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WELL SPACED INTEGERS GENERATED BY AN INFINITE SET OF PRIMES

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ABSTRACT. In this article we prove that there exists an infinite set of prime numbers with the property that the sequence $1 = n_1 < n_2 < \cdots$ of positive integers made up of primes from the set is well spaced. For example we prove that there is an infinite set of prime numbers for which

$$n_{i+1} - n_i > n_i / \exp((\log n_i)^{1/2})$$

for $i = 1, 2, \ldots$.

1. Introduction

Let T be a set of prime numbers and let $1 = n_1 < n_2 < \cdots$ be the sequence of positive integers all of whose prime factors are from T. Let $|T|$ denote the cardinality of T. If T is a finite set, Tijdeman [15] proved in 1973, improving earlier work of Pólya [11] and Siegel [13], that there exist positive numbers $C_1 = C_1(T)$ and $C_2 = C_2(T)$, which are effectively computable in terms of T, such that

$$n_{i+1} - n_i > n_i/(\log n_i)^{C_1} \tag{1.1}$$

for $n_i \geq 3$ and

$$n_{i+1} - n_i < n_i/C_2(\log n_i)^{|T|-1} \tag{1.2}$$

for infinitely many positive integers i. Further, Tijdeman [15] proved in 1974 that there are positive numbers C_3 and C_4, which are effectively computable in terms of T, such that if $n_i > C_3$, then

$$n_{i+1} - n_i < n_i/(\log n_i)^{C_4}.$$

Prior to this Wintner had asked Erdős [3] if there is an infinite set of primes T for which

$$\lim_{i\to\infty} n_{i+1} - n_i = \infty.$$

Tijdeman [15] answered Wintner's question in the affirmative. He proved that for each real number α with $0 < \alpha < 1$ there exists an infinite set of primes $T(\alpha)$ for which

$$n_{i+1} - n_i > n_i^{1-\alpha} \tag{1.3}$$

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for $i = 1, 2 \ldots$. In this note we would like to refine (1.3) in two ways. The first refinement is an elaboration of Tijdeman's result and concerns the size of the primes in $T(\alpha)$. We shall prove that for each positive real number ε there is a positive number $C = C(\alpha, \varepsilon)$ and an infinite set of primes $T(\alpha) = \{p_1, p_2, \ldots\}$ with

$$p_r < Cr^{(2/\alpha+\varepsilon)r^2}$$

for which (1.3) holds. More precisely, we shall prove the following.

Theorem 1.1. *Let α be a real number with $0 < \alpha < 1$. There is a positive number $c = c(\alpha)$, which is effectively computable in terms of α, and an infinite set of prime numbers $T(\alpha) = \{p_1, p_2, \ldots\}$ with*

$$t_r/2 < p_r \leq t_r,$$

where

$$t_r = r^{2r^2/\alpha}(\log 3r)^{cr^2} \tag{1.4}$$

for $r = 1, 2, \ldots$ such that if $1 = n_1 < n_2 < \cdots$ is the sequence of positive integers all of whose prime factors are from $T(\alpha)$, then

$$n_{i+1} - n_i > n_i^{1-\alpha} \tag{1.5}$$

for $i = 1, 2, \ldots$.

Our second refinement involves strengthening inequality (1.3). We shall prove that it is possible to find infinite sets of primes T for which the positive integers generated by them are more widely spaced than (1.3).

In order to state our next result we introduce the set $\mathcal{E}$ of continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ for which $\varphi(\varphi(x)) = e^x$. $\mathcal{E}$ is an uncountable set and it is possible to characterize the elements of $\mathcal{E}$. In particular φ is injective since the exponential function is injective. Further, since the image of $\mathbb{R}$ under the exponential function is $(0, \infty)$ and φ is continuous, the image of $\mathbb{R}$ under φ is (θ, ∞) for some negative real number θ. Therefore the image of $(-\infty, \theta]$ is $(\theta, 0]$. But notice that once we have specified a negative real number θ and a continuous injective map φ from $(-\infty, \theta]$ to $(\theta, 0]$, we can extend φ to an element of $\mathcal{E}$ in the following recursive manner. Put $e_1(x) = e^x$ and

$$e_k(x) = e^{e_{k-1}(x)} \quad \text{for } k = 2, 3, \ldots.$$

Then put $\alpha_0 = \theta$, $\alpha_1 = 0$ and

$$\alpha_{2i} = e_i(\theta), \quad \alpha_{2i+1} = e_i(0) \quad \text{for } i = 1, 2, \ldots.$$

Next put $I_{-1} = (-\infty, \theta]$ and $I_j = (\alpha_j, \alpha_{j+1}]$ for $j = 0, 1, 2, \ldots$ and map φ from I_j to I_{j+1} for $j = 1, 2, \ldots$ by defining $\varphi(x)$ for x in I_j to be e^y, where y is in I_{j-1} and $\varphi(y) = x$; see Hardy [4] for such a construction.

