On the Schur index of Graded Representations

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A more accurate but less informative title might have been: "On a paper of Turull [**T**]".

Below is some theory for studying representations with extra structure (grading and Clifford algebra action) of groups with extra structure (sign homomorphism and central involution) over general fields of characteristic zero. We produce an invariant in the Brauer–Wall group which determines a Schur index. That invariant, a case of which is closely related to an invariant defined by Turull, is calculated here in a quite straightforward way for the basic irreducible representations of the essential covers of the symmetric groups. This method suggests a closed formula for the Brauer–Wall invariant of the general irreducible for these covers, which we give at the end of the paper.

Our motivation is twofold. In joint work with John Q. Huang, we are attempting to understand the structure, over fields other than $\mathbf{C}$ and $\mathbf{R}$, of the induction algebras whose elements are virtual representations of essential covers of the symmetric groups. To gauge the chances of success, it seemed a good idea to recast, into our language, some ideas from Turull's Annals of Math paper [**T**]. Furthermore, the theory below is now in a form so as to be readily applicable to other examples, such as the three families of essential covers of the hyperoctahedral groups whose representations can be described using PSH–algebras [**B–H**] [**H–H2**]. Secondly, it is likely that a generalization of the work below to the case of gradings over the product of more than one copy of $\mathbf{Z}/\mathbf{2}$ will eventually have applications, for example to other covers of monomial groups. So the case of one copy is practice for the general case. Such a development would involve using the Brauer–Wall functor constructed from algebras graded by such a product of copies of $\mathbf{Z}/\mathbf{2}$. Perhaps there is a third motive: to advertise the language used in [**H1**] (i.e. the categories $\mathcal{Z}_n$) for working with projective representations of families of classical groups.

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In Sections 1, 2, and 3, we give general properties of the Brauer–Wall invariant for objects in the categories $\mathcal{Z}_n(G)$ defined in [**H1**]. This is inspired by Turull's paper. Some of it can be interpreted as generalizations of parts of his work. Many proofs here are rather different. If one doesn't regard the theory in [**H1**; Sect.1,2,3] as part of these proofs, some turn out to be shorter. In Section 4 is the motivating example: we use the periodicity theorem, $\mathcal{Z}_n(G) \simeq \mathcal{Z}_{n+2}(G)$, from the first section of [**H1**], to give transparent definitions over general fields of characteristic zero for the smallest realizable sums of the basic complex projective representations of the symmetric groups, or at least for their alter egos in $\mathcal{Z}_{n\pm1}(\tilde{S}_n)$. When $\sqrt{2} \in F$, this makes almost trivial the calculation (as graded quaternion algebras) of their corresponding endo–algebras. When $\sqrt{2} \notin F$, a bit more work is necessary. To treat arbitrary irreducibles, the products used by Turull on representations and in the Brauer–Wall group (both denoted $\vee$) are replaced by the natural tensor product from [**H1**] for the categories $\mathcal{Z}_n$ and the usual operation in the Brauer–Wall group, respectively.

Note that the paper is written so as to be independent of the literature on the Schur index, making the first two sections longer than strictly necessary. Experts will need only to skim these two sections. Dependence on the theory of central graded algebras involves only its most basic aspects, covered very nicely by Lam [**L**; Ch.4, 5]. We have included a few necessary facts concerning *division graded algebras* in and after 1.11. These appear in detail in [**C–H**].

I'm very grateful to the referee, who saved me from a couple of bad errors, as well as making a lot of excellent suggestions. The latter included several simplifications to formulae at the end of the paper, using a superior knowledge of the Brauer group.

1. Endo–Algebras from $\mathcal{Z}_n(G)$

Fix a field F of characteristic 0. For $n \geq 0$, denote as $\mathcal{Z}_n(G)$ the category (redefined below) from [**H1**], with $\mathbf{C}$ replaced by F. In the second section, we shall use the notation $\mathcal{Z}_n^F(G)$ usually, since several fields enter the discussion. Objects of the form (G, z, σ), consisting of a finite group G, a central element z of order 2, and a homomorphism $\sigma : G \to \mathbf{Z/2}$ sending z to 0, will be denoted simply G. These objects occur in Sergeev [**Se**]. In [**H–H1**] the representations of the $\hat{\curlyvee}$–product (see Section 3 below) of two such objects is determined; the treatment in [**H1**] is more straightforward. These 'decorated groups' also occur in Stembridge [**St**], and the $\hat{\curlyvee}$ –product in Turull [**T**] (where it also is denoted $\vee$). For any such G, the objects of $\mathcal{Z}_n(G)$ are tuples $\mathcal{V} = (V, V', \eta_1, \cdots, \eta_n)$ such that (V, V') is a $\mathbf{Z/2}$–graded representation of G with z acting as -1, and the η_i are anti–commuting, gradation–reversing G–linear involutions of (V, V'). Morphisms in $\mathcal{Z}_n(G)$ are gradation–preserving G–maps which commute with the η_i. The *associate* of the object $\mathcal{V}$, denoted $\rho\mathcal{V}$, is obtained by interchanging V and V', and multiplying each η by -1.

It will be convenient also to consider the categories $\mathcal{Z}_{-n}(G)$ for $n > 0$. The only change from the above definition is to require each η_i^2 to be -1 rather than $+1$.

Definitions. Below we review definitions from pp. 76–77 of Lam [**L**]:
Graded algebra (abbreviated 'GA');
Graded centre, $\hat{Z}$, of a graded algebra;
CGA, central graded algebra;
SGA, simple graded algebra;
CSGA, central simple graded algebra.
We shall always think of a graded algebra as an ordered pair $A = (A_0,\ A_1)$ of F–vector spaces (plus multiplication maps sending $A_i \times A_j$ to A_{i+j}); that is, we never bother with inhomogeneous elements. Thus $\hat{Z}_0(A)$ consists of those elements in A_0 which commute with everything in $A_0 \cup A_1$, whereas $\hat{Z}_1(A)$ consists of those elements in A_1 which commute with everything in A_0 but anti–commute with everything in A_1. A CGA is a GA with $\hat{Z}_1 = 0$ and $\hat{Z}_0 = F$. An SGA is one for which there is no pair $(I_0,\ I_1) \subset (A_0,\ A_1)$ for which $I_0 \oplus I_1$ is a proper non–zero 2–sided ideal in the ungraded algebra $A_0 \oplus A_1$. A CSGA has both properties.

Definition. For graded algebras $A = (A_0, A_1)$ and $B = (B_0, B_1)$, define

$$A \times B \ = \ (A_0 \times B_0\ ,\ A_1 \times B_1)$$

with coordinatewise operations.

Proposition 1.1. $\hat{Z}(A \times B\ ) \ = \ \hat{Z}(A) \times \hat{Z}(B)$.

Corollary 1.2. *$A \times B$ is never a* CGA.

Definition. For $\mathcal{W}$ in $\mathcal{Z}_n(G)$, let $End^*\mathcal{W}$ be the GA over F whose ith component is $Hom(\mathcal{W},\ \rho^i\mathcal{W})$, the set of maps in $\mathcal{Z}_n(G)$ from $\mathcal{W}$ to $\rho^i\mathcal{W}$, specializing the definition in [**H1**; Sect. 5] to the case $H = \{1,\ z\}$, and ignoring the action of the trivial object $\{1, z\}$. The algebra multiplication is composition, using the facts that $\rho^2 = \mathrm{id}$ and that

$$Hom(\rho\mathcal{V},\ \rho\mathcal{W}) \ \cong\ Hom(\mathcal{V},\ \mathcal{W})$$

by a canonical isomorphism (essentially equality).

Proposition 1.3. *For all $\mathcal{W}$, the graded algebra $End^*\mathcal{W}$ is well defined, isomorphic to $End^*(\rho\mathcal{W})$, and invariant under isomorphism (in fact, $Hom(\text{—}, \text{—})$ is a functor).*

Proposition 1.4. a) *Given $\mathcal{W}_1$ and $\mathcal{W}_2$, assume that, for all irreducibles $\mathcal{V}_i \subset \mathcal{W}_i$, we have $\mathcal{V}_1 \ncong \mathcal{V}_2 \ncong \rho\mathcal{V}_1$. Then*

$$End^*(\mathcal{W}_1 \oplus \mathcal{W}_2) \ \cong\ End^*(\mathcal{W}_1) \ \times\ End^*(\mathcal{W}_2)$$

as graded algebras.

b) *On the other hand, assuming only $\mathcal{V}_1 \ncong \mathcal{V}_2$, it follows that*

$$End^0(\mathcal{W}_1 \oplus \mathcal{W}_2) \ \cong\ End^0(\mathcal{W}_1) \ \times\ End^0(\mathcal{W}_2)$$

as 'just plain algebras'.

The proof is the same as in the classical (ungraded) case.

Proposition 1.5. *If $\mathcal{V}$ is irreducible, then $End^0\mathcal{V}$ is a division algebra over F, and $End^*\mathcal{V}$ is a division graded algebra over F.*

A *division graded algebra* is a GA, $(D_0,\ D_1)$, in which each non–zero element of $D_0 \cup D_1$ is invertible. Note that the ungraded algebra $D_0 \oplus D_1$ might not be a division algebra. See 1.11 ahead.

The proof of 1.5 is the same as the usual for the main part of Schur's lemma.

Note. In a division algebra (since the characteristic is not 2), $x^2 = -x^2$ clearly implies that $x = 0$. Thus $\hat{Z}_1(End^*\mathcal{V}) = 0$ for an irreducible $\mathcal{V}$; that is, $\hat{Z}(End^*\mathcal{V}) = (\hat{Z}_0(End^*\mathcal{V})\ ,\ 0)$ There are field extensions

$$F \subset \hat{Z}_0(End^*\mathcal{V}) \subset Z(End^0\mathcal{V})\ ,$$

both inclusions being strict in general.

Definitions. We'll say that an irreducible $\mathcal{V}$ has *ample scalars* if and only if there is no proper field extension K of F and object $\mathcal{U}$ in $\mathcal{Z}_n^K(G)$ such that $\mathcal{V}$ is simply $\mathcal{U}$ with scalar multiplication by elements of $K \setminus F$ ignored (i.e. such that $\mathcal{U}_F = \mathcal{V}$), and such that any automorphism of $\mathcal{V}$ is also an automorphism of $\mathcal{U}$. This is easily seen to be equivalent to requiring that $Z(End^0\mathcal{V}) = F$ as F–algebras.

We'll say that an irreducible $\mathcal{V}$ has *adequate scalars* if and only if $\hat{Z}_0(End^*\mathcal{V})$ coincides with F (as F–algebras), i.e. by the note above, if and only if $End^*\mathcal{V}$ is a CGA. This is readily seen to be equivalent to the non–existence of any proper field extension K of F and object $\mathcal{U} \in \mathcal{Z}_n^K(G)$ such that both $\mathcal{U}_F = \mathcal{V}$ and any isomorphism $\mathcal{V} \cong \rho^j\mathcal{V}$ is also an isomorphism $\mathcal{U} \cong \rho^j\mathcal{U}$. (See also 2.3 below.)

Clearly ample implies adequate. The converse doesn't hold since there are examples with $F = \mathbf{R}$ where End^* is isomorphic to that grading of the quaternions which has $\mathbf{C}$ in degree 0 and the span of $\{j, k\}$ in degree 1. This graded algebra would be denoted $< \frac{-1,-1}{\mathbf{R}} >$ in Lam's notation for generalized graded quaternion algebras. There is also a grading of the full matrix algebra $\mathbf{R}^{2\times 2}$ which yields a division graded algebra with $\mathbf{C}$ in degree zero.

Corollary 1.6. If $\mathcal{V}$ *is irreducible with ample scalars, then* $End^0\mathcal{V}$ *is a central division algebra over* F. If $\mathcal{V}$ *is irreducible with adequate scalars, then* $End^*\mathcal{V}$ *is a central (division) graded algebra over* F, *hence a* CSGA.

Proposition 1.7. a) *If* $\mathcal{W}$ *has an irreducible summand without adequate scalars, then* $End^*\mathcal{W}$ *is not a* CGA.

b) *If* $\mathcal{W}$ *has an irreducible summand without ample scalars, then* $End^0\mathcal{W}$ *is not a central algebra.*

In a), this is because one can give a central element outside F by multiplying in all summands isomorphic to that irreducible $\mathcal{V}$ or its associate by an element of $\hat{Z}_0(End^*\mathcal{V}) \setminus F$, and multiplying by zero on other summands (if they exist, in which case the result follows alternatively from 1.4 and 1.2). See also 1.16a) below. Part b) is proved in a similar way.

