ON THE REALIZABILITY OF STEENROD MODULES

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LET p be any prime, and let $\mathscr{A}$ be the mod p Steenrod algebra (8). A problem which arises is to give an algebraic characterization of those graded $\mathscr{A}$-modules M which can occur as the cohomology of some space. If we define the rank of M to be its dimension as an ungraded $\mathbf{Z}_p$-space, the mod p Hopf invariant problem [see (1), (3), (6), and (7)] can be seen as the special case of this problem for modules of rank 2. Using the stable version of the K-theoretic method in (3), we shall prove that the 'finiteness' aspect of the solution to the Hopf invariant problem generalizes to modules of any finite rank.

DEFINITION. *A TFR module is a graded $\mathscr{A}$-module M such that $M \cong \tilde{\mathbf{H}}^*(X; \mathbf{Z}_p)$ for some space X whose integral cohomology has no p-torsion, where the isomorphism may raise or lower degree.*

A direct sum of TFR modules is obviously TFR.

THEOREM 1. *Any collection of non-isomorphic TFR modules of finite rank k, none of which is a non-trivial direct sum of TFR modules, is finite.*

This will follow easily from the following more technical result.

THEOREM 2. *For any positive integer l there exist at most finitely many sequences $0 = w_0 < w_1 < \ldots < w_l$ of positive integers for which there is a space X and an integer n such that*

(i) $\mathbf{H}^*(X; \mathbf{Z})$ *is finitely generated and free of p-torsion;*

(ii) $\tilde{\mathbf{H}}^{2q}(X; \mathbf{Q}) = 0$ *unless $q = n + w_i$ for some i;*

(iii) *there exists t such that*

$$\mathscr{P}^t\colon \mathbf{H}^{2n}(X; \mathbf{Z}_p) \to \mathbf{H}^{2n+2w_l}(X; \mathbf{Z}_p)$$

is defined and non-zero.

To apply K-theory we use the relation between the Steenrod and Adams operations developed by Atiyah in (4).

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THEOREM 3. (Atiyah.) *Let p be a prime and let X be a finite CW-complex with $\mathbf{H}^*(X;\mathbf{Z})$ torsion free. Then if $x \in \mathbf{K}_{2n}(X)$, there exist*

$$x_t \in \mathbf{K}_{2n+2t(p-1)}(X) \quad \textit{for } 0 \leqslant t \leqslant n$$

such that

$$\psi^p(x) = \sum_{t=0}^{n} p^{n-t} x_t.$$

If σ_r is the composite

$$\mathbf{K}_{2r}(X) \to \mathbf{K}_{2r}(X)/\mathbf{K}_{2r+2}(X) \cong \mathbf{H}^{2r}(X) \to \mathbf{H}^{2r}(X;\mathbf{Z}_p)$$

then

$$\mathscr{P}^t \sigma_n(x) = \sigma_{n+t(p-1)}(x_t).$$

Here ψ^p is the Adams operation introduced in (**2**), $\mathscr{P}^t = \mathscr{S}q^{2t}$ if $p = 2$, and the filtration in complex $\mathbf{K}$-theory is defined by

$$\mathbf{K}_{2r}(X) = \operatorname{Ker}[\mathbf{K}(X) \to \mathbf{K}(X^{2r-1})] \quad (r \geqslant 0).$$

The tactic then is to show that a large power of p divides $\psi^p(x)$ most of the time. Let $\nu_p(a)$ denote the power of p in a, where a is a non-zero element of a free abelian group. As in (**3**), define, for any r-tuple $(m_1,\ldots,m_r)$ of distinct non-negative integers, and for $1 \leqslant i \leqslant r$,

$$d_i(m_1,\ldots,m_r) = \text{g.c.d.}\Big\{\prod_{j\neq i}(k_j^{m_j}-k_j^{m_i}) \mid k_j \geqslant 2\Big\}.$$

LEMMA 4. *Let X be a torsion free finite CW-complex such that $\mathbf{H}^{2k}(X;\mathbf{Q}) = 0$ except for $k = 0, m_1, m_2,\ldots, m_r$. Then for any non-zero element $x \in \tilde{\mathbf{K}}(X)$*

$$\nu_p[\psi^p(x)] \geqslant \min_{1\leqslant i\leqslant r}\{m_i - \nu_p[d_i(m_1,\ldots,m_r)]\}.$$