Observe that since φ is a continuous and injective map from $(-\infty, \theta]$ to $(\theta, 0]$ it is increasing on $(-\infty, \theta]$ and in fact on all of $\mathbb{R}$. Let $\varphi^{-1} : (\theta, \infty) \to \mathbb{R}$ denote the function which is the compositional inverse of φ so that $\varphi \circ \varphi^{-1}$ is the identity function on (θ, ∞) and $\varphi^{-1} \circ \varphi$ is the identity function on $\mathbb{R}$. Then φ^{-1} is injective and increasing. In particular, if $0 < x < y$, then $\varphi^{-1}(x) < \varphi^{-1}(y)$. While there is no function φ in $\mathcal{E}$ which can be extended to an analytic function on $\mathbb{C}$, Kneser [6] showed that there are such functions which are real analytic. Szekeres [14] proposed a way to pick an element from $\mathcal{E}$ in a canonical manner; see also [2].

We shall be interested in the growth of functions from $\mathcal{E}$, a topic studied by Hardy [4]. He proved that their rate of growth cannot be measured by the logarithmic-exponential scale. Put $\ell_1(x) = \log x$ for $x > 1$ and put

$$\ell_k(x) = \log(\ell_{k-1}(x)) \quad \text{for } x > e_{k-1}(1)$$

for $k = 2, 3, \ldots$. Then for φ in $\mathcal{E}$ and k a positive integer,

$$e_k((\ell_k(x))^2) < \varphi(x) < e_{k+1}((\ell_k(x))^{1/2}) \tag{1.6}$$

for x sufficiently large, as is readily checked. Functions from $\mathcal{E}$ have arisen in studies of computational complexity [9].

Let φ be in $\mathcal{E}$. We denote by $F_\varphi(x)$ the function from $\mathbb{R}^+$ to $\mathbb{R}^+$ which is 1 for x at most e^e and is given by

$$F_\varphi(x) = \varphi(\log x \log\log x)^{\log\log x} \tag{1.7}$$

for $x > e^e$.

Theorem 1.2. *Let $\varphi \in \mathcal{E}$. There exists an infinite set of primes T such that if $1 = n_1 < n_2 < \cdots$ is the sequence of positive integers all of whose prime factors are from T, then*

$$n_{i+1} - n_i > n_i / F_\varphi(n_i). \tag{1.8}$$

Let k be a positive integer and let θ be a real number with $0 < \theta < 1$. It can be checked that when φ is in $\mathcal{E}$, then

$$F_\varphi(x) < e_k((\ell_k(x))^\theta)$$

for x sufficiently large. As a consequence we are able to deduce the following result.

Corollary 1.3. *Let k be a positive integer. There is an infinite set of primes T_k such that if $1 = n_1 < n_2 < \cdots$ is the sequence of positive integers all of whose prime factors are from T_k, then*

$$n_{i+1} - n_i > n_i / e_k((\ell_k(n_i))^{1/2}) \tag{1.9}$$

for $i > 1$.

It follows from (1.2) that we cannot replace $e_k((\ell_k(n_i))^{1/2})$ in (1.9) by $(\log n_i)^c$ with c a positive real number for any infinite set of primes T. We remark that the exponent $1/2$ in (1.9) can be replaced by θ for any real number θ with $0 < \theta < 1$, provided that T_k is replaced by $T_{k,\theta}$. Further, on taking $k = 1$ in Corollary 1.3 one finds that there is an infinite set of primes for which

$$n_{i+1} - n_i > n_i / \exp((\log n_i)^{1/2}) \tag{1.10}$$

for $i = 1, 2, \ldots$.

For the proofs of Theorems 1.1 and 1.2 we shall employ the strategy Tijdeman used to prove (1.3). The key ingredient in the proofs is a lower bound for linear forms in the logarithms of rational numbers. Whereas Tijdeman appealed to an estimate of Baker [1], we shall apply an estimate of Matveev [7, 8].