Proposition 1.8. *If* $\mathcal{V}$ *is a special irreducible, then* $End^*\mathcal{V} = (End^0\mathcal{V},\ 0)$;

that is, $End^1\mathcal{V} = 0$*. Furthermore, such a* $\mathcal{V}$ *has ample scalars if and only if it has adequate scalars.*

This is clear by the easier part of Schur's lemma, since an irreducible is *special* precisely when it is not isomorphic to its own associate.

Proposition 1.9. a) *If* A *is a central simple algebra over* F*, then the graded algebra defined below is a* CSGA :

$$(A^{(N,M)\times(N,M)})_0 \;:=\; \left(\begin{array}{c|c} A^{N\times N} & 0 \\ \hline 0 & A^{M\times M} \end{array}\right) \;;$$

$$(A^{(N,M)\times(N,M)})_1 \;:=\; \left(\begin{array}{c|c} 0 & A^{N\times M} \\ \hline \mathrm{A}^{M\times N} & 0 \end{array}\right) \;.$$

b) *More generally, for any ungraded algebra* A*, the graded centre of* $A^{(N,M)\times(N,M)}$ *is isomorphic to* $(Z(A),\ 0)$*. Note however that the centre of* $(A^{(N,M)\times(N,M)})_0$ *is* $Z(A)$ *only when* $NM = 0$*. If both* N *and* M *are non–zero, one gets* $Z(A)\times Z(A)$ *.*

The proof is a simple calculation using the corresponding fact concerning (ungraded) matrix algebras.

Proposition 1.10. *If* $\mathcal{V}$ *is a special irreducible, then*

$$End^*[\mathcal{V}^{\oplus N} \;\oplus\; (\rho\mathcal{V})^{\oplus M}] \;\cong\; (End^0\mathcal{V})^{(N,M)\times(N,M)} \;,$$

(*and so it is a* CSGA *if* $\mathcal{V}$ *has adequate scalars, by* 1.6 and 1.9a)).

The proof is straightforward from 1.8 (which is a special case).

Definition Given a division algebra D over F, a non–zero element d of D, and an F–automorphism θ of D which fixes d, define $D < \sqrt{d};\ \theta >$ to be the GA which is (D, eD) as a graded F–module, and where multiplication is determined by that of D, plus requiring that $e^2 = d$ and $xe = e\theta(x)$ for all $x \in D$.

The following theorem is proved as Lemma 2 in [**C–H**].

Theorem 1.11. *a) For any such* $(D,\ d,\ \theta)$*, this defines a division graded algebra. Its graded centre is given by :*

$$\hat{Z}_0(D < \sqrt{d};\ \theta >) = Z(D) \cap (+1\text{-}eigenspace\ of\ \ \theta)\ ,$$

of which $Z(D)$ *is an extension of degree at most 2; and*

$$\hat{Z}_1(D < \sqrt{d};\ \theta >) = 0\ .$$

b) Any division graded algebra either has the form $D < \sqrt{d};\ \theta >$ *for some such* $(D,\ d,\ \theta)$*, or else has the form* $(D,\ 0)$*.*

c) If D is a central division algebra over F, then $D < \sqrt{d};\ \theta >$ *is a central division graded algebra, hence a* CSGA.

d) Any central division graded algebra over F is isomorphic to exactly one of the GA's in the four possibilities below (see the table before 4.1 ahead for the example F=**R**) :

i) $(D,\ 0)$ *for a central division algebra* D *over* F*;*

ii) $D < \sqrt{f};\ id >$ *for a central division algebra* D *over* F *and a non–zero element* $f \in F$*, uniquely determined as an element of* $\dot{F}/\dot{F}^2$*;*

iii) $(C \otimes_F F[\sqrt{f}\ ]) < \sqrt{1};\ 1 \otimes \gamma >$*, where:*
C *is a central division algebra over* F*, unique up to isomorphism;*
f *is a non–zero element in F which is not a square in C, uniquely determined as a (non–trivial) element of* $\dot{F}/\dot{F}^2$*; and*
γ *is the unique non–trivial* F*–automorphism of* $F[\sqrt{f}\ ]$*;*

iv) $(D_0,\ D_1)$*, depending on* $(D,\ f,\ s)$*, where:*
D *is a central division algebra over* F*, uniquely determined up to isomorphism of (ungraded) algebras;*
f *is a non–zero element in F which is not a square in F, but is a square in D, and where f is uniquely determined as a (non–trivial) element of* $\dot{F}/\dot{F}^2$*;*
$s \in Z(D_0)$*, and* $(D_0,\ D_1)$ *is a grading of* D *with* $s^2 = f$ *and* D_1 *non–zero.*

Remarks. The types in part *d)* are partially distinguishable by their dimensions, which take the forms:

i) $(m^2,\ 0)$; *ii)* $(m^2,\ m^2)$; *iii)* and *iv)* $(2m^2,\ 2m^2)$.

The classes in the Brauer–Wall group of the division graded algebras, A, in part *d)* are as follows.

i) $[D\ ;\ 0\ ;\ 1]$.

ii) $[D\ ;\ 1\ ;\ f]$. In this case, $A_0 \oplus A_1$ is not a central F–algebra, and is a division algebra if and only if $f \notin \dot{D}^2$.

iii) $[C\ ;\ 0\ ;\ f]$, with $f \notin \dot{C}^2$. In this case, $A_0 \oplus A_1$ is a central F–algebra, and is not a division algebra. It is $C^{2\times 2}$.

iv) $[D\ ;\ 0\ ;\ f]$, with $f \in \dot{D}^2 \setminus \dot{F}^2$. Here, $A_0 \oplus A_1$ is a central F–algebra, and is a division algebra.

It follows, as proved in Theorem 1 of [**C–H**], that every element of the Brauer–Wall group is uniquely representable as a division graded algebra.

Proposition 1.12. *If $\mathcal{V}$ is an irreducible, and $\mathcal{V} \;\cong\; \rho\mathcal{V}$, then*

$$End^*\mathcal{V} \;\cong\; (End^0\mathcal{V}) < \sqrt{d};\ \theta >$$

for some non–zero $d \in End^0\mathcal{V}$ and some θ fixing d. Furthermore,

$$\hat{Z}_0(End^*\mathcal{V}) \subset Z(End^0\mathcal{V})$$

is an extension of degree at most 2, by 1.11a). *If also $\mathcal{V}$ has adequate scalars, then $End^*\mathcal{V}$ is a* CSGA *over F, by* 1.6.

This is immediate from 1.5.

Definition. If A is a GA, let $A^{N\times N} \;:=\; (A_0^{N\times N},\ A_1^{N\times N})$.

Proposition 1.13. a) *If A is a CSGA, then so is $A^{N\times N}$.*
b) *More generally, for any graded algebra A, the graded centre of $A^{N\times N}$ coincides with that of A; that is, $\hat{Z}(A^{N\times N}) = (\hat{Z}_0(A)\cdot I_N\ ,\ \hat{Z}_1(A)\cdot I_N)$.*

This is again almost immediate from the corresponding fact for ungraded matrix algebras.

Proposition 1.14. *If $\mathcal{V}$ is an irreducible, and $\mathcal{V} \;\cong\; \rho\mathcal{V}$, then*

$$End^*(\mathcal{V}^{\oplus N}) \;\cong\; (End^*\mathcal{V})^{N\times N}\ ,$$

(*and so it is a* CSGA *if $\mathcal{V}$ has adequate scalars, by* 1.6 and 1.13a)).

This is similar to 1.10.

Gathering some of these bite–size snacks into a meal, we obtain:

Theorem 1.15. *Let $\mathcal{W}$ be any object in $\mathcal{Z}_n(G)$. Then:*
a) *$End^*\mathcal{W}$ is a* CSGA *if and only if there is an irreducible $\mathcal{V}$ with adequate scalars, and integers N and M, non-negative (and at least one positive), such that*

$$\mathcal{W} \;\cong\; \mathcal{V}^{\oplus N} \;\oplus\; (\rho\mathcal{V})^{\oplus M} \;;$$

b) *$End^0\mathcal{W}$ is a central simple algebra if and only if there is an irreducible $\mathcal{V}$ with ample scalars, and a positive integer N, such that*

$$\mathcal{W} \;\cong\; \mathcal{V}^{\oplus N} \;.$$

Proof. In a) the 'if' part is the conjunction of the bracketed phrases of 1.10 and 1.14. Conversely, if $\mathcal{W}$ did not satisfy the condition, then either 1.7 applies, or else $\mathcal{W}$ can be written as a direct sum as in 1.4, which completes the proof, in view of 1.2. The proof of b) is similar.

Note. It follows from the proof that $End^*\mathcal{W}$ is a CSGA if and only if it is a CGA; that is, no non–simple CGA occurs as $End^*\mathcal{W}$.

Proposition 1.16. *For all $\mathcal{W}$, we have*
a) $\hat{Z}_0(End^*\mathcal{W})$ is the product of the fields $\hat{Z}_0(End^*\mathcal{V})$, one for each 'isomorphism up to associates' class of irreducible, $\mathcal{V}$, occurring in $\mathcal{W}$; and
b) $Z(End^0\mathcal{W})$ is the product of the fields $Z(End^0\mathcal{V})$, one for each isomorphism class of irreducible, $\mathcal{V}$, occurring in $\mathcal{W}$.

By 1.4 and 1.1, it suffices to prove a) when $\mathcal{W}$ is 'quasi–homogeneous' i.e. of the form $\mathcal{V}^{\oplus N} \;\oplus\; (\rho\mathcal{V})^{\oplus M}$ with $\mathcal{V}$ irreducible. For special $\mathcal{V}$, use 1.10 and 1.9b). In the other case, use 1.14 and 1.13b). The proof of b) is similar.

Proposition 1.17. a)*For any $\mathcal{W} \in \mathcal{Z}_n(G)$, the following two conditions are equivalent* :

i) $\hat{Z}(End^*\mathcal{W}) \;\cong\; (\; F^d \;,\; 0 \;\;)$;

ii) *We have $\mathcal{W} \;\cong\; \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_d$ for non–zero $\mathcal{W}_i \in \mathcal{Z}_n(G)$ which: (A) have all irreducible summands with adequate scalars; (B) contain irreducibles from either only one isomorphism class or from two which* *are associates; and (C) do not contain any irreducible which occurs, or whose associate occurs, in $\mathcal{W}_j$ for any $j \neq i$.*

b)*For any $\mathcal{W} \in \mathcal{Z}_n(G)$, the following two are equivalent* :

i)′ $Z(End^0\mathcal{W}) \ \cong \ F^e$.

ii)′ *We have $\mathcal{W} \ \cong \ \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_e$ for non–zero $\mathcal{W}_i \in \mathcal{Z}_n(G)$ which: (A)′ have all irreducible summands with ample scalars; (B)′ contain irreducibles from only one isomorphism class ; and (C)′ do not contain any irreducible which occurs in $\mathcal{W}_j$ for any $j \neq i$.*

Proof. a) To deduce i) from ii), we may assume $d = 1$, by 1.4a). Then it follows from half of 1.15a). Conversely, if $\mathcal{W}$ is written as a direct sum of irreducibles which are then bundled together as specified in (B) and (C) (i.e. as a direct sum of 'quasi–homogeneous' $\mathcal{W}_i$), then 1.16a) gives $\hat{Z}_0(End^*\mathcal{W})$ as a product of fields (extensions of F), one for each i. It follows from i) that there must be "d" such fields, all isomorphic to F, which shows that the number of $\mathcal{W}_i$ is correct, and that each satisfies (A), as required.

b) To deduce i)′ from ii)′, we may assume $e = 1$, by 1.4b). Then it follows from half of 1.15b). Conversely, if $\mathcal{W}$ is written as a direct sum of irreducibles which are then bundled together as specified in (B)′ and (C)′ (i.e. as a direct sum of 'isotypical' or 'homogeneous' $\mathcal{W}_i$), then 1.16b) gives $Z(End^0\mathcal{W})$ as a product of fields (extensions of F). Since we are assuming i)′, there must be "e" such fields, all isomorphic to F, which shows that the number of $\mathcal{W}_i$ is correct, and that each satisfies (A)′, as required.

Remark. In the theory with gradings over $(\mathbf{Z}/\mathbf{2})^c$, there will be a total of 2^c adjectives weakly between the adjectives 'adequate' and 'ample' (including the case $c = 0$!).

For $n > 0$ and $\delta = \pm$, we have the functor

$$\kappa_\delta \ : \ \mathcal{Z}_{\delta n}(G) \to \mathcal{Z}_{\delta(n-1)}(G) \ ,$$

given by

$$(V, \ V', \ \eta_1 \cdots \eta_n) \mapsto (V, \ V', \ \eta_1 \cdots \eta_{n-1}) \ .$$

Given a non–zero element $f \in F$, recall the (central, yet 'ungraded–commutative') division graded algebra $F < \sqrt{f}\ ;\ id >$ (denoted $F\langle\sqrt{f}\rangle$ in [**L**]), which is one–dimensional in each grading, with an element r in the 1–grading such that $r^2 = f$.