Proof. As in (**3**), and using their numbering, decompose

$$\tilde{\mathbf{K}}(X)\otimes\mathbf{Q} \quad \text{as} \quad \bigoplus_1^r V_i, \tag{2.3}$$

where V_i is the eigenspace of ψ^k corresponding to eigenvalue k^{m_i} for any $k > 1$. Then

$$\psi^p(x) = \sum_1^r p^{m_i}\pi_i(x),$$

where

$$\pi_i = \prod_{j\neq i}\left[\frac{\psi^{k_j}-k_j^{m_j}}{k_j^{m_i}-k_j^{m_j}}\right] \tag{2.4}$$

for any sequence $k_1,\ldots, k_{i-1}, k_{i+1},\ldots, k_r$ of integers larger than 1. Then if x is non-zero, and lies in the image of $\tilde{\mathbf{K}}(X)$ in $\tilde{\mathbf{K}}(X)\otimes\mathbf{Q}$,

$$\nu_p[\psi^p(x)]$$
$$\geqslant \min\{m_i - \nu_p[\pi(k_{ij}^{m_i}-k_{ij}^{m_j})] \mid 1 \leqslant i \leqslant r;\ \pi_i(x) \neq 0;\ k_{ij} > 1,\ i \neq j\}.$$

The conclusion is immediate.

PROPOSITION 5. *For any prime p and positive integer l, there exist at most finitely many sequences $0 = w_0 < w_1 < \ldots < w_l$ of integers such that for some n,*

$$\frac{w_l}{p-1} \leqslant \max_{1\leqslant i\leqslant l+1}\{\nu_p[d_i(n+w_0,\ldots,n+w_l)] - w_{i-1}\}.$$

Proof. Given p and l, choose an integer a such that

$$p^{b-1} \geqslant l(p-1)(b+2) \quad \text{for all } b > a.$$

Suppose given a sequence w_i as above. Choose c such that $p^{c-1} < w_l \leqslant p^c$. Choose d prime to p such that $\nu_p(d^e-1) \leqslant \nu_p(e)+2$ for any non-zero e. Then

$$\frac{p^{c-1}}{p-1} < \frac{w_l}{p-1} \leqslant \nu_p[d_i(n+w_0,\ldots,n+w_l)] \quad (\text{for some } i)$$
$$\leqslant \nu_p\Big[\prod_{j\neq i-1}(d^{n+w_j}-d^{n+w_{i-1}})\Big] = \sum_{j\neq i-1}\nu_p(d^{w_j-w_{i-1}}-1)$$
$$\leqslant \sum_{j\neq i-1}[\nu_p(w_j-w_{i-1})+2] \leqslant (c+2)l.$$

Thus $c \leqslant a$, w_l is bounded by p^a, and the proof is complete.

Given 'extended' integers $a \leqslant b$ and a graded $\mathscr{A}$-module M, define another module $M^{a,b}$:

$$M_k^{a,b} = \begin{cases} M_k & \text{if } a \leqslant k \leqslant b,\\ 0 & \text{otherwise,}\end{cases}$$

where the action of $\theta \in \mathscr{A}$ on $M^{a,b}$ is given by the composite

$$M^{a,b} \xrightarrow{\alpha} M \xrightarrow{\theta} M \xrightarrow{\beta} M^{a,b},$$

α and β being the canonical injection and projection.

LEMMA 6. *If M is a finite TFR module, then for any a, b there exists a CW-complex X such that*

(i) $\tilde{\mathbf{H}}^*(X;\mathbf{Z}_p) \cong M^{a,b}$;

(ii) *$\mathbf{H}^*(X;\mathbf{Z})$ is torsion free;*

(iii) *the number of cells of dimension n in X is the rank of $\mathbf{H}^n(X;\mathbf{Z})$.*

Proof. By the p-primary approximation theorem [(**5**) Theorem 2.2.1], we can replace a p-torsion free space which realizes M by a simply connected torsion-free CW-complex Y which realizes M and satisfies (iii). So the theorem holds for $a = -\infty$, $b = \infty$ with $X = Y$. Since $\mathbf{H}^c(Y/Y^{c-1})$ is torsion free, and has rank no larger than the rank of $\mathbf{H}^c(Y)$ by (iii) for Y, we have $\mathbf{H}^c(Y/Y^{c-1}) \cong \mathbf{H}^c(Y)$ and $\mathbf{H}^{c-1}(Y) \cong \mathbf{H}^{c-1}(Y^{c-1})$. Thus Y^{c-1} and Y/Y^{c-1} realize $M^{-\infty,c-1}$ and $M^{c,\infty}$ respectively, and satisfy (ii) and (iii). Now for general a, b, let $X = Y^b/Y^{a-1}$.