2. Lower bounds for linear forms in the logarithms of rational numbers

For any rational number α we have $\alpha = r/s$ with r and s coprime integers with s positive. We define $H(\alpha)$, the height of α, by

$$H(\alpha) = \max(|r|, |s|).$$

Let $\alpha_1, \dots, \alpha_n$ be positive rational numbers with heights at most $A_1, \dots, A_n$ respectively. Suppose that $A_i \geq 3$ for $i = 1, \dots, n$ and that $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over the rationals where log denotes the principal value of the logarithm. Let $b_1, \dots, b_n$ be non-zero integers of absolute value at most B with $B \geq 3$ and put

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n.$$

Lemma 2.1. *There exists an effectively computable positive number $c_0 (\geq 1)$ such that*

$$\log |\Lambda| > -c_0^n \log A_1 \cdots \log A_n \log B.$$

Proof. This follows from Theorem 2.2 of Nesterenko [10]. Nesterenko's result is a special case of the work of Matveev [7, 8]. ☐

We remark that while there is an extensive literature on estimates for linear forms in the logarithms of algebraic numbers due to Baker and others, Matveev was the first to establish an estimate for Λ with the factor c_0^n in place of $n^{\delta n}$ for some positive real number δ. Such weaker estimates would lead to a version of Theorem 1.1 with $2r^2/\alpha$ in the exponent in (1.4) replaced by $(2 + \delta)r^2/\alpha$. We also note that the work of Matveev and of Nesterenko is completely explicit, and so c_0 may be made explicit.

3. Proof of Theorem 1.1

Let α be a positive real number and let $c_1, c_2, \dots$ denote positive numbers which are effectively computable in terms of α. Suppose that c is a positive number which exceeds $c_1, \dots, c_5$, where $c_1, \dots, c_5$ are defined below, and define t_r for $r = 1, 2, \dots$ as in (1.4). We shall construct primes $p_1, p_2, \dots$ with

$$t_i/2 < p_i \leq t_i \tag{3.1}$$

for $i = 1, 2, \dots$ so that (1.5) holds for the integers generated by $p_1, \dots, p_r$ for $r = 1, 2, \dots$. The result then follows by taking $T(\alpha)$ to be $\{p_1, p_2, \dots\}$.

For each real number x let $\pi(x)$ denote the number of primes of size at most x. By Corollary 3 of [12],

$$\pi(x) - \pi(x/2) > 3x/(10 \log x) \tag{3.2}$$

for $x > 41$. Let c_1 satisfy

$$(\log 3)^{c_1} > 41.$$

Since $t_1 = (\log 3)^c$ and c exceeds c_1, there is, by (3.2), a prime p_1 with

$$20 < t_1/2 < p_1 \leq t_1. \tag{3.3}$$

The sequence $1 = n_1 < n_2 < \cdots$ of integers generated by p_1 satisfies $n_i = p_1^{i-1}$ for $i = 1, 2, \dots$ and certainly

$$n_{i+1} - n_i > n_i \geq n_i^{1-\alpha}$$

for $i = 1, 2, \dots$ and any α with $0 < \alpha < 1$.

Suppose now that r is a positive integer and $p_1, \ldots, p_r$ have been constructed to satisfy (3.1) and so that (1.5) holds for the integers generated by $p_1, \ldots, p_r$. Then we shall prove that there is a prime p_{r+1} satisfying (3.1) with $i = r+1$ and so that (1.5) holds for the integers generated by $p_1, \ldots, p_{r+1}$. We do so by bounding the number of primes p with

$$t_{r+1}/2 < p \leq t_{r+1} \tag{3.4}$$

for which this is not the case. Accordingly suppose that p is a prime satisfying (3.4) for which the sequence $1 = n_1 < n_2 < \cdots$ of integers generated by $p_1, \ldots, p_r, p$ does not satisfy (1.5) for all positive integers i. Put $b = n_{i+1}$ and $a = n_i$, where i is the smallest index for which (1.5) fails. Observe that a and b are coprime and also that

$$1 < b/a \leq 1 + a^{-\alpha}. \tag{3.5}$$

Further, p divides one of a and b since (1.5) holds for the integers generated by $p_1, \ldots, p_r$. Put

$$a = p_1^{\alpha_1} \cdots p_r^{\alpha_r} p^{\alpha} \quad \text{and} \quad b = p_1^{\beta_1} \cdots p_r^{\beta_r} p^{\beta}$$

with $\alpha_1, \ldots, \alpha_r, \alpha, \beta_1, \ldots, \beta_r, \beta$ non-negative integers. By (3.5) b is at most $2a$ and by (3.3) $a \geq p_1 \geq 23$, so

$$\max\{\alpha_1, \ldots, \alpha_r, \alpha, \beta_1, \ldots, \beta_r, \beta\} \leq \frac{\log b}{\log 23} \leq \frac{\log 2a}{\log 23} < \log a. \tag{3.6}$$

We shall now bound $\log a$ from above by Lemma 2.1 with $n = r+1$ and

$$\Lambda = (\beta_1 - \alpha_1)\log p_1 + \cdots + (\beta_r - \alpha_r)\log p_r + (\beta - \alpha)\log p.$$