Proposition 1.18. *For any object $\mathcal{V}$ in $\mathcal{Z}_{\delta n}(G)$, the map*

$$(End^*\mathcal{V})\ \hat{\otimes}\ F\langle\sqrt{\delta}\rangle \longrightarrow End^*(\kappa_\delta\mathcal{V})\ ,$$

determined by

$$\alpha \otimes 1 \mapsto \alpha$$

and

$$\alpha \otimes r \mapsto \alpha\eta_n\ ,$$

is an isomorphism of GA's over F.

Proof. Call the map Γ. It is elementary to see that Γ is a well defined map of graded vector spaces. To check that it is a map of algebras, we note, with exponents in $\mathbf{Z/2}$, that $r^j r^k = \delta^{jk} r^{j+k}$, and similarly with η_n in place of r. Then

$$\Gamma[(\alpha \otimes r^j)(\beta \otimes r^k)] \text{ and } \Gamma(\alpha \otimes r^j)\Gamma(\beta \otimes r^k)$$

both work out to be

$$\delta^{jk}(-1)^{j|\beta|}\alpha\beta\eta_n^{j+k}\ .$$

To check surjectivity of Γ, given γ in its codomain, we have

$$\Gamma(\alpha \otimes 1\ +\ \beta \otimes r)\ =\ \gamma\ ,$$

where

$$\alpha = \frac{1}{2}(\gamma + (-1)^{|\gamma|}\delta\eta_n\gamma\eta_n)\ ,$$

and

$$\beta = \frac{\delta}{2}(\gamma\eta_n + (-1)^{|\gamma|+1}\eta_n\gamma)\ .$$

To check injectivity, suppose that

$$\alpha \otimes 1\ +\ \beta \otimes r \mapsto 0 = \alpha + \beta\eta_n\ ,$$

with $|\beta| = 1 + |\alpha|$. Then

$$0 \;=\; (-1)^{|\alpha|}\delta\eta_n 0\eta_n \;=\; (-1)^{|\alpha|}\delta(\eta_n\alpha\eta_n + \eta_n\beta\eta_n^2) \;=\; \alpha + (-1)^{|\alpha|}\eta_n\beta \;=\; \alpha - \beta\eta_n \;.$$

Thus $\alpha = 0 = \beta$, as required.

The proof of the final result in this section is just sketched, since the industrious reader will mostly need to consult the references. Each graded algebra A has a 'twin' $\bar{A}$, which coincides with A in all respects except that its multiplication, $*$, is defined in terms of that, $\bullet$, on A by

$$x * y \;:=\; (-1)^{|x||y|} x \bullet y \;.$$

Twinning preserves the subset of CGA's (resp. SGA's, CSGA's). The inverse of the class of A in the Brauer–Wall group is formed by using the class of $(\bar{A})^{opp}$, which equals $\overline{(A^{opp})}$. The map determined by $a \otimes b \mapsto b \otimes a$ shows that $\overline{A \hat{\otimes} B}$ is isomorphic to $\overline{B} \;\hat{\otimes}\; \overline{A}$.

Proposition 1.19. *Suppose that $\mathcal{V} \in \mathcal{Z}_1^F(G)$ corresponds to the FG–module M, under the equivalence of categories between $\mathcal{Z}_1^F(G)$ and the category of (ungraded) FG–modules in which z acts as -1. Then $End^*\mathcal{V}$ is isomorphic as a GA to $\overline{A(M)}$,* where $A(M)$ is the graded algebra associated to M by Turull [**T**].

Remark. Both definitions of these graded algebras seem perfectly natural. They don't coincide up to isomorphism because we found it more convenient to use the functor

$$\rho(V,\; V',\; \eta_1,\; \cdots,\; \eta_n) = (V',\; V,\; -\eta_1,\; \cdots,\; -\eta_n)\;,$$

rather than the naturally isomorphic functor

$$\rho_1([V,\; V',\; \bullet],\; \eta_1,\; \cdots,\; \eta_n) = ([V',\; V,\; *],\; \eta_1,\; \cdots,\; \eta_n)\;,$$

where the G–action, $*$, is defined in terms of the given action, $\bullet$, by

$$g * v \;:=\; (-1)^{|g|} g \bullet v \;.$$

Sketch Proof. We may take

$$A(M)_0 \;=\; End_{FG}M \;=\; End_{FG}(M^{ass})\;,$$

and

$$A(M)_1 \;=\; Hom_{FG}(M\ ,\ M^{ass}) \;=\; Hom_{FG}(\ M^{ass}\ ,\ M)\ .$$

Here the associate, M^{ass}, coincides with M in all respects except that its G–action is $*$, as defined in the remark above.
Define a map of graded vector spaces,

$$\Omega\ :\ (End^0\mathcal{V}\ ,\ End^1\mathcal{V}) \to (A(M)_0\ ,\ A(M)_1)\ ,$$

as follows. If $\mathcal{V} = (V,\ V',\ \eta)$, we may take M to be the $+1$–eigenspace of η regarded as a linear operator on $V \oplus V'$. The latter inherits its G–action from that on $(V,\ V')$. For $\alpha \in End^0\mathcal{V}$, think of α as an operator on $V \oplus V'$ and define $\Omega(\alpha)$ to be the restriction of α to M. For $\beta \in End^1\mathcal{V}$, define $\Omega(\beta)$ to be the restriction of β' to M, where

$$\beta'(v) \;:=\; (-1)^{|v|}\eta\beta(v)\ .$$

In both gradings we have a bijection Ω, since we are using an equivalence of categories [**H1**].
It remains only to check the behaviour with respect to the algebra multiplications. If also $\gamma \in End^1\mathcal{V}$, then

$$\Omega\beta[\Omega\gamma(v)] = (-1)^{|v|}\eta\beta[(-1)^{|v|}\eta\gamma(v)] = -\eta^2\beta\gamma(v) = -\Omega(\beta\circ\gamma)(v)\ .$$

In the three cases where at least one of the endomorphisms has grading 0, the corresponding calculation is even easier and the sign doesn't appear, as required.

Continuation of Remark. For computations in Section 4, it will be useful to have another algebra, $\overline{End}^*\mathcal{V}$, defined using ρ_1, rather than ρ:

$$\overline{End}^0\mathcal{V} := End^0\mathcal{V} = Hom(\mathcal{V},\ \mathcal{V}) = Hom(\rho_1\mathcal{V},\ \rho_1\mathcal{V})\ ;$$

$$\overline{End}^1\mathcal{V} := Hom(\mathcal{V},\ \rho_1\mathcal{V}) = Hom(\rho_1\mathcal{V},\ \mathcal{V})\ .$$

Then we have an isomorphism of algebras $\overline{End^*\mathcal{V}} \cong \overline{End}^*\mathcal{V}$ via the map

$$(End^0\mathcal{V}\ ,\ End^1\mathcal{V}) \to (\overline{End}^0\mathcal{V}\ ,\ \overline{End}^1\mathcal{V})$$

which is the identity on the zero part, and takes $\alpha \in End^1\mathcal{V}$ to the mapping $[v \mapsto (-1)^{|v|}\alpha(v)]$. The proof is mechanical, as above.

2. Top–Down and Bottom–Up Schur Index

Given a field extension $F \subset L$ in characteristic 0, we'll prove (mimicking a character–free treatment of the ungraded theory) that the map

$$\mathcal{Z}_n^F(G) \longrightarrow \mathcal{Z}_n^L(G)$$
$$\mathcal{V} \longmapsto L \otimes_F \mathcal{V}$$

has the following property: For each irreducible $\mathcal{V}$ over F,

$$\mathcal{V} \longmapsto \bigoplus_i (\mathcal{V}_{1,i} \oplus \cdots \oplus \mathcal{V}_{d_i,i})^{\oplus m_i} \ ,$$

for some positive integers d_i and m_i, and some distinct irreducibles $\mathcal{V}_{j,i}$ over L, *and such that each irreducible over L occurs as a $\mathcal{V}_{j,i}$ precisely once, as we vary $\mathcal{V}$.* In particular, the induced map between the Grothendieck groups of these categories is injective.

Proposition 2.1. *If the extension $F \subset L$ is finite, then the composition*

$$\mathcal{Z}_n^F(G) \ \rightarrow \ \mathcal{Z}_n^L(G) \ \rightarrow \ \mathcal{Z}_n^F(G)$$

is multiplication by $[L : F]$; that is, $\mathcal{V} \mapsto \mathcal{V}^{\oplus [L:F]}$.

Note Whenever maps as above are written down without being more specific, we mean either tensoring with the larger field, or restricting scalars to the smaller (the latter only for finite extensions, of course).

The proof of 2.1 is the same as in the ungraded case: just pick any basis for the field extension in order to write down the required subspaces.

Corollary 2.2. *The map $\mathcal{Z}_n^F(G) \rightarrow \mathcal{Z}_n^L(G)$ is injective on isomorphism classes; more precisely, in the first paragraph of this section, each irreducible over L can occur as a $\mathcal{V}_{j,i}$ for at most one $\mathcal{V}$* (since $(\mathcal{V}_{j,i})_F$ is isomorphic to a direct sum of copies of $\mathcal{V}$).

Given a field extension $F \subset C$, such that $\mathcal{Z}_n^C(G) \rightarrow \mathcal{Z}_n^{\bar{C}}(G)$ is an isomorphism ($\bar{C}$ being the algebraic closure of C), it is straightforward to see that there exist minimal S with $F \subset S \subset C$ such that $\mathcal{Z}_n^S(G) \to \mathcal{Z}_n^{\bar{S}}(G)$ is also an isomorphism. The extension $F \subset S$ is finite. Furthermore, $\mathcal{Z}_n^S(G) \to \mathcal{Z}_n^T(G)$ is an isomorphism for any extension, T, of S, so $\mathcal{Z}_n^S(G)$ is independent of F. One can call such an S a $\mathcal{Z}_n$*–splitting field* for G over F. Now the statements in the first paragraph above follow easily for general L from the case when L is a splitting field over F. In the latter case, only one i is needed, as proved and expanded in Theorem 2.6 below. In this case the integer $m_i = m$ is called the $\mathcal{Z}_n$*–Schur index*. Below, the latter is related to the Brauer–Wall group, in analogy to the Brauer group connection for ungraded representations. Later we'll see that, for two values of n, this becomes the ordinary Schur index with respect to G and to Ker σ, respectively. By the 8–fold periodicity (see the next two sections), there are six other cases in general. But if $\sqrt{-1}$ is in F, these collapse to the above two cases, by 2–fold periodicity.

Now fix n and F, and fix also a splitting field S for $\mathcal{Z}_n(G)$ over F. Given an irreducible $\mathcal{V} \in \mathcal{Z}_n^F(G)$, there are intermediate fields $A = A(\mathcal{V})$ and $M = M(\mathcal{V})$, unique up to F-isomorphism, for which $\mathcal{V}$ *has ample scalars in* M and *adequate scalars in* A, respectively, i.e. for which the two sets of four equivalent conditions in the next result hold, respectively. Existence and uniqueness of A and M follow from conditions *iii)* and *iii)′*. These also show that we may take $A \subset M$. By 1.12, this extension has degree 2 or 1.

