SETS GENERATED BY FINITE SETS OF ALGEBRAIC NUMBERS

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For Professor Robert Tijdeman on the occasion of his seventy-fifth birthday.

1. Introduction

Let $S = \{p_1, \ldots, p_r\}$ be a finite set of prime numbers with $r \geq 2$ and let $(n_i)_{i=1}^{\infty}$ be the increasing sequence of positive integers composed of the primes from S. In 1973 [10] and 1974 [11] Tijdeman proved that there exist positive numbers c_1, c_2 and c_3, effectively computable in terms of S, such that for $n_i \geq c_3$

$$\frac{n_i}{(\log n_i)^{c_1}} < n_{i+1} - n_i < \frac{n_i}{(\log n_i)^{c_2}}. \tag{1}$$

Tijdeman [10] also resolved a question of Wintner by proving that there exist infinite sets of primes S for which the associated sequence $(n_i)_{i=1}^{\infty}$ satisfies

$$\lim_{i\to\infty} n_{i+1} - n_i = \infty,$$

see also [5].

In this note we shall study the distribution of the numbers formed when we take S to be a finite set of multiplicatively independent algebraic numbers of absolute value larger than 1 instead of a finite set of primes. Our first result corresponds to the lower bound in (1) and shows that such numbers are not close to each other.

Theorem 1. *Let $\alpha_1, \ldots, \alpha_r$ be multiplicatively independent algebraic numbers with $|\alpha_i| > 1$ for $i = 1, \ldots, r$. Put*

$$T = \{\alpha_1^{h_1} \cdots \alpha_r^{h_r} \mid h_i \geq 0 \quad \textit{for} \quad i = 1, \ldots, r\}.$$

There exists a positive number c, which is effectively computable in terms of $\alpha_1, \ldots, \alpha_r$, such that if t and t' are in T with $|t| \geq 3$ then

$$|t - t'| > |t|/(\log |t|)^c.$$

Theorem 1 follows directly from lower bounds for linear forms in the logarithms of algebraic numbers, see [1, 2, 3, 6, 7].

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We next obtain generalizations of the upper bound in (1). We consider two cases. For the first case we restrict our attention to sets of real algebraic numbers.

Theorem 2. *Let α_1 and α_2 be multiplicatively independent real algebraic numbers. Suppose $\alpha_i > 1$ for $i = 1, 2$ and put*

$$T = \{\alpha_1^{h_1}\alpha_2^{h_2} \mid h_i \geq 0 \quad \textit{for} \quad i = 1, 2\}.$$

There exists a positive number c_1, which is effectively computable in terms of α_1 and α_2, such that for any real number x with $x \geq 3$ there exists an element t of T with

$$|x - t| < x/(\log x)^{c_1}.$$

For the proof of Theorem 2 we modify the argument given by Tijdeman in [11].

Finally we consider the case when the elements of T are not all real.

Theorem 3. *Let α_1, α_2 and α_3 be multiplicatively independent algebraic numbers with $|\alpha_i| > 1$ for $i = 1, 2, 3$. Suppose that α_1 and α_2 are positive real numbers and that $\alpha_3/|\alpha_3|$ is not a root of unity. Put*

$$T = \{\alpha_1^{h_1}\alpha_2^{h_2}\alpha_3^{h_3} \mid h_i \geq 0 \quad \textit{for} \quad i = 1, 2, 3\}.$$

There exists a positive number c_2, which is effectively computable in terms of α_1, α_2 and α_3, such that for any complex number z with $|z| \geq 3$ there exists an element t of T with

$$|z - t| < |z|/(\log |z|)^{c_2}.$$

Observe that if α_1 and α_2 are real numbers and $\alpha_3/|\alpha_3|$ is a root of unity then there is a positive number c_4 and complex numbers z of arbitrarily large modulus for which

$$|z - t| > c_4|z|$$

for all elements t in T.