Proof of Theorem 2. Suppose given such a sequence, with corresponding n and X. We may suppose that X is a finite CW-complex with torsion-free integral cohomology, by Lemma 6. Then $w_l = t(p-1)$, and since $\mathscr{P}^t \neq 0$, we deduce from Theorem 3 that for some non-zero $x \in \mathrm{K}_{2n}(X)$

$$\nu_p[\psi^p(x)] = n-t = n-\frac{w_l}{p-1}.$$

But by Lemma 4

$$\nu_p[\psi^p(x)] \geqslant \min_{1\leqslant i\leqslant l+1} \{n+w_{i-1}-\nu_p[d_i(n+w_0,\ldots, n+w_l)]\}.$$

The conclusion then follows from Proposition 5.

Proof of Theorem 1. Given a non-zero graded $\mathscr{A}$-module M of finite rank, define

$$\operatorname{span}(M) = \max\{j-i \mid M_i \neq 0 \neq M_j\}.$$

We note that any collection of non-isomorphic $\mathscr{A}$-modules bounded in rank and span is necessarily finite.

The theorem certainly holds for $k = 0$ and $k = 1$. We proceed by induction on k. By letting l vary from 2 to k in Theorem 2, it follows that any collection of non-isomorphic TFR modules of rank k, each of which has the property that $\mathscr{P}^t \neq 0$ on M for $2t(p-1) = \operatorname{span}(M)$, must necessarily be finite. So we may assume that we are dealing with a collection of non-isomorphic TFR modules M each of which satisfies

(i) M has rank k and is not the direct sum of TFR modules of rank less than k;

(ii) if $\mathscr{P}^t\colon M \to M$ is non-zero, $\operatorname{span}(M) > 2t(p-1)$. We wish to show that this collection is finite. It suffices to find an upper bound for the spans of modules in the collection. By our inductive hypothesis, there is an integer t_0 such that $\mathscr{P}^t$ acts trivially on any TFR module of rank less than k for all $t \geqslant t_0$. We show that $2kt_0(p-1)$ is the upper bound we seek.

Suppose M is a TFR module of rank k satisfying (ii) such that

$$\operatorname{span}(M) > 2kt_0(p-1).$$

Then there exist integers c and d such that

$$M_c \neq 0 \neq M_d, \quad d-c > 2t_0(p-1), \quad \text{and} \quad M_e = 0 \quad \text{for } c < e < d.$$

By (ii), if $\mathscr{P}^t$ is non-zero on M, then $\mathscr{P}^t$ is non-zero on $M^{a,b}$ for a and b such that $\operatorname{span}(M^{a,b}) < \operatorname{span}(M)$, and so $\operatorname{rank}(M^{a,b}) < \operatorname{rank}(M)$. Since, by Lemma 6, $M^{a,b}$ is also TFR, we must have $t < t_0$. So there are no non-zero operations from M_f to M_g if $f \leqslant c$ and $g \geqslant d$. So M decomposes

$$M \cong M^{-\infty,c} \oplus M^{d,\infty}$$

as a direct sum of non-zero TFR modules, and the proof is complete.

Remarks. If n is a power of 2, there is sometimes a CW-complex $S^n \cup_\phi CY$, where $Y = S^{2n-2} \cup_2 e^{2n-1}$, on which $\mathscr{S}q^n$ acts non-trivially. It is not yet known whether this happens for infinitely many n. If not, then one might be so courageous as to conjecture the generalization of Theorem 1 where the condition that the realizing spaces be torsion free is dropped. These methods can be used to make some progress in classifying TFR modules (up to degree-changing isomorphism, of course), but we have no general results in this direction.

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