Proposition 2.3. *Let $\mathcal{V}$ be an irreducible in $\mathcal{Z}_n^F(G)$.*

a) The following are equivalent conditions on an intermediate field A (i.e. $F \subset A \subset S$):

i) There exists a $\mathcal{W} \in \mathcal{Z}_n^A(G)$ with adequate scalars (in A) where $\mathcal{W}_F \cong \mathcal{V}$ and every map from $\mathcal{V}$ to $\rho^j\mathcal{V}$ is also a map from $\mathcal{W}$ to $\rho^j\mathcal{W}$.

ii) There exists a $\mathcal{W} \in \mathcal{Z}_n^A(G)$ such that $\mathcal{W}_F \cong \mathcal{V}$, and any such $\mathcal{W}$ both has adequate scalars (in A) and also has the property that every map from $\mathcal{V}$ to $\rho^j\mathcal{V}$ is in fact a map from $\mathcal{W}$ to $\rho^j\mathcal{W}$.

iii) The graded F–algebras $\hat{Z}(End^*\mathcal{V})$ and $(A\ ,\ 0\ )$ are isomorphic.

iv) If d is the degree $[A:F]$, then $A \otimes_F \mathcal{V} \cong \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_d$ for distinct non–associated irreducibles $\mathcal{W}_i$ over A, *each with adequate scalars (in A) and satisfying the mapping property of $\mathcal{W}$ in i) and ii).*

b) *Furthermore, the following are also equivalent conditions on an intermediate field M:*

i)′ There exists a $\mathcal{W} \in \mathcal{Z}_n^M(G)$ with ample scalars (in M) where $\mathcal{W}_F \cong \mathcal{V}$ and every endomorphism of $\mathcal{V}$ is also an endomorphism of $\mathcal{W}$.

ii)′ There exists a $\mathcal{W} \in \mathcal{Z}_n^M(G)$ such that $\mathcal{W}_F \cong \mathcal{V}$, and any such $\mathcal{W}$ both has ample scalars (in M) and also has the property that every endomorphism of $\mathcal{V}$ is in fact an endomorphism of $\mathcal{W}$.

iii)′ $Z(End^0\mathcal{V}) \;\cong\; M$ as an F–algebra.

iv)′ If e is the degree $[M : F]$, then $M \otimes_F \mathcal{V} \;\cong\; \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_e$ for distinct irreducibles $\mathcal{W}_i$ over *M, each with ample scalars (in M) and each satisfying the assertion about endomorphisms in i)′ and ii)′.*

Proof. The proof for b) is similar to, but easier than, that for a), so we'll give only the latter.

i) implies iii): An isomorphism $\mathcal{W}_F \to \mathcal{V}$ gives an action of A on $\mathcal{V}$, and so a map $\alpha : A \to End^0\mathcal{V}$ of F–algebras. It maps into $Z(End^0\mathcal{V})$ by the assumed map extendability for $j = 0$. Being a map between fields, α is injective. It maps into $\hat{Z}_0(End^*\mathcal{V})$ by the assumed map extendability for $j = 1$. It maps *onto* $\hat{Z}_0(End^*\mathcal{V})$, since otherwise the action on $\mathcal{V}$ could be extended to the action of a proper extension of A so as to contradict adequateness.

iii) implies iv): We have isomorphisms of F–algebras:

$$\hat{Z}[End^*_{AG}(A \otimes_F \mathcal{V})] \;\cong\; \hat{Z}[A \otimes_F End^*_{FG}(\mathcal{V})]$$

$$\cong\; A \otimes_F \hat{Z}(End^*_{FG}\mathcal{V}) \;\cong\; A \otimes_F (A,\ 0) \;\cong\; (A^d,\ 0)\ .$$

Thus $A \otimes_F \mathcal{V} \cong \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_d$, where each $\mathcal{W}_i$ is as specified in 1.17a)ii). But then, using 2.1,

$$\mathcal{V}^{\oplus d} \;\cong\; (A \otimes_F \mathcal{V})_F \;\cong\; \bigoplus (\mathcal{W}_i)_F\ .$$

Since $\mathcal{V}$ is irreducible and each $\mathcal{W}_i$ is non–zero, we must have $(\mathcal{W}_i)_F \cong \mathcal{V}$ for all i, and so $\mathcal{W}_i$ is irreducible. Thus the $\mathcal{W}_i$ form a collection of distinct non–associated irreducibles with adequate scalars. The isomorphism between $\hat{Z}(End^*\mathcal{V})$ and $(A,\ 0)$ immediately implies the required mapping property of $\mathcal{W}_i$.

iv) implies i): We have, using 2.1,

$$\bigoplus(\mathcal{W}_i)_F \;\cong\; (A \otimes_F \mathcal{V})_F \;\cong\; \mathcal{V}^{\oplus d},$$

so each $(\mathcal{W}_i)_F \cong \mathcal{V}$, just as above. It is immediate that A satisfies *i)* for $(\mathcal{W}_i)_F$, and so it also satisfies *i)* for $\mathcal{V}$.

Since *ii)* trivially implies *i)*, it remains only to do the following.

iv) implies ii): By the proof that *iv) implies i)*, it remains only to prove that the $\mathcal{W}_i$ in *iv)* are the only possible $\mathcal{W}$ as in *i)* and *ii)*. But to show that such a $\mathcal{W}$ must occur in $A \otimes_F \mathcal{V}$, a non–zero $\mathcal{Z}_n^A(G)$–map from the latter to the former is given by $a \otimes v \mapsto a\theta(v)$ for any $\mathcal{Z}_n^F(G)$– isomorphism $\theta : \mathcal{V} \to \mathcal{W}_F$.

Corollary 2.4. *If $\mathcal{V}$ is an irreducible in $\mathcal{Z}_n^F(G)$, and has either adequate or ample scalars in K (i.e. assume that K is one of the possibilities for A or M), then $F \subset K$ is a (finite) Galois extension, and*

$$K \otimes_F \mathcal{V} \;\cong\; \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_{[K:F]}$$

where the $\mathcal{W}_i$ are a complete set of distinct irreducibles which are mutually conjugate via the Galois group $\mathcal{G}al(K//F)$.

Note. Given an automorphism θ of K, and $\mathcal{W} \in \mathcal{Z}_n^K(G)$, its *conjugate* $\mathcal{W}^\theta$ is defined to be the same as $\mathcal{W}$ in all respects except that the new scalar multiplication, $*$, of K is defined in terms of the old one, $\bullet$, by $k*w = \theta(k)\bullet w$.

Proof. Two different $\mathcal{W}_i$ in *iv)* or *iv)'* of 2.3 are given by two isomorphisms $K \to Z(End^0\mathcal{V})$ *or* $\hat{Z}_0(End^*\mathcal{V})$, respectively, of F–algebras. These two isomorphisms differ by an element of $\mathcal{G}al(K//F)$. It follows that $\mathcal{G}al(K//F)$ has at least (and so exactly) $[K : F]$ elements, and so the extension is normal, as required.

Proposition 2.5. *Recall that S is a $\mathcal{Z}_n$–splitting field of G over F. Let K be any intermediate field, and let $\mathcal{W}$ be an irreducible in $\mathcal{Z}_n^K(G)$.*

a) *The following are equivalent:*
i) $\mathcal{W}$ has adequate scalars (in K);
ii) $S \otimes_K \mathcal{W}$ has the form $\mathcal{U}^{\oplus N} \oplus (\rho\mathcal{U})^{\oplus M}$ for an irreducible $\mathcal{U} \in \mathcal{Z}_n^S(G)$;
iii) $S \otimes_K \mathcal{W}$ has the form $\mathcal{U}^{\oplus N}$ or $(\mathcal{U} \oplus \rho\mathcal{U})^{\oplus M}$ for an irreducible $\mathcal{U} \in \mathcal{Z}_n^S(G)$.

b) *So are the following:*

iv) $\mathcal{W}$ *has ample scalars (in* K*);*

v) $S \otimes_K \mathcal{W}$ *has the form* $\mathcal{U}^{\oplus N}$ *for an irreducible* $\mathcal{U} \in \mathcal{Z}_n^S(G)$.

Proof. As usual, we'll give it only for a). Condition *i)* is equivalent by definition to saying that $\hat{Z}_0(End^*_{KG}\mathcal{W}) = K$. By 1.17 and since everything over S has adequate scalars, condition *ii)* is equivalent to the assertion $\hat{Z}_0[End^*_{SG}(S \otimes_K \mathcal{W})] = S$. The proof of the equivalence of *i)* and *ii)* is completed by observing that the latter centre is isomorphic to S tensored over K into the former centre (as at the start of the proof that *iii)* implies *iv)* in 2.3). To prove that *ii)* implies *iii)*, consider the two cases according to whether $\mathcal{W}$ is special or not. If it is, then we must have $NM = 0$, since, if both N and M are non–zero, restricting scalars back to K in the equation of *ii)* gives (by 2.1) that both $\mathcal{U}_F$ and $(\rho\mathcal{U})_F$ are direct sums of copies of $\mathcal{W}$. But since $(\rho\mathcal{U})_F = \rho(\mathcal{U}_F)$, this contradicts specialness of $\mathcal{W}$. On the other hand, if $\mathcal{W} \cong \rho\mathcal{W}$, then either $\mathcal{U} \cong \rho\mathcal{U}$ in which case we can obviously take either N or M to be 0, or else $\mathcal{U}$ is special, which implies $N = M$ by applying ρ to both sides of the equation of *ii)*.

Theorem 2.6. *For any* G, n *and* F, *denote, as above, by* S *a fixed* $\mathcal{Z}_n$*–splitting field of* G *over* F. *Then the functor*

$$\mathcal{Z}_n^F(G) \longrightarrow \mathcal{Z}_n^S(G)$$

$$\mathcal{V} \longmapsto S \otimes_F \mathcal{V}$$

is described as follows: for each irreducible $\mathcal{V}$ *over* F,

$$\mathcal{V} \longmapsto (\mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_e)^{\oplus m} \ ,$$

where the following hold.

i) The $\mathcal{U}_i$ *are distinct irreducibles, and each irreducible in* $\mathcal{Z}_n^S(G)$ *appears exactly once as a* $\mathcal{U}_i$, *as we vary* $\mathcal{V}$ *over all the irreducibles in* $\mathcal{Z}_n^F(G)$.

ii) The integer e *is the degree over* F *of any ample scalar field,* M, *of* $\mathcal{V}$.

iii) (BOTTOM UP) Given $\mathcal{V}$, *the integer* m, *the Schur index, is the index over* M *of the central division algebra* $End^0_M\mathcal{W}$ *(i.e. the square root of its dimension over* M*) , where* $\mathcal{V}$ *has ample scalars in* M, *and* $\mathcal{W}$ *is any choice* *of an object in* $\mathcal{Z}_n^M(G)$ *for which* $\mathcal{W}_F \cong \mathcal{V}$. *Equivalently,* m *is the index of* $End_F^0\mathcal{V}$ *over its centre (which is* M*). In terms of* End^* *and the adequate scalar field,* A*, of* $\mathcal{V}$*, we have the following three cases which determine* m*: for any* $\mathcal{W}_i$ *occurring in* 2.3a)*iv),*

$$dim_A(End^*\mathcal{W}_i) = \begin{cases} (m^2,\ 0) & \text{if } \mathcal{V} \text{ is special, so } A = M \text{ and } \mathcal{U}_i \ncong \rho\mathcal{U}_i\ ; \\ (m^2,\ m^2) & \text{if } \mathcal{V} \cong \rho\mathcal{V} \text{and } \mathcal{U}_i \cong \rho\mathcal{U}_i,\ \text{so } A = M\ ; \\ (2m^2,\ 2m^2) & \text{if } \mathcal{V} \cong \rho\mathcal{V} \text{and } \mathcal{U}_i \ncong \rho\mathcal{U}_i,\ \text{so } [M:A] = 2\ . \end{cases}$$

(The same case will occur for all indices i. Note that these are the dimensions which occur in the remark after 1.11.*)*

iv) (TOP DOWN) Starting from an irreducible $\mathcal{U}$ *in* $\mathcal{Z}_n^S(G)$*, its Schur index relative to* F *may be calculated by choosing an intermediate field* K *between* F *and* S *and an object* $\mathcal{W}_{large}$ *in* $\mathcal{Z}_n^K(G)$ *for which*

$$S \otimes_K (\mathcal{W}_{large}) \cong \text{ either } \mathcal{U}^{\oplus N} \quad \text{or} \quad (\mathcal{U} \oplus \rho\mathcal{U})^{\oplus P}$$

for some integer N *or* P*, and such that* K *has no proper subfield containing* F *for which such a choice can be made. Then*

$$(\mathcal{W}_{large})_F \cong \mathcal{V}^{\oplus r} \quad \text{or} \quad (\mathcal{V} \oplus \rho\mathcal{V})^{\oplus p}$$

for some r *and/or* p*, where* $\mathcal{V}$ *is the unique irreducible summand in* $\mathcal{U}_F$*. The trichotomy in part iii) becomes the following:*

If $\mathcal{U} \ncong \rho\mathcal{U}$ *and* $\mathcal{V} \ncong \rho\mathcal{V}$*, then* $A = M = K$ *is the only possibility for* K*, and* $\mathcal{U}^{\oplus N}$ *is the only choice.*

If $\mathcal{U} \cong \rho\mathcal{U}$*, then* $\mathcal{V} \cong \rho\mathcal{V}$*, and* $A = M = K$*, and* $\mathcal{U}^{\oplus N}$ *is the only sensible choice (but if* N *were even, one could pretend to make the other choice with* $P = N/2$*).*

If $\mathcal{U} \ncong \rho\mathcal{U}$ *and* $\mathcal{V} \cong \rho\mathcal{V}$*, then* $[M : A] = 2$*. One can choose* K *to be either* A *with the choice* $(\mathcal{U} \oplus \rho\mathcal{U})^{\oplus P}$*, or to be* M *with the choice* $\mathcal{U}^{\oplus N}$*.*

In each case, the choices for $\mathcal{W}_{large}$ *are direct sums of "* r *" and/or "* $2p$ *" copies of any one of the* $\mathcal{W}_i$ *in* 2.3 *iv) and/or iv)'.*

If K *is chosen to be* A*, then* $End_K^*(\mathcal{W}_{large})$ *is a CSGA over* K*. The Schur index is calculated as in part iii), using the formulas in* 1.9 *to* 1.13*; that is,* *applying End^* to $\mathcal{W}_i$ or $\mathcal{W}_{large}$ yields the same element in the Brauer–Wall group, and this element determines m.*

When the choice $K = M$ is made, one can use instead the ungraded algebra $End^0_K(\mathcal{W}_{large})$ in the Brauer group of K, as indicated in part iii).