With Min Sha and Igor Shparlinski [8] we have applied both (1) and Theorem 3 in order to study the distribution of multiplicatively dependent vectors of algebraic numbers.

2. Linear forms in the logarithms of algebraic numbers

For any algebraic number α the height of α is the maximum of the absolute values of the relatively prime integer coefficients of the minimal polynomial of α. Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers of heights at most

$A_1, \ldots, A_n$ respectively. Let $b_1, \ldots, b_n$ be non-zero integers of absolute value at most B with $B \geq 2$. Put

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n$$

and

$$d = [\mathbb{Q}(\alpha_1, \ldots, \alpha_n) : \mathbb{Q}].$$

Baker [1, 2] and Feldman [3] proved the following result.

Lemma 4. *There is a positive number c, which depends on $A_1, \ldots, A_n, n$ and d, such that if $\Lambda \neq 0$ then*

$$|\Lambda| > B^{-c}.$$

For a sharp explicit dependence of c in Lemma 4 on the parameters $A_1, \ldots, A_n$, n and d, see Matveev [6, 7].

3. Proof of Theorem 1

Let $c_1, c_2, \ldots$ be positive numbers which are effectively computable in terms of $\alpha_1, \ldots, \alpha_r$. Let t be in T with $|t| \geq 3$. Then

$$t = \alpha_1^{h_1} \cdots \alpha_r^{h_r}$$

with $h_i \geq 0$ for $i = 1, \ldots, r$. Suppose t' is in T with $t' \neq t$. We have

$$t' = \alpha_1^{j_1} \cdots \alpha_r^{j_r}$$

with $j_i \geq 0$ for $i = 1, \ldots, r$.

Then

$$|t - t'| = |t| \left| \alpha_1^{j_1 - h_1} \cdots \alpha_r^{j_r - h_r} - 1 \right|.$$

Since $t \neq t'$ we may apply Lemma 4, as in Theorem A of [9], to give

$$|t - t'| > |t| B^{-c_1}, \tag{2}$$

where

$$B = \max(4, |j_1 - h_1|, \ldots, |j_r - h_r|).$$

We may suppose that $|t'| \leq 2|t|$ since otherwise the result holds and thus

$$B < c_2 \log |t|. \tag{3}$$

Our result now follows from (2) and (3) since $|t| \geq 3$.

4. A preliminary result for the proof of Theorem 2

Let α_1 and α_2 be multiplicatively independent real algebraic numbers with $\alpha_i > 1$ for $i = 1, 2$. Let $\ell_0/k_0, \ell_1/k_1, \ldots$ be the sequence of convergents to $\log\alpha_1/\log\alpha_2$. Our next result gives a bound on the growth of the k_i's. The proof depends upon Lemma 4 and is due to Tijdeman [11] when α_1 and α_2 are distinct primes.

Lemma 5. *There exists a positive number c, which is effectively computable in terms of α_1 and α_2, such that*

$$k_{j+1} < k_j^c$$

for $j = 2, 3, \ldots$.

Proof. Replacing $\log p/\log q$ by $\log\alpha_1/\log\alpha_2$ in the proof of the Lemma in [11] and noting that $k_j \geq 2$ for $j \geq 2$ we obtain the result. □

5. Proof of Theorem 2

Let $c_1, c_2, \ldots$ denote positive numbers which are effectively computable in terms of α_1 and α_2. Let x be a real number with $x \geq 3$ and let t be the largest element of T with $t \leq x$. Then

$$\frac{x}{\max(\alpha_1, \alpha_2)} \leq t$$

and so

$$\frac{1}{2}\log x < \log t, \tag{4}$$

for $x > c_1$.

