10). □

We also require an Archimedean estimate for linear forms in the logarithms of algebraic numbers.

LEMMA 3

Suppose that $\alpha_1, \ldots, \alpha_n$ *are positive rational numbers, and put*

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n,$$

where log *denotes the principal branch of the logarithm. If* $\Lambda \neq 0$, *then there exists an effectively computable positive number* c_{10} *such that*

$$|\Lambda| > \exp\left(-(c_{10}n)^{2n}\log B \prod_{j=1}^{n} \max\left(h_0(\alpha_j), 1\right)\right).$$

Proof

This is a consequence of [2, Theorem]. □

LEMMA 4

Let $\alpha_1, \ldots, \alpha_n$ *be prime numbers with* $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. *Let* $q = 2$ *and* $\alpha_0 \in \{-1, i\}$ *or* $q = 3$ *and* $\alpha_0 = e^{2\pi i/3}$, *and put* $K = \mathbb{Q}(\alpha_0)$. *Then*

$$\left[K\left(\alpha_0^{1/q}, \alpha_1^{1/q}, \ldots, \alpha_n^{1/q}\right) : K\right] = q^{n+1}$$

except when $q = 2$, $\alpha_0 = i$, *and* $\alpha_1 = 2$, *and in this case*

$$\left[K\left(\alpha_0^{1/2}, (1+i)^{1/2}, \alpha_2^{1/2}, \ldots, \alpha_n^{1/2}\right) : K\right] = 2^{n+1}.$$

Proof

This follows from [12, Lemma 3] except when $q = 2$ and $\alpha_0 = -1$. In this case, the proof of [12, Lemma 3] again applies. □

LEMMA 5

Let $2 = p_1, p_2, \ldots$ be the sequence of prime numbers in increasing order. There is an effectively computable positive constant c_{11} such that, for every positive integer r, we have

$$\prod_{j=1}^{r} \frac{p_j}{\log p_j} > \left(\frac{r+3}{c_{11}}\right)^{r+3}.$$

Proof

This is [12, Lemma 4]. □

3. Proofs of Theorems 1 and 2

Note that Theorem 1 holds for $z = 2$ trivially. Henceforth, let x, y, and z be positive integers with $x + y = z$, $(x, y, z) = 1$, and $z > 2$. We may suppose, without loss of generality, that $x \leq y$. Since $z > 2$, we see that $x < y < z$ and $G \geq 6$. Note that

$$\max\{\mathrm{ord}_p x, \mathrm{ord}_p y, \mathrm{ord}_p z\} \leq \frac{\log z}{\log 2}. \tag{11}$$

Put

$$\tilde{G} = \max\left(\frac{G}{p_x p_y p_z}, 16\right)$$

and

$$r = \omega(xyz),$$

the number of distinct prime factors of xyz.

Let $c_{12}, c_{13}, \ldots$ denote effectively computable positive constants. By Lemma 5,

$$\tilde{G} > \left(\frac{r}{c_{12}}\right)^{r},$$

and so

$$r < c_{13}\frac{\log \tilde{G}}{\log_2 \tilde{G}}. \tag{12}$$

Put $m = r - 2$ if $x = 1$ (whence $p_x = 1$) and $m = r - 3$ otherwise. Notice that, by the arithmetic-geometric mean inequality,

$$\prod_{\substack{p \mid xyz \\ p \notin \{p_x, p_y, p_z\}}} \log p \leq \left(\frac{1}{m} \sum_{\substack{p \mid xyz \\ p \notin \{p_x, p_y, p_z\}}} \log p\right)^{m} \leq \left(\frac{\log \tilde{G}}{m}\right)^{m}, \tag{13}$$

provided that m is positive. From (12) and (13), we deduce that

$$\prod_{\substack{p \mid xyz \\ p \notin \{p_x, p_y, p_z\}}} \log p < \exp\left(c_{14}\frac{\log \tilde{G} \log_3 \tilde{G}}{\log_2 \tilde{G}}\right), \tag{14}$$

with the usual convention that the empty product is 1. It also follows from (12) that

$$(\log r)^{2r} < \exp\left(c_{15}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right). \tag{15}$$