The other irreducibles occurring along with $\mathcal{U}$ in the image of an irreducible from $\mathcal{Z}^F_n(G)$ are precisely those in the orbit of $\mathcal{U}$ under $\mathcal{G}al(S//F)$. The subgroup leaving $\mathcal{U}$ invariant is $\mathcal{G}al(S//M)$.

Remark. Besides being annoyed at a gargantuan theorem statement, the reader may at this point be wondering why we bother at all with A and End^*, since M and End^0 do everything we need more simply. See however Sections 3 and 4 ahead.

Proof. Given $\mathcal{V}$, let K be either an ample scalar field M or an adequate scalar field A inside S. Let

$$K \otimes_F \mathcal{V} \;\cong\; \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_{[K:F]}$$

as in 2.4, and let

$$S \otimes_K \mathcal{W}_i \cong \quad \text{either} \;\; \mathcal{U}_i^{\oplus N_i} \quad \text{or} \quad (\mathcal{U}_i \oplus \; \rho\mathcal{U}_i)^{\oplus P_i}$$

as in 2.5. If θ is an automorphism of K, we have

$$S \otimes_K (\mathcal{W}^\theta) \cong (S \otimes_K \mathcal{W})^\phi$$

for any automorphism ϕ of S which extends θ: use the map $s \otimes w \mapsto \phi(s) \otimes w$. It follows that all N_i and all P_i are equal (say, to m), and that $\mathcal{U}_1, \cdots, \mathcal{U}_{[K:F]}$ are a complete set of conjugate irreducibles under the action of $\mathcal{G}al(S//F)$. Furthermore

$$S \otimes_F \mathcal{V} \cong S \otimes_K (K \otimes_F \mathcal{V}) \cong S \otimes_K (\bigoplus_i \mathcal{W}_i) \cong (\bigoplus_i [\mathcal{U}_i \;\; \textit{or} \;\; \mathcal{U}_i \oplus \; \rho\mathcal{U}_i])^{\oplus m}.$$

In the case where we choose $K = M$, for each i,

$$S^{m\times m} \cong (End^0_{SG}\, \mathcal{U}_i)^{m\times m} \cong End^0_{SG}(\mathcal{U}_i^{\oplus m})$$

$$\cong End^0_{SG}(S \otimes_K \mathcal{W}_i) \cong S \otimes_K End^0_{KG}(\mathcal{W}_i)\ .$$

Taking dimensions over K, we get

$$[S:K]\ m^2 \ = \ [S:K]\ dim_K End^0_{KG}(\mathcal{W}_i)\ ,$$

as required.
When one has $K = A$ and works with End^* rather than End^0, the analogous calculation goes as follows. The three cases are in the order given in the theorem statement. In the third case, where $e = 2d$, the indices are chosen so that

$$S \otimes_A \mathcal{W}_i \cong (\mathcal{U}_{2i-1} \oplus \mathcal{U}_{2i})^{\oplus m} \qquad \text{with} \qquad \mathcal{U}_{2i} = \rho\mathcal{U}_{2i-1}$$

for $1 \leq i \leq d$:

$$dim_A End^*_{AG}\mathcal{W}_i = dim_S(S \otimes_A End^*_{AG}\mathcal{W}_i) = dim_S End^*_{SG}(S \otimes_A \mathcal{W}_i)$$

$$= \begin{cases} dim_S End^*_{SG}(\mathcal{U}_i^{\oplus m}) & = & dim_S[(End^0_{SG}\mathcal{U}_i)^{(m,0)\times(m,0)}] \\ dim_S End^*_{SG}(\mathcal{U}_i^{\oplus m}) & = & dim_S[(End^*_{SG}\mathcal{U}_i)^{m\times m}] \\ dim_S End^*_{SG}[(\mathcal{U}_{2i-1} \oplus \mathcal{U}_{2i})^{\oplus m}] & = & dim_S[(End^0_{SG}\mathcal{U}_{2i-1})^{(m,m)\times(m,m)}] \end{cases}$$

$$= \begin{cases} dim_S(S^{(m,0)\times(m,0)}) & = & (m^2,\ 0) & \text{if } \mathcal{V} \ncong \rho\mathcal{V} \textit{ and } \mathcal{U}_i \ncong \rho\mathcal{U}_i\ ; \\ dim_S[(S,\ S)^{m\times m}] & = & (m^2,\ m^2) & \text{if } \mathcal{V} \cong \rho\mathcal{V} \textit{ and } \mathcal{U}_i \cong \rho\mathcal{U}_i\ ; \\ dim_S(S^{(m,m)\times(m,m)}) & = & (2m^2,\ 2m^2) & \text{if } \mathcal{V} \cong \rho\mathcal{V} \textit{ and } \mathcal{U}_i \ncong \rho\mathcal{U}_i\ . \end{cases}$$

Together with 2.3, this proves all of *i), ii)* and *iii*), except to show that each irreducible $\mathcal{U}$ in $\mathcal{Z}^S_n(G)$ is a summand of $S \otimes_F \mathcal{V}$ for at least one (and therefore exactly one) $\mathcal{V}$. Given such a $\mathcal{U}$, the map

$$S \otimes_F (\mathcal{U}_F) \to \mathcal{U}$$

$$s \otimes u \mapsto su$$

is easily seen to be a map in $\mathcal{Z}^S_n(G)$, and is clearly surjective. Thus $\mathcal{U}$ appears as a summand in $S \otimes_F \mathcal{V}$ for at least one irreducible summand $\mathcal{V}$ of $\mathcal{U}_F$, as required.
It remains to prove *iv*). By what we just proved, a pair $(K\ ,\ \mathcal{W}_{large})$ certainly exists: one takes K to be as in the first part of the proof, for the $\mathcal{V}$ in

the previous paragraph, and takes $\mathcal{W}_{large}$ to be $\mathcal{W}_i^{\oplus r}$ for any irreducible $\mathcal{W}_i$ appearing in $K \otimes_F \mathcal{V}$ and any $r \geq 1$.
Conversely, given K and $\mathcal{W}_{large}$ as indicated in the statement of *iv*), the latter must contain copies of only one irreducible, together possibly with its associate, since $S \otimes_K \mathcal{W}_{large}$ does. When the associate doesn't occur (the other case is similar), let $\mathcal{W}_{large} \cong \mathcal{W}^{\oplus s}$ for some irreducible $\mathcal{W}$. Then

$$(\mathcal{W}_F)^{\oplus s} \cong (\mathcal{W}_{large})_F \cong \mathcal{V}^{\oplus r}.$$

Thus

$$(K \otimes_F \mathcal{V})^{\oplus r} \cong K \otimes_F (\mathcal{V}^{\oplus r}) \cong K \otimes_F (\mathcal{W}_F^{\oplus s}) \cong [K \otimes_F (\mathcal{W}_F)]^{\oplus s}$$

The argument as above shows that $K \otimes_F (\mathcal{W}_F)$ contains a copy of $\mathcal{W}$, and therefore so does $K \otimes_F \mathcal{V}$. Also $\mathcal{W}$ has ample scalars since $\mathcal{W}_{large}$ does, and so $\mathcal{V}$ has ample scalars in K. Thus $K \otimes_F \mathcal{V}$ contains $\mathcal{W}$ exactly once, and $\mathcal{W}_F \cong \mathcal{V}$ (and $r = s$). So

$$End_K^0(\mathcal{W}_{large}) \cong End_K^0(\mathcal{W}^{\oplus s}) \cong (End_K^0\mathcal{W})^{s \times s}.$$

By the proof of *iii)*, $End_K^0\mathcal{W}$ is a division algebra of dimension m^2, as required. The remaining statements are clear from 2.4 and the fact stated above about θ and ϕ.

Referring to the clauses in the previous theorem using A and End^*, we can make the following definitions.

Definitions. Given F, n, and an irreducible $\mathcal{V} \in \mathcal{Z}_n^F(G)$, let A be an adequate field of scalars for $\mathcal{V}$ inside some splitting field. Define

$$bw_A(\mathcal{V}) \in BW(A)$$

to be the class of $End^*(\mathcal{W})$ for any $\mathcal{W} \in \mathcal{Z}_n^A(G)$ for which $\mathcal{W}_F \cong \mathcal{V}$. This is a well defined element in the Brauer–Wall group because of 1.10, 1.14, and 1.15 (as well as 2.4, since conjugate representations over F evidently have isomorphic End^*-algebras, so $bw_A(\mathcal{V})$ is independent of the choice of $\mathcal{W}$).
Given a splitting field S for $\mathcal{Z}_n(G)$ over F, and an irreducible $\mathcal{U} \in \mathcal{Z}_n^S(G)$, let $\mathcal{U}_F$ be a direct sum of copies of the irreducible $\mathcal{V}$. Choose A, an adequate field for $\mathcal{V}$ between F and S. Define

$$bw(\mathcal{U},\ F,\ A)\ :=\ bw_A(\mathcal{V})\ .$$

Then, by the last theorem, $bw(\mathcal{U},\ F,\ A)$ is the class of $End^*(\mathcal{W}_{large})$ for any choice of $\mathcal{W}_{large}$ in that theorem when K is chosen to be A, and it determines the Schur index of $\mathcal{U}$ over F.
If $\mathcal{U}' \in \mathcal{Z}_n^T(G)$ is irreducible, where $S \subset T$, define

$$bw(\mathcal{U}',\ F,\ A)\ :=\ bw(\mathcal{U},\ F,\ A)\ ,$$

for the unique $\mathcal{U} \in \mathcal{Z}_n^S(G)$ for which $\mathcal{U}' \cong T \otimes_S \mathcal{U}$.

Note that this coincides with the previous definition when T *is* a splitting field, and in general is independent of which splitting field is chosen intermediate between A and T.

3. Products, equivalences, inducing and restricting

Up to now, the need for End^* as opposed to End^0 has not been obvious. The calculations in Section 4 should help to dissolve this impression. In addition , behaviour with respect to the natural tensor product,

$$\mathcal{Z}_a(G) \times \mathcal{Z}_b(H) \longrightarrow \mathcal{Z}_{a+b}(G\ \hat{\Upsilon}\ H)$$

$$(\mathcal{V}, \mathcal{W}) \longmapsto \mathcal{V} \otimes \mathcal{W}$$

defined in [**H1**; Sect. 2], is simpler using the graded algebra. Adjointness properties of this tensor operation lead quickly to

Theorem 3.1. *For all $\mathcal{V}$ and $\mathcal{W}$ as above, we have*

$$End^*(\mathcal{V} \otimes \mathcal{W})\ \cong\ (End^*\mathcal{V})\ \hat{\otimes}\ (End^*\mathcal{W})\ \ ,$$

where the isomorphism, and the tensor product on the right, are for $\mathbf{Z}/\mathbf{2}$*–graded algebras.*

Corollary. *If $End^*(\mathcal{V}_j)$ is a CGA for each j, then $End^*(\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots)$ is also a CGA.*

Proof. This is immediate, by iterating 3.1, from the fact that a tensor product of CGA's is again a CGA.

Proof of 3.1. We have

$$
\begin{aligned}
End^*(\mathcal{V}\otimes\mathcal{W}) &= Hom^*(\mathcal{V}\otimes\mathcal{W}\ ,\ \mathcal{V}\otimes\mathcal{W}) \\
&\cong Hom^*[\mathcal{V}\ ,\ Hom^*(\mathcal{W}\ ,\ \mathcal{V}\otimes\mathcal{W})] \\
&\cong Hom^*[\mathcal{V}\ ,\ \mathcal{V}\otimes Hom^*(\mathcal{W}\ ,\ \mathcal{W})] \\
&\cong Hom^*(\mathcal{V}\ ,\ \mathcal{V})\ \hat{\otimes}\ Hom^*(\mathcal{W}\ ,\ \mathcal{W}) \\
&= End^*(\mathcal{V})\ \hat{\otimes}\ End^*(\mathcal{W})\quad ,
\end{aligned}
$$

as required, using adjointness [**H1**;5.3] once, and 'co–adjointness' [**H1**; end of Section 5] twice, in that order. A mechanical check verifies that the isomorphism commutes with the algebra multiplications.