We have

$$t = \alpha_1^{h_1}\alpha_2^{h_2}$$

with h_1 and h_2 non-negative integers. We may assume, without loss of generality, that

$$\alpha_1^{h_1} \geq t^{1/2}$$

and so

$$h_1 \geq \frac{1}{2\log\alpha_1}\log t. \tag{5}$$

Let $\frac{\ell_0}{k_0}, \frac{\ell_1}{k_1}, \ldots$ be the sequence of convergents from the continued fraction expansion of $\log\alpha_1/\log\alpha_2$. Recall that the convergents with even index are smaller that $\log\alpha_1/\log\alpha_2$ and those with odd index are larger. Choose j to be the largest odd integer for which

$$k_j \leq h_1; \tag{6}$$

certainly $k_1 \leq h_1$ for $x > c_2$. Then k_{j+2} exceeds h_1 and by (4) and (5)

$$k_{j+2} > \frac{1}{4\log\alpha_1}\log x$$

for $x > c_3$. By Lemma 5, $k_{j+2} < k_{j+1}^{c_4}$ and so

$$k_{j+1} > \left(\frac{1}{4\log\alpha_1}\log x\right)^{1/c_4}$$

hence

$$k_{j+1} > (\log x)^{c_5} \tag{7}$$

for $x > c_6$.

Put

$$t' = \alpha_1^{h_1-k_j}\alpha_2^{h_2+\ell_j}$$

and note that by (6) t' is in T. Further since $t < t'$ we have $x < t'$. By Theorem 167 and Theorem 171 of [4],

$$0 < \frac{\ell_j}{k_j} - \frac{\log\alpha_1}{\log\alpha_2} < \frac{1}{k_j k_{j+1}}$$

and so

$$\log(t'/t) < \frac{\log\alpha_2}{k_{j+1}}. \tag{8}$$

It follows from (7) and (8) that $\log(t'/t) < \frac{1}{4}$ hence

$$\log(t'/t) = \log(1 + (t'-t)/t) > \frac{t'-t}{2t} \tag{9}$$

and thus, by (8) and (9),

$$t' - t < \frac{(2\log\alpha_2)t}{k_{j+1}}, \tag{10}$$

for $x > c_6$. Recall (7) and that $t \leq x < t'$. We see from (10) that

$$x - t < c_7\frac{x}{(\log x)^{c_8}} < \frac{x}{(\log x)^{c_9}}$$

for $x > c_{10}$. Note that this suffices to prove Theorem 2 since

$$x - t \leq x - 1 \leq \frac{x}{\left(1 + \frac{1}{x-1}\right)} \leq \frac{x}{(\log x)^{c_{11}}}$$

for $3 \leq x \leq c_{10}$.

6. Preliminaries for the proof of Theorem 3

For any non-zero complex number z let $\mathrm{Im}(z)$ denote the imaginary part of z, let $\mathrm{Arg}(z)$ denote the argument of z chosen so that $0 \le \mathrm{Arg}(z) < 2\pi$ and let $\log z$ denote the principal branch of the logarithm so that $0 \le \mathrm{Im}(\log z) < 2\pi$.

Let ν be a real number with $0 \le \nu < 2\pi$ and let α be an algebraic number with $|\alpha| > 1$ for which $\alpha/|\alpha|$ is not a root of unity. For each positive integer k let b_k be the smallest positive integer for which

$$\left|\mathrm{Arg}(\alpha^{b_k}) - \nu\right| \le \frac{2\pi}{k}.$$

Lemma 6. *There exists a positive number c, which is effectively computable in terms of α, such that*

$$b_k < k^c$$

for $k \ge 2$.

Proof. Suppose $k \ge 2$. By Dirichlet's box principle there exists an integer r_k with $1 \le r_k \le k$ and an integer m_k so that

$$\left| r_k \log \frac{\alpha}{|\alpha|} - m_k 2\pi i \right| \le \frac{2\pi}{k}.$$

Notice that we have $0 \le m_k \le r_k$.