We now estimate $\operatorname{ord}_p(xyz)$ for each prime p that divides xyz and satisfies

$$p < e^{(\log r)^2}. \tag{16}$$

First suppose that $p \mid z$. Since $(x, y, z) = 1$ and $x + y = z$, we have $(x, y) = (x, z) = (y, z) = 1$. Thus, for each prime p that divides z,

$$\operatorname{ord}_p z = \operatorname{ord}_p\left(\frac{z}{-y}\right) = \operatorname{ord}_p\left(\frac{x}{-y} - 1\right) \le \operatorname{ord}_p\left(\left(\frac{x}{y}\right)^4 - 1\right). \tag{17}$$

Let $\alpha_1 < \cdots < \alpha_n$ be the primes that divide either x or y except in the case when $p \equiv 1 \bmod 4$ and $\alpha_1 = 2$. In that case, we take $\alpha_1 = 1 + i$ in place of $\alpha_1 = 2$. Note that $2^4 = (1+i)^8$. Write

$$\left(\frac{x}{y}\right)^4 = \alpha_1^{b_1}\cdots\alpha_n^{b_n},$$

with $b_1, \ldots, b_n$ rational integers. We choose $q, \alpha_0, K = \mathbb{Q}(\alpha_0)$, as in §2. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ lying above p. Since $p \mid z$ and $(x, z) = (y, z) = 1$, we have

$$\operatorname{ord}_{\mathfrak{p}}\alpha_j = 0,$$

for $j = 1, \ldots, n$. Let B denote the maximum of the absolute values of the b_j's. Then, by (11),

$$\log B \le \log\left(8\frac{\log z}{\log 2}\right). \tag{18}$$

Put

$$\Theta = \left(\frac{x}{y}\right)^4 - 1 = \alpha_1^{b_1}\cdots\alpha_n^{b_n} - 1,$$

and note that

$$\operatorname{ord}_p\Theta = \operatorname{ord}_{\mathfrak{p}}\Theta. \tag{19}$$

By (16), (18), and Lemmas 1 and 4,

$$\operatorname{ord}_{\mathfrak{p}}\Theta < pc_{16}^n(\log r)^{2n}\log\log z \prod_{\substack{l \mid xy \\ l \text{ a prime}}} \log l. \tag{20}$$

It follows from (12), (14)–(17), (19), and (20) that

$$\operatorname{ord}_p z < \exp\left(c_{17}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right)\log(2p_x)\log p_y\log\log z. \tag{21}$$

In a similar fashion, we deduce, for p satisfying (16), that

$$\operatorname{ord}_p y < \exp\left(c_{18}\frac{\log \tilde{G}\log_3 \tilde{G}}{\log_2 \tilde{G}}\right)\log(2p_x)\log p_z \log\log z \tag{22}$$

and that

$$\operatorname{ord}_p x < \exp\left(c_{19}\frac{\log \tilde{G}\log_3 \tilde{G}}{\log_2 \tilde{G}}\right)\log p_y \log p_z \log\log z. \tag{23}$$

We now define R, S, and T by

$$R = \prod_{\substack{l|x, l\neq p_x\\ l<e^{(\log r)^2}}} l^{\operatorname{ord}_l x}, \qquad S = \prod_{\substack{l|y, l\neq p_y\\ l<e^{(\log r)^2}}} l^{\operatorname{ord}_l y}, \qquad T = \prod_{\substack{l|z, l\neq p_z\\ l<e^{(\log r)^2}}} l^{\operatorname{ord}_l z},$$

where l runs through primes. Observe that

$$h_0\left(\frac{R}{-S}\right) < r(\log r)^2 \max\left(\max_{\substack{l|x\\ l<e^{(\log r)^2}}} \operatorname{ord}_l x,\ \max_{\substack{l|y\\ l<e^{(\log r)^2}}} \operatorname{ord}_l y\right);$$

hence, by (12), (22), and (23),

$$h_0\left(\frac{R}{-S}\right) < \exp\left(c_{20}\frac{\log \tilde{G}\log_3 \tilde{G}}{\log_2 \tilde{G}}\right)\log\max(p_x, p_y)\log p_z \log\log z. \tag{24}$$