In the first section of [**H1**], the main result is a periodicity equivalence $\mathcal{Z}_n^{\mathbf{C}}(G)\simeq\mathcal{Z}_{n+2}^{\mathbf{C}}(G)$ for $n\geq 0$. Examination of the proof shows that it holds over any extension F of $\mathbf{Q}$ which contains $\sqrt{-1}$. More generally, John Q. Huang in [**QH1**, **QH2**] has extended this to all n, and has proved an 8–fold periodicity, $\mathcal{Z}_n^F(G)\simeq\mathcal{Z}_{n+8}^F(G)$, which holds over *all* fields of characteristic 0. (There is also an intermediate case, giving period 4 when -1 is a sum of two squares in F. See also [**H3**] for the generalization to gradings over arbitrary finite abelian groups.) The following is proved directly in the same manner as 1.19.

Proposition 3.2. *End^* commutes with the periodicity equivalences.*

There are also equivalences (over any field of characteristic zero) of $\mathcal{Z}_1(G)$ with the category of (ungraded) representations of G in which z acts negatively; and, when $\sigma\neq 0$, of $\mathcal{Z}_0(G)$ with the category of (ungraded, z–negative) representations of $Ker\ \sigma$. The behaviour of End^* with respect to the first of these is explained in 1.19. The interpretation of End^0 in these 'ungraded' categories is obvious in both cases, and also of End^1 once one knows what the operation ρ amounts to. On G–modules, it is multiplication by the sign representation. On $Ker\ \sigma$–modules, it is conjugation, using any element not in $Ker\ \sigma$.

From this information, it is clear how to extract, from knowledge of End^*, the usual Schur index of an ungraded representation which corresponds to some given element in $\mathcal{Z}_n(G)$ under iteration of the above category equivalences.

As with the well known facts concerning linear representations of S_n, there is a 'unitriangular connection' between the family of irreducible complex projective representations of S_n and A_n, and a certain family of induced representations (where what is induced are products, as in the first paragraph above, of basically Clifford modules—various versions of the latter are given in the next section). Thus we'll need the following result concerning End^* of irreducibles which occur with multiplicity one in representations induced from irreducibles on smaller groups (or, equivalently by reciprocity, the similar statement with 'restricted' in place of 'induced' and 'larger' replacing 'smaller').

Remark. Assumption $(*)$ in 3.3 below holds in the application to the double covers of the symmetric groups in the next section, as explained at the end of this section. However it would be desirable to have a more general result relating to Schur indices of induced representations. Note also that $(*)$ would fail in these examples if 'adequate' were replaced by 'ample', for example, when $F = \mathbf{Q}$. This may be seen by noting that an irreducible projective representation of the symmetric (resp. alternating) group doesn't necessarily have a rational (integer) valued character, but it does once its associate (resp. conjugate) is added on.

Theorem 3.3. *Let $f : G' \to G$ be an injective map of objects (a monomorphism preserving z and commuting with σ). Let T be an extension of F containing $\mathcal{Z}_n$–splitting fields S and S' over F for G and G' respectively. Let $\mathcal{U}$ be an irreducible in $\mathcal{Z}_n^T(G)$ and $\mathcal{U}'$ an irreducible in $\mathcal{Z}_n^T(G')$. Assume that $\mathcal{U}'$ appears with multiplicity 1 in the restriction of $\mathcal{U}$ from G to G', and that $\mathcal{U}'$ is special if and only if $\mathcal{U}$ is, in which case $\rho\mathcal{U}'$ does not appear in that restriction. Assume also :*

$$A(\mathcal{U}',\ F) \ = \ F \ = \ A(\mathcal{U},\ F) \qquad (*)$$

(That is, F itself is a choice—and so, the unique choice—for the adequate scalar field for the irreducible occurring when either object is restricted to F. This will then hold for all choices of S and S'.) Then the Brauer-Wall invariants agree: $bw(\mathcal{U},\ F,\ F) = bw(\mathcal{U}',\ F,\ F)$. In particular, the Schur indices over F of $\mathcal{U}$ and $\mathcal{U}'$ are equal.

Proof. By $(*)$, there is a unique irreducible $\mathcal{W}' \in \mathcal{Z}_n^F(G')$ [respectively $\mathcal{W} \in \mathcal{Z}_n^F(G)$] corresponding to $\mathcal{U}'$ [resp. $\mathcal{U}$] as in 2.5a). Let $\mathcal{W}'' \in \mathcal{Z}_n^F(G')$ be the $\mathcal{W}'$–isotypical component of the restriction of $\mathcal{W}$ from G to G'. It is certainly non–zero, since $\mathcal{U}'$ occurs in the restriction of $\mathcal{U}$ from G to G', and since restriction to subgroups commutes with both tensoring up to a larger field and with restricting scalars to a smaller field. Clearly $\mathcal{W}''$ is invariant under any element of $End^*(\mathcal{W})$. Thus we have, by restriction, a map

$$\Omega \ : \ End^*(\mathcal{W}) \to End^*(\mathcal{W}'')$$

of GA's, which is injective since the domain of Ω is a division graded algebra and its codomain is non–zero.
Below we show that the domain and codomain of Ω have the same dimensions in each grading, so that Ω is an isomorphism. This will complete the proof, since these two isomorphic CSGA's over F have classes in $BW(F)$ equal to $bw(\mathcal{U},\ F,\ F)$ and $bw(\mathcal{U}',\ F,\ F)$.
Using the assumption concerning the multiplicities of $\mathcal{U}'$ and $\rho\mathcal{U}'$ in the restriction of $\mathcal{U}$ from G to G', it follows that if $\mathcal{W}''$ maps under $T\otimes_F(—)$ to $\mathcal{U}'^{\oplus N}$ [to $(\mathcal{U}' \oplus \rho\mathcal{U}')^{\oplus P}$ respectively], then $\mathcal{W}$ maps to $\mathcal{U}^{\oplus N}$ [to $(\mathcal{U} \oplus \rho\mathcal{U})^{\oplus P}$, respectively]. But the dimensions pair of End^* over F of the first of these agrees with the dimensions of End^* over T of the second, and that for the third agrees with the fourth. The second and fourth dimensions are equal, as required.

The application of this theorem sketched below should logically be at the end of the next and final section; but psychologically it seems better here.

In the next section, we calculate End^* for certain 'basic' irreducibles of symmetric group covers. It follows immediately from the constructed modules in the next section that $(*)$ in 3.3 (i.e. that $A = F$) holds with $\mathcal{U}$ or $\mathcal{U}'$ equal to any of these basic objects. Then, by the corollary to 3.1, condition $(*)$ holds with $\mathcal{U}'$ equal to any tensor product of such basic objects, say for symmetric group covers indexed by $k_1, \cdots, k_\ell$. This will be the $\mathcal{U}'$ in our application. More specifically, we take $k_1, \cdots, k_\ell$ to be a strict partition, and in 3.3 we take G as a cover of $S_{k_1+\cdots+k_\ell}$, and G' as the cover of $S_{k_1} \times \cdots \times S_{k_\ell}$, using the $\hat{\mathsf{Y}}$–product. Then we may take $\mathcal{U}$ equal to the special irreducible indexed by the strict partition $k_1, \cdots, k_\ell$, and $\mathcal{U}'$ equal to the tensor product above. Except for checking that $(*)$ holds for $\mathcal{U}$, the other hypotheses of 3.3 are standard facts about these groups.
The proof of $(*)$ for $\mathcal{U}$ is by induction on the reverse of the usual lexicographic order on the set of strict partitions of a given integer. The initial case follows

from the constructions of the next section, as noted above. What we must prove is that there is a non–zero object $\mathcal{V}$, defined over F, which maps under extension of scalars to a direct sum of copies of $\mathcal{U}$ and/or its associate $\rho\mathcal{U}$. (So we might as well take $F = \mathbf{Q}$.) It suffices to be able to write an equation $\mathcal{U}_1 = \mathcal{U} \oplus \mathcal{U}_2$ for which this assertion holds for both $\mathcal{U}_1$ and $\mathcal{U}_2$. But this is once again the "standard fact about these groups" above, taking $\mathcal{U}_1$ to be the result of inducing the tensor product above, from the $\hat{\curlyvee}$–product cover of $S_{k_1} \times \cdots \times S_{k_\ell}$, to the cover of $S_{k_1+\cdots+k_\ell}$. The truth of "this assertion" for the leftover piece $\mathcal{U}_2$ is immediate from the inductive hypothesis.
Thus the Schur index over F for the general irreducible may be calculated 'easily', starting from that for the basic irreducible, as done by Turull [**T**]. Besides avoiding characters, we have: i) replaced his new binary operation $\vee$ in $BW(F)$ (which amounts to 'translating the origin') by the usual operation; and ii) replaced his $\vee$–product of representations (which is a special case of the product $\tilde{\otimes}$ in [**H–H1**; Ch.6]) by the tensor product of graded Clifford representations. Explicitly, the bw–invariant for the irreducible indexed by the strict partition above is the product in the Brauer–Wall group of the invariants for each of the basic objects indexed by the parts of the partition.

4. Covers of the Symmetric Group

In this section we shall do three things. The first is to exhibit some particularly simple objects (in categories $\mathcal{Z}_n$) which are 'alter egos' for the basic irreducible projective representations of the symmetric and alternating groups (i.e. the former map to the latter under iterations of the category equivalences before and after 3.2). Then we calculate the Brauer–Wall invariant $bw(\mathcal{U},\ F,\ F)$ for all F, where $\mathcal{U}$ ranges over these basic elements in $\mathcal{Z}_n^{\mathbf{C}}(G)$, with G being either of the essential double covers of the symmetric group. There are eight relevant values of n by periodicity, despite the fact that the period is two over $\mathbf{C}$. For $n \equiv 1 \bmod 8$, these values agree with those of Turull [**T**] (taking into account 1.19), although the method is different. For $n \equiv 0 \bmod 8$, the values give invariants for the alternating groups, and an alternative method to calculate their Schur indices. This provides a very straightforward way to get the Brauer–Wall invariants which determine the Schur indices of the basic irreducible projective characters of the symmetric and the alternating groups. Finally, the theory of the previous section then gives the invariants for all the irreducibles. We write down a closed formula for them at the end of the section.

Recall the groups $S_k^{\pm 1}$ in [**T**]. These are double covers of the symmetric groups, denoted $\tilde{S}_k$ and $\hat{S}_k$ in previous papers of this author. Note that the generator t_i projecting to the transposition $(i \quad i+1)$ has square $-\epsilon$, not $+\epsilon$ in the group S_k^ϵ. To avoid confusing ourselves, we shall redo the notation, defining $S_{k,\delta}$ to be Turull's $S_k^{-\delta}$ (so that, in our notation, t_i^2 acts as -1 on $S_{k,-}$-modules, whereas t_i^2 acts as $+1$ on $S_{k,+}$-modules). These double covers of the symmetric group are such that the categories $\mathcal{Z}_{2r+1}^{\mathbf{C}}(S_{k,\pm})$ are all equivalent to the category of complex projective representations of S_k, whereas changing $2r+1$ to $2r$ gives the projective representations for the alternating group A_k. (There are a few exceptions for small values of k.)

Below we shall do three calculations—first over $\mathbf{R}$, then over those F with $\sqrt{2} \in F$, and finally over F with $\sqrt{2} \notin F$. The first is redundant, but gives a particularly transparent method to recover the real Schur index. (This was first calculated by Schur [**S**] using manipulations with characters and Q–functions. John Q. Huang [**QH2**] has given another, short proof using the induction algebra. Of course it also follows by specializing Turull's results.) In each of the three cases, we first find an object defined over the field in question in some category $\mathcal{Z}_n$, choosing n to make life easy. Then we apply a functor κ a number of times to get into $\mathcal{Z}_{1 \text{ or } 0}$, these categories being equivalent to the (ungraded) module categories for the symmetric or alternating group covers, respectively. Via 1.18 this immediately gives the element in the Brauer–Wall group required to get the Schur index of the corresponding basic projective representation of a symmetric or alternating group.

Before starting, it may be helpful to note the following. There is a 'basic' triple of irreducible complex projective representations of S_k and A_k. *Asymptotically in k, these are by far the non-zero representations of smallest dimension.* Below we construct objects

$$B_{k-1} \quad , \quad C_k \oplus \rho C_k \quad , \quad C_{k+1} \quad , \quad C'_{k+1}$$

in $\mathcal{Z}_{k\mp 1}$. It is necessary to check that these are quasihomogeneous with respect to one from the above basic triple under the category equivalences mentioned before and after 3.2. (A character calculation here would be a last resort!) For all but the first few values of k, this follows from the smallness of the dimension. (Now apply the principle of aesthetic continuation?)

Continue with a fixed base field F. Let $C_k^{\pm}$ be the Clifford algebra for the standard (± 1)–ive quadratic form on F^k. Let $B_{k-1}^{\pm}$ be the sub–Clifford algebra for the restriction of the form to the subspace,

$$\{\ (a_1, \cdots,\ a_k)\ :\ \sum\ (\mp 1)^j a_j = 0\ \}\ \ ,$$

of codimension 1 in F^k. When $\sqrt{2} \in F$, we have embeddings

$$\begin{aligned} S_{k,\pm} \quad &\longrightarrow \qquad B_{k-1}^{\pm} \\ t_j \quad &\mapsto \quad (e_j \pm e_{j+1})/\sqrt{2} \end{aligned}$$

where t_j is the j^{th} generator of the group, and e_j is the j^{th} standard basis vector. Note that z maps to -1.

