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A NOTE ON LARGE GAPS BETWEEN CONSECUTIVE PRIME NUMBERS

by Cameron L. STEWART (*)

In 1938, RANKIN [7], improving on an earlier result of ERDÖS [3], showed that for each $\epsilon > 0$ there exist infinitely many integers n such that

(1) $$p_{n+1} - p_n > (1/3 - \epsilon) \log p_n \log_2 p_n \log_4 p_n/(\log_3 p_n)^2 ,$$

where p_n denotes the n-th largest prime number and $\log_i = \log(\log_{i-1})$ denotes the i-th iteration of the logarithm function. In 1963, SCHÖNHAGE [9] showed that it is possible to replace $1/3$ in (1) by $e^\gamma/2$ where γ is Euler's constant, and RANKIN [8] in the same year improved $e^\gamma/2$ to e^γ $(= 1.78...)$. In this note, I would like to give a simple proof of (1) with $1/3$ replaced by $1/2$.

Let t be a positive integer and let ϵ be a real number with $0 < \epsilon < 1/2$. Denote the interval $(t^{(1-\epsilon)\log_3 t/\log_2 t} , t/\log t)$ by T, and put $k = \prod_{p \in T} p$, where the product is taken over primes p. Note that by the prime number theorem

(2) $$k = e^{(1+o(1))t/\log t} .$$

The number of integers from $1, \ldots, t$ which are coprime with k is at most $N_1 + N_2$ where N_1 is the number of positive integers less than or equal to t all of whose prime factors are less than $t^{(1-\epsilon)\log_3 t/\log_2 t}$ and where N_2 is the number of positive integers less than or equal to t and having a prime factor greater than $t/\log t$. It follows from work of De BRUIJN (in particular (1.3) and (1.4) of [1] and (1.8) of [2]), that $N_1 = t/u^{u(1+o(1))}$ with $u = \log_2 t/(1-\epsilon)\log_3 t$, hence $N_1 = o(t/\log t)$. Further,

$$N_2 = \sum_{t/\log t < p \leq t} \left[\frac{t}{p}\right] \leq t \sum_{t/\log t < p \leq t} \frac{1}{p} ,$$

hence, (by 22.7.3 and 22.7.4 of [4]),

$$N_2 \leq t(\log_2 t - \log(\log t - \log_2 t) + O(1/\log t))$$

$$\leq t(- \log(1 - (\log_2 t/\log t)) + O(1/\log t))$$

$$\leq t \log_2 t/\log t + O(t/\log t) .$$

Therefore

(3) $$N_1 + N_2 \leq (1 + o(1))\, t \log_2 t/\log t .$$

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Let S denote the number of primes which are of the form $kz + \ell$ with $1 \leqslant \ell \leqslant t$ and $1 \leqslant z \leqslant k^{[\log_2 t]}$. By the Brun-Titchmarsh theorem (see Theorem 2 of [6]),

(4) $$S \leqslant (N_1 + N_2)\ 2k^{1+[\log_2 t]}/\Phi(k) \log k^{[\log_2 t]} .$$

Since $\Phi(k) = k \prod_{p \in T} (1 - \frac{1}{p})$ we have, by Merten's theorem (Theorem 429 of [4]),

(5) $$\Phi(k) = k(1 - \epsilon + o(1)) \log_3 t/\log_2 t .$$

We find, from (2), (3), (4) and (5), that

$$S \leqslant (2/(1 - \epsilon) + o(1))(\log_2 t/\log_3 t)k^{[\log_2 t]} .$$

Thus for some integer z_0 with $1 \leqslant z_0 \leqslant k^{[\log_2 t]}$ the interval $(kz_0 + 1 , kz_0 + t)$ will contain at most $(2/(1 - \epsilon) + o(1))(\log_2 t/\log_3 t)$ primes and so in this interval there will be a gap between consecutive primes, p_n and p_{n+1} say, of size at least $((1 - \epsilon)/2 + o(1))\ t \log_3 t/\log_2 t$. But $p_n \leqslant k^{1+[\log_2 t]} + t$ hence, recall (2),

(6) $$\log p_n \leqslant (1 + o(1))\ t \log_2 t/\log t ,$$

and so, $p_{n+1} - p_n \geqslant (\frac{1}{2} - \frac{\epsilon}{2} + o(1)) \log p_n \log t \log_3 t/(\log_2 t)^2$. Our result now follows since, by (6), $\log_2 p_n \leqslant (1 + o(1)) \log t$.

The key estimates in the above argument is the estimate for N_1 which allows us to sieve the integers from $1 , \ldots , t$ by primes from T in an efficient manner. A similar estimate also appears in the proofs of RANKIN ([7], [8]) and of SCHÖNHAGE [9]. RANKIN and SCHÖNHAGE construct, by a sieving process and the Chinese Remainder Theorem, long intervals free of primes whereas we use the Brun-Titchmarsh theorem and an averaging argument to this end. MAIER [5] has used a related averaging argument to show that for each positive integer j there is a positive number c_j such that for infinitely many integers n,

$$p_{n+i+1} - p_{n+i} > c_j \log p_n \log_2 p_n \log_4 p_n/(\log_3 p_n)^2 , \quad \text{for } i = 0 , \ldots , j - 1 .$$

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