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ON IRREGULARITIES OF DISTRIBUTION IN SEQUENCES OF INTEGERS

by

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Throughout this article, $c_1, c_2, \dots$ will denote effectively computable positive absolute constants. In 1964 [4], see also [5] and [6], Roth investigated how well distributed a sequence of integers can be among arithmetical progressions. The following theorem is a special case of one of his results.

THEOREM 1 (Roth [4]). - *Let* N *be a positive integer and let* $\varepsilon_1, \dots, \varepsilon_N$ *be real numbers of absolute value one. Then there exist positive integers* a *and* q *such that*

$$\left| \sum_{\substack{j \geq 1 \\ 1 \leq a+jq \leq N}} \varepsilon_{a+jq} \right| > c_1 N^{\frac{1}{4}} , \qquad (1)$$

for $N > c_2$, *where* c_1 *and* c_2 *are effectively computable positive numbers*.

Sárközy showed that Theorem 1 remains valid with the weaker hypothesis that $\varepsilon_1, \dots, \varepsilon_N$ are complex numbers of absolute value one, see Corollary 4 of [7]. Further as an immediate consequence of Theorem 1, we see that no matter how we partition $\{1, \dots, N\}$ into two sets there will always be an arithmetical progression lying in $\{1, \dots, N\}$ which contains a preponderance of terms from one of the two sets. In fact we have at least $c_3 N^{\frac{1}{4}}$ more terms from one set than from the other. This may be contrasted with van der Waerden's theorem which states that for any positive integer k there is a smallest integer $N(k)$ such that whenever N is greater than $N(k)$ any partition of $\{1, \ldots, N\}$ into two sets has the property that at least one of the two sets contains an arithmetical progression of length k. The behaviour of $N(k)$ as a function of k is not well understood although Schmidt [9] proved that

$$N(k) > 2^{k - c_4 (k \log k)^{\frac{1}{2}}}$$

and Berlekamp [2] proved that if k is a prime then

$$N(k) > k\, 2^k .$$

Roth suspected that Theorem 1 could be improved considerably. In particular he thought, see [10], that given $\delta > 0$ there should exist positive numbers $C_1(\delta)$ and $C_2(\delta)$, which depend on δ, such that Theorem 1 holds with $c_1 N^{\frac{1}{4}}$ in (1) replaced by $C_1(\delta) N^{\frac{1}{2} - \delta}$ and c_2 replaced by $C_2(\delta)$. Spencer [10] showed that there exists, for any positive integer N, a sequence $\varepsilon_1, \ldots, \varepsilon_N$ of plus and minus ones such that

$$\max_{a, q \in \mathbb{Z}^+} \Big| \sum_{\substack{j \geq 1 \\ 1 \leq a+jq \leq N}} \varepsilon_{a+jq} \Big| < c_5 \left(\frac{N \log\log N}{\log N} \right)^{\frac{1}{2}} \qquad (2)$$

and Sárközy (see [3], Chapter 8), disproved Roth's hypothesis by showing that for any positive integer N there is a sequence $\varepsilon_1, \ldots, \varepsilon_N$ of plus and minus ones for which (2) remains valid, with the right hand side of (2) replaced by $c_6 (N \log N)^{\frac{1}{3}}$. Recently Beck [1] showed that Roth's result is essentially best possible by proving that for any positive integer N there is a sequence, $\varepsilon_1, \ldots, \varepsilon_N$, of plus and minus ones for which (2) holds with the right hand side of (2) replaced by $c_7 N^{\frac{1}{4}} (\log N)^{5/2}$.

While Roth's lower bound is obtained by analytic means, Beck's result is a sophisticated piece of non-constructive combinatorics. Bech deduces his result from the following theorem. A hypergraph $\mathcal{H}$ is a finite family of finite sets. Given $\mathcal{H}$ put $S(\mathcal{H}) = \bigcup_{A \in \mathcal{H}} A$,

$$\mathrm{Deg}(\mathcal{H}, x) = |\{A \in \mathcal{H} \mid x \in A\}| ,$$

$$\mathrm{Deg}(\mathcal{H}) = \max_{x \in S(\mathcal{H})} \mathrm{Deg}(\mathcal{H}, x)$$

and

$$\mathrm{Disc}(\mathcal{H}) = \min_{g} \max_{A \in \mathcal{H}} \left| \sum_{x \in A} g(x) \right| ,$$

where the minimum above is taken over all functions g from $S(\mathcal{H})$ to $\{1, -1\}$.

THEOREM 2 (Beck [1]). - Let $\mathcal{H}$ be a hypergraph and assume that one can find a real number t such that

$$\mathrm{Deg}(\{ A \in \mathcal{H} : |A| \geq t \}) \leq t .$$

Then

$$\mathrm{Disc}(\mathcal{H}) \leq c_8 t^{\frac{1}{2}} (\log |\mathcal{H}|)^{\frac{1}{2}} \log |S(\mathcal{H})| .$$

Sarkőzy and I have recently generalized Roth's result. In place of arithmetical progressions we consider shifts and dilations of general sequences. Perhaps the most interesting such case is that of shifts and dilations of the sequence of squares.

THEOREM 3 (Sarkőzy and Stewart [8]). - Let δ be a positive real number, let N be a positive integer and let $\varepsilon_1, \ldots, \varepsilon_N$ be complex numbers of absolute value one. Then there exists an integer a and a positive integer q such that

$$\left| \sum_{\substack{j \geq 1 \\ 1 \leq a+j^2 q \leq N}} \varepsilon_{a+j^2 q} \right| > N^{1/6-\delta} , \qquad (3)$$

for $N > C_3(\delta)$, where $C_3(\delta)$ is a real number which is effectively computable in terms of δ .

We believe that the exponent $1/6$ in (3) cannot be replaced by a larger number.

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