Similarly, we find that

$$h_0\left(\frac{T}{R}\right) < \exp\left(c_{21}\frac{\log \tilde{G}\log_3 \tilde{G}}{\log_2 \tilde{G}}\right)\log\max(p_x, p_z)\log p_y \log\log z \tag{25}$$

and that

$$h_0\left(\frac{T}{S}\right) < \exp\left(c_{22}\frac{\log \tilde{G}\log_3 \tilde{G}}{\log_2 \tilde{G}}\right)\log\max(p_y, p_z)\log(2p_x) \log\log z. \tag{26}$$

We are now in a position to estimate $\operatorname{ord}_p(xyz)$ for each prime p that divides xyz. In particular, we no longer require condition (16). We first estimate $\operatorname{ord}_p z$ for $p \mid z$. As in (17), we have

$$\operatorname{ord}_p z = \operatorname{ord}_p\left(\frac{x}{-y} - 1\right). \tag{27}$$

Put $\alpha_1 = R/(-S)$, and let

$$\frac{x}{-y} = \alpha_1 \alpha_2^{b_2} \cdots \alpha_n^{b_n},$$

where $\alpha_2, \ldots, \alpha_n$ are distinct prime numbers and where $b_2, \ldots, b_n$ are nonzero rational integers. Since

$$\alpha_2 \cdots \alpha_n \mid G$$

and

$$\alpha_j \geq e^{(\log r)^2},$$

for $j = 2, \ldots, n$ with $\alpha_j \notin \{p_x, p_y\}$, we deduce that

$$n-3 \le \frac{\log \tilde{G}}{(\log r)^2}. \tag{28}$$

Next observe that

$$n^{2n} < \exp\left(c_{23}\frac{\log \tilde{G}}{\log_2 \tilde{G}}\right) \tag{29}$$

since if r is at most $(\log \tilde{G})^{1/2}$, then the result is immediate on noting that n is at most r, while if r exceeds $(\log \tilde{G})^{1/2}$, then (29) follows from (28).

Let $B = \max(|b_2|, \ldots, |b_n|, 3)$, and note that (18) follows from (11) as before. Put

$$\Theta = \frac{x}{-y} - 1 = \alpha_1\alpha_2^{b_2}\cdots\alpha_n^{b_n} - 1,$$

and observe that (19) holds. Next put

$$W_p = \exp\left(c_{24}\frac{\log \tilde{G}\log_3 \tilde{G}}{\log_2 \tilde{G}}\right)\frac{p}{(\log p)^3}\prod_{l\in\{p_x,p_y,p_z\}}\log\max(l,\, p)\cdot(\log\log z)^2. \tag{30}$$

We now apply Lemma 2, taking into account (12), (14), (24), (27), and (29), to conclude that

$$(\mathrm{ord}_p z)\log p < W_p \log\max(p_x,\, p_y). \tag{31}$$

Similarly, if $p \mid y$, then, by considering $\mathrm{ord}_p\big(z/x - 1\big)$ and applying Lemma 2, we find that

$$(\mathrm{ord}_p y)\log p < W_p \log\max(p_x,\, p_z); \tag{32}$$

while if $p \mid x$, then, by considering $\mathrm{ord}_p\big(z/y - 1\big)$ and applying Lemma 2, we obtain

$$(\mathrm{ord}_p x)\log p < W_p \log\max(p_y,\, p_z). \tag{33}$$

Certainly,

$$\log z = \sum_{p\mid z}(\mathrm{ord}_p z)\log p \le r\left(\max_{p\mid z}(\mathrm{ord}_p z)\log p\right). \tag{34}$$

Put

$$L = \log\max(p_x,\, p_y)\cdot\log\max(p_x,\, p_z)\cdot\log\max(p_y,\, p_z).$$

By (12), (30), (31), and (34), we find that

$$\frac{\log z}{(\log\log z)^2} < \exp\left(c_{25}\frac{\log \tilde{G}\log_3 \tilde{G}}{\log_2 \tilde{G}}\right)\frac{p_z}{(\log p_z)^2}L. \tag{35}$$

