# Spin Geometry

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## 1 Course Outline

Originally arising from Dirac's relativistic equation for fermions and later developed in the work of Atiyah and Singer, spin geometry provides a unifying framework connecting differential geometry, global analysis, topology, and mathematical physics. Dirac operators play a central role in many areas of mathematics, with applications ranging from obstructions to positive scalar curvature, to Hodge theory, index theory, and the study of geometric structures such as Kähler and special holonomy manifolds.

This course offers a systematic introduction to spin geometry with a particular emphasis on index theory and geometric applications of Dirac operators. The aim is to illustrate how analytic objects defined locally on a manifold can encode global topological information and lead to strong geometric consequences.

The course begins with the algebraic and geometric foundations of spin geometry, including Clifford algebras, spin and spin$^c$ structures, and the construction of spinor bundles on Riemannian manifolds. Within this framework, Dirac operators are introduced as natural first-order elliptic differential operators associated to Riemannian geometry. Their basic properties are discussed, along with geometric identities relating Dirac operators to curvature, which already lead to striking applications such as obstructions to the existence of certain metrics.

A central theme of the course is the index of Dirac-type operators. After introducing the necessary analytic background, we study how the index can be computed using heat kernel methods and how it is related to topological invariants of the underlying manifold and vector bundles. This culminates in the Atiyah-Singer index theorem for Dirac operators, illustrating in a concrete and geometric way how local differential-geometric data gives rise to global topological information.

Depending on time and audience, the course may include further examples and applications, such as the interpretation of classical results from Hodge theory in the language of Dirac operators, geometric obstructions arising from spin geometry, or brief outlooks toward more advanced topics.

The course aims to provide both a solid foundation in spin geometry and a conceptual understanding of index theory as a fundamental tool in modern geometric analysis.

## 2 Requirements

The course is intended for master's and graduate students with a background in differential geometry and basic functional analysis. Students should be familiar with

- smooth manifolds, differential forms, vector bundles, and connections,
- basic Riemannian geometry (Riemannian metrics, Levi–Civita connection, curvature),
- Lie groups and Lie algebras,
- basic functional analysis, including Banach and Hilbert spaces and bounded linear operators,
- de Rham cohomology.

## 3 Grading

There will be several graded homework assignments.