Now, given two GA's A and B, let ${}_A\mathcal{M}_B$ denote the category of graded $(A,\ B)$–bimodules (everything should be finite–dimensional over the ground field). When $\sqrt{2} \in F$, we then have an embedding of categories

$${}_{B_{k-1}^{\delta}}\mathcal{M}_{C_l^{\delta}} \quad \longrightarrow \quad \mathcal{Z}_{\delta l}(S_{k,\delta})$$

by sending a module to itself as a graded vector space, with the action of $S_{k,\delta}$ coming from the left action of B_{k-1} via the above embedding, and the map η_j being the right action of the j^{th} standard basis element e_j.

First take $F = \mathbf{R}$ (or, for fixed k, any field with enough square roots so that B_{k-1} has an orthonormal basis in its defining quadratic space. There is a formula for such a basis in [**H–H**; A6.1]).Then

$$B_{k-1}^{\delta}\ \in\ \ {}_{B_{k-1}^{\delta}}\mathcal{M}_{B_{k-1}^{\delta}}\ \cong\ {}_{B_{k-1}^{\delta}}\mathcal{M}_{C_{k-1}^{\delta}}\ .$$

By the paragraph above, we may therefore regard B_{k-1}^{δ} as an object in $\mathcal{Z}_{\delta(k-1)}(S_{k,\delta})$ over either $\mathbf{R}$ or $\mathbf{C}$. Over $\mathbf{C}$, it is 'the' basic irreducible projective representation of the symmetric (resp. alternating) group for k even (resp. k odd). ($\rightarrow$For those who find the above dimension argument insufficient, refer to 4.1 and 4.2 below, noting that, e.g. when $\delta = 1$, the object B_{k-1} is isomorphic to the $\sqrt{-1}$–eigenspace of $(\cdot e_k)(\cdot e_{k+1})$ acting on C_k via the map taking x to

$$x(1 + e_1e_2 + e_1e_3 + \cdots + e_1e_{k-1} + (1+\sqrt{k})e_1e_k)\ .\leftarrow)$$

Now it is clear that

$$bw(B_{k-1}) = \textit{class of } (\mathbf{R},\ 0) = [1;\ 0;\ 1]\ ,$$

the identity element in $BW(\mathbf{R})$, since a central algebra, regarded as a bimodule over itself, has no interesting endomorphisms.

Thus, when $\delta = +1$, the element $\kappa_+^{k-2} B_{k-1}^+ \in \mathcal{Z}_1(S_{k,+})$ is quasihomogeneous for the irreducible corresponding to the ungraded basic irreducible complex projective representation(s) of S_k. Applying bw to it gives $[1;\ 1;\ 1]^{k-2}$ by 1.18, since the class of $F < \sqrt{1} >$ is $[1;\ 1;\ 1]\ \in\ BW(F)$ for any F.

When $\delta = -1$, we must apply κ_-^k and get $[1;\ 1;\ -1]^k$, since the class of $F < \sqrt{-1} >$ is $[1;\ 1;\ -1]\ \in\ BW(F)$, and since we must get into $\mathcal{Z}_1$, not $\mathcal{Z}_{-1}$. (See [**H1**, Sect. 1] for how to define

$$\kappa_- : \mathcal{Z}_n \to \mathcal{Z}_{n+1}$$

when $n \geq 0$.)

For those who really like to see some blood and gore, we recall that $BW(\mathbf{R})$ is cyclic of order 8, generated by either $[1;\ 1;\ 1]$ or $[1;\ 1;\ -1]$, which are inverses of each other. Explicitly, with $f(n) = n(n-1)(n-2)(n-3)/4!$,

$$[1;\ 1;\ 1]^n = [(-1,-1)^{f(n+1)};\ n\ ;\ (-1)^{\frac{n(n-1)}{2}}]$$

and

$$[1;\ 1;\ -1]^n = [(-1,-1)^{f(n+2)};\ n\ ;\ (-1)^{\frac{n(n+1)}{2}}]\ .$$

Substituting these into the last paragraph gives explicit formulae. These agree with what Turull obtains, but don't forget 1.19 and that his ϵ is $-\delta$. (Of course, over $\mathbf{R}$, one may erase all the occurrences of 2 and of n, which is our k, from the formulae in [**T**; 5.6 and repeated in 6.1].)

To recover Schur's full result on the real Schur index, we apply the final few paragraphs of the previous section. The pair of associated fundamental objects indexed by a given strict partition λ are in $\mathcal{Z}_{\delta(|\lambda|-\mathrm{length}\lambda)}$. One of them is obtained by tensoring together the basic objects, B_{k-1}, one for each part, k, of λ. Its bw–invariant is obtained by multiplying the invariants of its components, so is still the identity element of the Brauer–Wall group. Multiplying by a power of κ_δ to get into $\mathcal{Z}_1$, we get the invariant

$$[1\ ;\ 1\ ;\ \delta]^{|\lambda|-\mathrm{length}\lambda-\delta}\ .$$

Below is a table listing all the elements, $[1\ ;\ 1\ ;\ 1]^n$, of $BW(\mathbf{R})$; the unique graded division algebras representing them (see 1.11); their dimensions; and the associated Schur indices. (Here $\mathbf{H}$ denotes the original quaternions of Hamilton, giving the element $(-1,\ -1)$ in the Brauer group of $\mathbf{R}$) :

$n(mod 8)$	$[1\ ;\ 1\ ;\ 1]^n$	div. gr. alg. in class	dims	m
0	$[1\ ;\ 0\ ;\ 1]$	$(\mathbf{R}, 0)$	$(1,0)$	1
1	$[1\ ;\ 1\ ;\ 1]$	$\mathbf{R} < \sqrt{1} >$	$(1,1)$	1
2	$[1\ ;\ 0\ ;\ -1]$	$\mathbf{R}[\sqrt{-1}] < \sqrt{1}; c >$	$(2,2)$	1
3	$[(-1,-1); 1; -1]$	$\mathbf{H} < \sqrt{-1} >$	$(4,4)$	2
4	$[(-1,-1); 0; 1]$	$(\mathbf{H}, 0)$	$(4,0)$	2
5	$[(-1,-1); 1; 1]$	$\mathbf{H} < \sqrt{1} >$	$(4,4)$	2
6	$[(-1,-1); 0; -1]$	$\mathbf{R}[\sqrt{-1}] < \sqrt{-1}; c >$	$(2,2)$	1
7	$[1\ ;\ 1\ ;\ -1]$	$\mathbf{R} < \sqrt{-1} >$	$(1,1)$	1

Thus $m = 2$ for the invariant of the object in $\mathcal{Z}_1$ above precisely when

$$|\lambda| - \text{length}\lambda - \delta \ \equiv\ 3,\ 4,\ 5\ \ (mod\ 8)\ .$$

This agrees with Schur's table [**S**; p. 68]. (Note that the third row of Schur's table should have been labeled s', not s. It refers to the groups $S_{k,+}$). Schur's table actually gives the more precise 'signature', i.e. the division of complex characters into real, complex and quaternionic cases. That agrees, as it must, with the zero part of the graded division algebras in our table above, taking $n = |\lambda| - \text{length}\lambda - \delta$.

Now assume that F is any field containing $\sqrt{2}$. There is a blindingly obvious object C_k in ${}_{C_k}\mathcal{M}_{C_k}$. When $\sqrt{-1}$, $\sqrt{k} \in F$ as well, by dropping down to the subalgebra B_{k-1} of C_k for the left action, we can pop up to the superalgebra (*not* the physicists' use of that word!) C_{k+1} of C_k for the right action, as follows: Let the right action of the additional basis element e_{k+1} be given by

$$v \cdot e_{k+1}\ :=\ \frac{(-1)^{|v|}}{\sqrt{-k}}\Big(\sum_{j=1}^{k}(\mp 1)^j e_j\Big) \cdot v\ \ .$$

Theorem 4.1. *This makes C_k into an object in ${}_{B_{k-1}}\mathcal{M}_{C_{k+1}}$ when $\sqrt{k}$ and $\sqrt{-1}$ are in F. When $\sqrt{2}$ also is, the corresponding object in $\mathcal{Z}_{\delta(k+1)}(S_{k,\delta})$ becomes 'the' basic irreducible projective representation of the symmetric (resp. alternating) group for k even (resp. k odd) after tensoring with* $\mathbf{C}$.

Sketch Proof. For $\delta = +$, this is a reformulation of [**H1**; Theorem 4.1]. The second assertion follows as well for all but a few values of k by the dimension argument.

There is another obvious element, C_{k+1} , in ${}_{B_{k-1}}\mathcal{M}_{C_{k+1}}$, this time over any field, but only defining a graded representation of a symmetric group cover over fields containing $\sqrt{2}$.

Theorem 4.2. *When F contains $\sqrt{-1}$, $\sqrt{2}$ and $\sqrt{k}$, as an element of $\mathcal{Z}_{\delta(k+1)}(S_{k,\delta})$, the module C^{δ}_{k+1} is either $C_k^{\oplus 2}$ or $C_k \oplus \rho C_k$.* (In fact it is the latter.) *Furthermore, over any F with $\sqrt{2} \in F$, the algebra $\overline{End}^* C^{\delta}_{k+1}$ is the graded quaternion algebra $< \frac{\delta,\delta k}{F} >$.* (See the definition at the very end of Section 1.)

Proof. The dimension argument, ($\rightarrow$ or, for example when $\delta = +1$, the map from C_k to C_{k+1} sending x to

$$x[e_{k+1}(e_1 + \cdots + e_k) - \sqrt{-k}] \quad \leftarrow)$$

shows that C^{δ}_k embeds into C^{δ}_{k+1} as objects in $\mathcal{Z}_{\delta(k+1)}(S_{k,\delta})$. The last assertion of 4.2 combined with 1.15 shows that C_{k+1} is 'quasihomogeneous', so, by a dimension count, the two stated decompositions are the only possibilities. (*But one may easily find a second map which proves the assertion in brackets in 4.2.*)
To prove the last assertion, note that an element of $\overline{End}^* C_{k+1}$ commutes with the η_j, i.e. with right multiplication by e_j, for $1 \leq j \leq k+1$. Thus it commutes with right multiplication by every element of C_{k+1}. Therefore it is left multiplication by some element of C_{k+1}. It 'graded–commutes' with the action of the group generators of $S_{k,\delta}$. They generate B_{k-1} as an algebra. Thus $\overline{End}^* C_{k+1}$ can be identified with the graded centralizer of B_{k-1} in C_{k+1}. It is straightforward to determine the latter explicitly. Taking d_k to be $\sum_1^k (\mp 1)^j e_j$, a homogeneous basis is $\{1 \ , \ e_{k+1}d_k\} \cup \{e_{k+1} \ , \ d_k\}$. The latter

two elements anti–commute and square to ± 1 and $\pm k$ respectively, giving the quaternion algebra, as stated. (Although a slightly tedious elementary argument exists to show that the graded centralizer at issue is no bigger than this, it's probably easier to argue as follows: Since C_{k+1} is only 'twice as big' as an irreducible over $\mathbf{C}$, we know beforehand that the dimension will be 4.)

Thus $End^* C_{k+1} \cong \overline{< \frac{\delta, \delta k}{F} >}$, whose class in the Brauer–Wall group is

$$\overline{[(\delta, \delta k);\ 0\ ;\ -k]} = [(-k, -\delta k);\ 0\ ;\ -k]\ .$$

To get from $\mathcal{Z}_{\delta(k+1)}$ to $\mathcal{Z}_{\delta(k-1)}$, apply κ_δ^2; i.e. multiply by $[1;\ 1;\ \delta]^2$, which equals $[(-1, \delta);\ 0;\ -1]$. This yields

$$\Theta_{\delta,k} = [(-k, \delta k)(-1, \delta);\ 0;\ k]\ .$$

Over $\mathbf{R}$, we had the identity element here, so over any F containing $\sqrt{2}$ one simply multiplies by Θ to get the *bw* invariants with respect to the various $\mathcal{Z}_n$'s from the ones over $\mathbf{R}$ (strictly speaking, from the eight Brauer–Wall elements in the second column of the previous table, which make sense over any field). When $n \equiv 1$ (mod 8), this again agrees with Turull's formulae, this time simplified only to the extent of removing the 2's.