Since $y > z/2$ and $z \ge 3$,

$$\log y > \log z - \log 2 > \frac{1}{4}\log z. \tag{36}$$

Plainly, (34) holds with z replaced by y, and so from (12), (30), (32), and (36), we deduce that

$$\frac{\log z}{(\log\log z)^2} < \exp\left(c_{26}\frac{\log \tilde{G}\log_3 \tilde{G}}{\log_2 \tilde{G}}\right)\frac{p_y}{(\log p_y)^2}L. \tag{37}$$

Next, either $x \geq y^{1/2}$, in which case

$$\log x \geq \frac{1}{2}\log y > \frac{1}{8}\log z, \tag{38}$$

or $x < y^{1/2}$, in which case

$$\log\left(\frac{x+y}{y}\right) = \log\left(1+\frac{x}{y}\right) < \log\left(1+\frac{1}{y^{1/2}}\right) < \frac{1}{y^{1/2}} < \left(\frac{2}{z}\right)^{1/2}. \tag{39}$$

In the former case, we may appeal to (34) with z replaced by x, and so from (12), (30), (33), and (38),

$$\frac{\log z}{(\log\log z)^2} < \exp\left(c_{27}\frac{\log \tilde{G}\log_3 \tilde{G}}{\log_2 \tilde{G}}\right)\frac{p_x}{(\log(2p_x))^2}L. \tag{40}$$

In the latter case, put $\alpha_1 = T/S$, and write

$$\frac{z}{y} = \alpha_1\alpha_2^{b_2}\cdots\alpha_n^{b_n},$$

where $\alpha_2, \ldots, \alpha_n$ are distinct prime numbers and where $b_2, \ldots, b_n$ are nonzero rational integers. Then

$$0 < \log\left(\frac{x+y}{y}\right) = \log\left(\frac{z}{y}\right) = \log\alpha_1 + b_2\log\alpha_2 + \cdots + b_n\log\alpha_n.$$

Note that we again have (29). Thus, on applying Lemma 3 and appealing to (11), (12), (14), (26), and (29), we obtain

$$\begin{aligned}\log\log\left(\frac{x+y}{y}\right) > -\exp\left(c_{28}\frac{\log \tilde{G}\log_3 \tilde{G}}{\log_2 \tilde{G}}\right)\log\max(p_y, p_z)\\ \cdot\log(2p_x)\log p_y\log p_z(\log\log z)^2.\end{aligned} \tag{41}$$

On comparing (39) and (41), we see that

$$\frac{\log z}{(\log\log z)^2} < \exp\left(c_{29}\frac{\log \tilde{G}\log_3 \tilde{G}}{\log_2 \tilde{G}}\right)\log\max(p_y, p_z)\log(2p_x)\log p_y\log p_z.$$

Therefore, in both cases $x \geq y^{1/2}$ and $x < y^{1/2}$, (40) holds.

Suppose that $\{p_x, p_y, p_z\} = \{p', p'', p'''\}$, and suppose that

$$p' < p'' < p'''.$$

It follows from (35), (37), and (40) that

$$\frac{\log z}{(\log\log z)^2} < \exp\left(c_{30}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right)\frac{p'}{(\log(2p'))^2}\log p''(\log p''')^2. \tag{42}$$

Just as for (14), we have

$$\prod_{p|xyz}\log p < \exp\left(c_{31}\frac{\log G\log_3 G^*}{\log_2 G}\right),$$

whence, by (42),

$$\frac{\log z}{(\log\log z)^2} < \exp\left(c_{32}\frac{\log G\log_3 G^*}{\log_2 G}\right)\frac{p'}{(\log(2p'))^2}. \tag{43}$$

Theorem 2 follows directly from (43). □

To prove Theorem 1, we remark that from (35), (37), and (40),

$$\left(\frac{\log z}{(\log\log z)^2}\right)^3 < \exp\left(c_{33}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right)\frac{p_x p_y p_z}{(\log(2p_x)\log p_y\log p_z)^2}\times(\log p'')^3(\log p''')^6.$$

Note that we may assume that

$$p' > G^{1/4}$$

since otherwise Theorem 1 follows from (43). Thus we have

$$\left(\frac{\log z}{(\log\log z)^2}\right)^3 < c_{34}\tilde{G}p_x p_y p_z(\log G)^3,$$

and so

$$\log z < c_{35}G^{1/3}(\log G)^3,$$

as required. □

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