Let $c_1, c_2 \dots$ denote positive numbers which are effectively computable in terms of α. By Lemma 4

$$\left| r_k \log \frac{\alpha}{|\alpha|} - 2m_k \log(-1) \right| \ge \frac{1}{(2r_k)^{c_1}} \ge \frac{1}{k^{c_2}}.$$

If

$$0 < \frac{1}{i}\left(r_k \log \frac{\alpha}{|\alpha|} - m_k 2\pi i \right) \le \frac{2\pi}{k}$$

then there exists an integer q_k with $1 \le q_k \le 2\pi k^{c_2}$ for which

$$\left| q_k \left(r_k \log \frac{\alpha}{|\alpha|} - 2m_k \log(-1) \right) - \nu i \right| \le \frac{2\pi}{k}.$$

On the other hand if $r_k \log \frac{\alpha}{|\alpha|} - 2m_k \log(-1)$ is of the form yi with $-\frac{2\pi}{k} \le y < 0$ then there exists an integer q_k with $1 \le q_k \le 2\pi k^{c_2}$ for which

$$\left| 2\pi i + q_k \left(r_k \log \frac{\alpha}{|\alpha|} - 2m_k \log(-1) \right) - \nu i \right| \le \frac{2\pi}{k}.$$

Therefore, since

$$\log(\frac{\alpha}{|\alpha|})^{q_k r_k} = i\mathrm{Arg}(\alpha^{q_k r_k})$$

and since $k \ge 2$,

$$b_k \le q_k r_k \le 2\pi k^{1+c_2} < k^{c_3}$$

as required. □

7. Proof of Theorem 3

Let z be a complex number with $|z| \geq 3$ and put $\nu = \mathrm{Arg}(z)$. Let $b_1, b_2, \ldots$ be defined as in §6 with α replaced by α_3. Let $c_1, c_2, \ldots$ denote positive numbers which are effectively computable in terms of α_1, α_2 and α_3. It suffices, as in the proof of Theorem 2, to establish our result for $|z| > c_1$. In particular we may suppose that $|z|$ exceeds $\max(9, |\alpha_3|^{2b_2})$. We now choose k so that

$$|\alpha_3|^{2b_k} \leq |z| < |\alpha_3|^{2b_{k+1}}; \tag{11}$$

since $|z|$ exceeds $|\alpha_3|^{2b_2}$ and since the sequence $(b_i)_{i=1}^{\infty}$ is non-decreasing k is well defined. By the definition of b_k we have

$$|\mathrm{Arg}(\alpha_3^{b_k}) - \mathrm{Arg}(z)| \leq \frac{2\pi}{k}.$$

Further

$$3 \leq |z|^{1/2} \leq |z|/|\alpha_3|^{b_k} < |z|. \tag{12}$$

We now choose non-negative integers j_1 and j_2 such that

$$\left| \alpha_1^{j_1} \alpha_2^{j_2} - \frac{|z|}{|\alpha_3|^{b_k}} \right| < \frac{|z|/|\alpha_3|^{b_k}}{(\log|z|)^{c_2}} \tag{13}$$

which is possible by (12) and Theorem 2. Put

$$t = \alpha_1^{j_1} \alpha_2^{j_2} \alpha_3^{b_k}.$$

Notice that by (13)

$$||z| - |t|| < \frac{|z|}{(\log|z|)^{c_2}}. \tag{14}$$

In addition, since $\mathrm{Arg}(t) = \mathrm{Arg}(\alpha_3^{b_k})$,

$$|\mathrm{Arg}(z) - \mathrm{Arg}(t)| \leq \frac{2\pi}{k}. \tag{15}$$

On the other hand, by (11),

$$\log|z| < 2b_{k+1} \log|\alpha_3|$$

and so by Lemma 6

$$\log|z| < k^{c_3} \tag{16}$$

since $k \geq 2$. Thus by (15) and (16)

$$|\mathrm{Arg}(z) - \mathrm{Arg}(t)| < \frac{2\pi}{(\log|z|)^{c_4}}. \tag{17}$$

Our result now follows from (14) and (17).

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