Now assume that F is any field *not* containing $\sqrt{2}$. Let C'_{k+1} be the Clifford algebra, *but over* $F(\sqrt{2})$. Regard it as an object in $\mathcal{Z}^F_{\delta(k+1)}(S_{k,\delta})$. As such, it is clearly quasihomogeneous for the basic irreducible object (containing, over $\mathbf{C}$, four irreducibles, counting repeats). We may calculate $\overline{End}^* C'_{k+1}$ as follows. The subalgebra B''_{k-1} of C'_{k+1} is defined to have 0–part $B^{(0)}_{k-1}$ and 1–part $\sqrt{2}B^{(1)}_{k-1}$, where the lack of a prime superscript on B_{k-1} indicates that we are defining the algebra part over F, not $F(\sqrt{2})$. It is clear that B''_{k-1} is generated as an algebra by that subgroup of invertibles which is the image of $S_{k,\delta}$ under the embedding we have been using. It is now immediate from the definitions that $\overline{End}^* C'_{k+1}$ consists of those graded vector space maps, $\alpha : C'_{k+1} \to C'_{k+1}$, of degree 0 or 1, which satisfy the conditions:

i) $\alpha(xy) = \alpha(x)y$ for all x and y in C'_{k+1} ; and

ii) $\alpha(wx) = (-1)^{|w||\alpha|} w\alpha(x)$ for all $x \in C'_{k+1}$ and $w \in B''_{k-1}$.

Writing $x = x' + \sqrt{2}x''$ for x' and x'' in C_{k+1}, it is immediate from i) that

$$\alpha(x) = c'x' + \sqrt{2}c''x''$$

for some c' and c'' of equal grading θ in C'_{k+1}. Then ii) yields that both of the elements c' and c'' commute with all $b \in B^{(0)}_{k-1}$; whereas, for $b \in B^{(1)}_{k-1}$, we have $c'b = (-1)^\theta bc''$ and $c''b = (-1)^\theta bc'$.
Now writing $c' = c'_0 + \sqrt{2}c'_1$ for $c'_j \in C_{k+1}$, and similarly for c'', it follows that our required endo–algebra can be identified with the algebra of matrices

$$\begin{pmatrix} c'_0 & \sqrt{2}c''_1 \\ \sqrt{2}c'_1 & c''_0 \end{pmatrix}$$

in which all four entries are elements of the same grading, θ, in the centralizer of $B^{(0)}_{k-1}$ in C_{k+1}, and satisfy, for $b \in B^{(1)}_{k-1}$, the equations $c'_j b = (-1)^\theta bc''_j$ and $c''_j b = (-1)^\theta bc'_j$.
To see the structure of this algebra, whose dimension we know beforehand to be 16, let $d = d_k$ (defined in the proof of 4.2); $e = e_{k+1}$; $p = e_1 \cdots e_k$; yielding elements

$$D = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \ ; \ E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \ ; \ P = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} \ ; \ T = \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} .$$

The set

$$\{D^\delta E^\epsilon P^\pi T^\tau \ : \ \text{exponents in } \mathbf{Z/2} \ \}$$

is linearly independent, therefore a basis for the algebra over F. The gradings of $D,\ E,\ P,\ T$ are $1,\ 1,\ k,\ 0$ respectively, and relations are

$$D^2 = \delta k \quad ; \quad E^2 = \delta \quad ; \quad P^2 = (-1)^{\frac{k(k-1)}{2}}\delta^k \quad ; \quad T^2 = 2 \quad ;$$

$$DE = -ED \quad ; \quad TP = -PT \quad ; \quad DT = TD \quad ; \quad TE = ET \quad ;$$

$$EP = (-1)^k PE \quad ; \quad DP = (-1)^{k+1} PD \ .$$

It is immediate that our algebra is

$$Alg\{E\} \ \hat{\otimes} \ Alg\{D,\ P,\ T\} \ .$$

Furthermore,

$$Alg\{E\} \cong F\langle\sqrt{\delta}\rangle \ \text{ whose class in } \ BW(F) \text{ is } [1;\ 1;\ \delta] \ .$$

When k is even, for suitable ψ,

$$Alg\{D,\ P,\ T\} \ \cong \ Alg\{P,\ T\}\ \langle\sqrt{\delta k}\ ;\ \psi\rangle \ \cong \ \left(\frac{(-1)^{\frac{k(k-1)}{2}}\delta^k, 2}{F}\right) \langle\sqrt{\delta k}\ ;\ \psi\rangle \ ,$$

whose class in $BW(F)$ is $[((-1)^{\frac{k(k-1)}{2}}\delta^k, 2)\ ;\ 1\ ;\ 2\delta k]\ =\ [1;\ 1;\ 2\delta k]$.

The last equality comes from the fact that $(-1,\ 2)$ and $(1,\ 2)$ are both the identity element in the Brauer group.
When k is odd, for suitable ϕ,

$$Alg\{D,\ P,\ T\}\ \cong\ Alg\{P,\ T\}\ \hat{\otimes}\ Alg\{D\}\ \cong Alg\{T\}\ \langle\sqrt{(-1)^{k+1}}\ ;\ \phi\rangle\ \hat{\otimes}\ F\langle\sqrt{\delta k}\rangle\ ,$$

whose class in $BW(F)$ is $[((-1)^{\frac{k(k-1)}{2}}\delta^{k+1}k, 2)\ ;\ 1\ ;\ \delta k]\ =\ [(k,\ 2);\ 1;\ \delta k]$.

This 2-piece 'answer' is then: 1) multiplied by $[1;\ 1;\ \delta]$ from $Alg\{E\}$ previous; 2) twinned; 3) multiplied by $[1;\ 1;\ \delta]^2$ to get from $\mathcal{Z}_{\delta(k+1)}$ to $\mathcal{Z}_{\delta(k-1)}$. As a result, we get an 'answer' from which it follows directly that to get the 'answer for fields not necessarily containing $\sqrt{2}$', one multiplies 'the answer for fields containing $\sqrt{2}$ ' (to be precise, the earlier result of multiplying $\Theta_{\delta,k}$ into a Brauer–Wall element from the previous table) by the element

$$[(k,2)\ ;\ 0\ ;\ 2]\quad \text{if } k \text{ is even}\ ,$$

and by the element

$$[(k,2)\ ;\ 0\ ;\ 1]\quad \text{if } k \text{ is odd}\ .$$

The case $n \equiv 1(mod\ 8)$, after twinning again, reproduces Turull's formula in its full splendour.

A Brauer–Wall invariant not appearing explicitly in [**T**] is that related to projective representations of the alternating group, and giving the Schur index for them. Since the category of ungraded negative modules for the kernel of the parity, σ, (if the latter is non–zero) is equivalent to $\mathcal{Z}_0$ of the enriched group, we simply need, in all the previous discussion, to apply the correct power of κ_δ to the objects B_{k-1} , C_{k+1} , C'_{k+1} (over $\mathbf{R}$, fields containing $\sqrt{2}$, fields not containing $\sqrt{2}$, respectively) to get into $\mathcal{Z}_0$ rather than $\mathcal{Z}_1$. Then use the same power of $[1\ ;\ 1\ ;\ \delta]$ in the Brauer–Wall calculation.
It is possible to define a graded algebra $B(N)$ for ungraded $ker(\sigma)$–modules N, using self–maps in grading zero, and maps to the conjugate module in grading one. This is the analogue of Turull's $A(M)$. The class of $B(N)$ in BW for the symmetric group covers is obtained as in the previous paragraph,

i.e. it agrees with the class of End^* applied to the appropriate object in $\mathcal{Z}_0$, up to 'twinning'.

An advantage to our procedure of creeping up, in three stages, on the *bw*–formula for the basic irreducible, is that the separate pieces of the formula are easily multiplied together to produce a closed formula for the general irreducible. Specifically, if λ is the strict partition

$$k_1 \;>\; k_2 \;>\; \cdots \;>\; k_\ell \;>\; 0\;,$$

with the number of *even* parts k_i being "M", then the *bw*–invariant of the pair of irreducibles in $\mathcal{Z}_{\delta(|\lambda|-\ell)}(S_{|\lambda|,\delta})$ indexed by λ is calculated as follows:

$$\prod_{even\ k_i}[(k_i,\ 2);\ 0;\ 2]\ \cdot\ \prod_{odd\ k_i}[(k_i,\ 2);\ 0;\ 1]\ \cdot\ \prod_i \Theta_{\delta,\ k_i}$$

$$= [1;\ 0;\ 2]^M\ \cdot\ \prod_i[(k_i,\ 2);\ 0;\ 1]\ \cdot\ \prod_i[(-k_i,\delta k_i)(-1,\ \delta);\ 0;\ k_i]$$

(using that $(2,\ 2)$ is the identity element)

$$= [1;\ 0;\ 2^M]\ \cdot\ [\prod_i(k_i,\ 2);\ 0;\ 1]\cdot[(-1,\ \delta)^\ell\cdot\prod_i(-k_i,\delta k_i)\cdot\prod_{i<j}(k_i,k_j)\ ;\ 0\ ;\ \prod_i k_i]$$

which multiplies out to be

$$[\ \ (-1,\delta)^\ell\cdot\prod_{i<j}(k_i,k_j)\cdot\prod_i(-k_i,\delta k_i)(k_i,\ 2^{M+1})\ \ ;\ \ 0\ \ ;\ \ 2^M\prod_i k_i\ \ ]\ .$$

If the base field F is an extension of $\mathbf{Q}(\sqrt{-1},\ \sqrt{2})$, this simplifies to

$$[\ \prod_{i<j}(k_i,k_j)\cdot\prod_i(k_i,k_i)\ ;\ 0\ ;\ \prod_i k_i\ ]\ .$$

Multiply by

$$[1\ ;\ 1\ ;\ \delta]^{|\lambda|-\ell-\delta}$$

to get the *bw*–invariant of the corresponding element(s) in $\mathcal{Z}_1(S_{|\lambda|,\delta})$, which match up with the irreducible projective representation(s) of $S_{|\lambda|}$. Twinning will produce the invariant which Turull describes using his $\vee$–product in the Brauer–Wall group [**T**; 6.2]. Multiply by

$$[1\ ;\ 1\ ;\ \delta]^{|\lambda|-\ell}$$

to get the bw–invariant of the corresponding element(s) in $\mathcal{Z}_0(S_{|\lambda|,\delta})$, which match up with the irreducible projective representation(s) of $A_{|\lambda|}$.

References

[**A – B – S**] Atiyah, M. F., Bott, R. and Shapiro, A. *Clifford modules.* Topology **3** (Supplement 1), (1964), 3–38; (also in Atiyah Collected Works, Vol.2, page 301).

[**B–H**] Bean, M. and Hoffman, P. N. *Zelevinski algebras related to projective representations.* Trans. Amer. Math. Soc. **309**(1988), 99–111.

[**C–H**] Coghlan, F. and Hoffman, P. *Division graded algebras in the Brauer–Wall group.* (preprint?????????).

[**H1**] Hoffman, P. N. *A projective analogue of Schur's tensor power construction.* Comm. Alg. **21(7)**(1993), 2211–2249.

[**H2**] Hoffman, P.N. *Projective and multigraded representations of monomial and multisigned groups. I. Graded representations of a twisted product.* Can.J.Math. **45(2)**(1993), 295–339.

[**H3**] Hoffman, P.N. *Generalizing Clifford module periodicity to graded representations.* J. Alg. **172**(1995), 61–77.

[**H–H1**] Hoffman, P. N. and Humphreys, J. F. *Projective Representations of the Symmetric Groups.* Oxford University Press, Oxford, 1992.

[**H–H2**] Hoffman, P. N. and Humphreys, J. F. *Projective representations of generalized symmetric groups using PSH-algebras.* Proc. London Math. Soc. (3)**59** (1989), 483–506.

[**QH1**] Huang, John Q. Ph.D. Thesis, University of Waterloo (1993).

[**QH2**] Huang, John Q. *Real projective representations of S_n and A_n.* Can.J.Math. **46(3)** (1994), 543–573.

[**L**] Lam, T.Y. *The algebraic theory of quadratic forms.* W.A. Benjamin, Reading, Mass., 1973.

[**S**] Schur, I. *Über die reelen Kollineationsgruppen die der symmetrischen oder der alternierenden Gruppe isomorph sind.* J. Reine Angew. Math. **158** (1927), 63–79.

[**Se**] Sergeev, A. N. *The tensor algebra of the identity representation as *a module over the Lie superalgebras GL(n, m) and Q(n).* Mat. Sbornik **123**(1984), 422—430 [A.M.S. Translations **51** (1985), 419–427].

[**St**] Stembridge, J. *Shifted tableaux and the projective representations of symmetric groups.* Adv. Math. **74** (1989), 87–134.

[**T**] Turull, A. *The Schur index of projective characters of symmetric and alternating groups.* Ann. Math. **135** (1992), 91–124.

[**W**] Wall, C.T.C. *Graded Brauer Groups.* J. Reine Angew. Math. **213** (1964), 187–199.

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