Since $c \geq c_1$ and $(\log 3)^{c_1}$ exceeds 41, we see that $t_{r+1} > 2t_r$ for $r = 1, 2, \ldots,$ and so

$$p_1 < p_2 < \cdots < p_r < p;$$

hence $\log p_1, \ldots, \log p_r, \log p$ are linearly independent over the rationals. We find that

$$\log(b/a) > \exp(-c_0^{r+1} \log t_1 \cdots \log t_{r+1} \log\log a). \tag{3.7}$$

By (3.5), since $\log(1+x) < x$ for any positive real number x,

$$\log(b/a) < a^{-\alpha}. \tag{3.8}$$

Put

$$\gamma = \alpha^{-1}(c_0 \log t_{r+1})^{r+1}. \tag{3.9}$$

It follows from (3.7), (3.8) and (3.9) that

$$\frac{\log a}{\log\log a} < \gamma.$$

If x and y are real numbers with $x \geq 3$ and $y \geq 1$ and $y/\log y < x$, then $y < 2x\log x$. Thus, since $c_0 \geq 1$,

$$\log a < 2\gamma \log \gamma. \tag{3.10}$$

Since a and b are coprime, at least one of α and β is zero and at least one of α_i and β_i is zero for $i = 1, \ldots, r$. Thus the number of possible $2r+2$-tuples $(\alpha_1, \ldots, \alpha_r, \alpha, \beta_1, \ldots, \beta_r, \beta)$ associated with a prime p satisfying (3.4) for which (1.5) does not hold for all positive integers i is at most

$$(1 + 2\log a)^{r+1}. \tag{3.11}$$

Each such $2r+2$-tuple determines an interval in which p must lie. In particular, if we put $A = p_1^{\alpha_1-\beta_1}\cdots p_r^{\alpha_r-\beta_r}$ we find from (3.5) that

$$A < p^{\beta-\alpha} \le A(1+a^{-\alpha}).$$

Put $A_1 = A^{1/(\beta-\alpha)}$. If $\beta > \alpha$, then

$$A_1 < p \le A_1(1+a^{-\alpha}),$$

whereas if $\alpha > \beta$, then

$$(1+a^{-\alpha})^{1/(\beta-\alpha)}A_1 \le p < A_1;$$

and in both cases p lies in an interval of length at most A_1/a^α. In the former case $A_1 < p$, while in the latter case $A_1 < 2p$ since $a^\alpha > 1$. Therefore each $2r+2$-tuple determines an interval in which p must lie of length at most $2t_{r+1}/a^\alpha$. Since $2a \ge b \ge p \ge t_{r+1}/2$ we see that the number of primes determined by each $2r+2$-tuple is at most $1 + 8t_{r+1}^{1-\alpha}$. Thus, since $t_{r+1} \ge 1$, the number of such primes is at most $9t_{r+1}^{1-\alpha}$. Therefore, by (3.10) and (3.11) there are at most

$$9t_{r+1}^{1-\alpha}(1+4\gamma\log\gamma)^{r+1} \tag{3.12}$$

primes from the interval $(t_{r+1}/2, t_{r+1}]$ which extend $\{p_1,\dots,p_r\}$ to give a sequence for which (1.5) does not hold. But by (3.2) there are at least $3t_{r+1}/(10\log t_{r+1})$ primes in $(t_{r+1}/2, t_{r+1}]$, and so provided that this number exceeds (3.12) we can choose p_{r+1} from $(t_{r+1}/2, t_{r+1}]$ so that (1.5) holds for the sequence of positive integers generated by $\{p_1,\dots,p_{r+1}\}$ as required.

Thus it suffices to check that

$$30(1+4\gamma\log\gamma)^{r+1}\log t_{r+1} < t_{r+1}^\alpha.$$

For $c > c_2$ we find that

$$\log t_r \le cr^2\log 3r \tag{3.13}$$

for $r = 1, 2, \dots$. Further, from (3.9) and (3.13), for $c > c_3$,

$$\gamma \le \alpha^{-1}(c_0c(r+1)^2\log 3(r+1))^{r+1} \le (r+1)^{2(r+1)}(\log 3(r+1))^{c\alpha(r+1)/4} \tag{3.14}$$

for $r = 1, 2, \dots$. By (3.14), for $c > c_4$,

$$1+4\gamma\log\gamma < (r+1)^{2(r+1)}(\log 3(r+1))^{c\alpha(r+1)/2} \tag{3.15}$$

for $r = 1, 2, \dots$. Finally, by (3.15), for $c > c_5$,

$$30(1+4\gamma\log\gamma)^{r+1}\log t_{r+1} < (r+1)^{2(r+1)^2}(\log 3(r+1))^{c\alpha(r+1)^2} = t_{r+1}^\alpha$$

for $r = 1, 2, \dots$ as required.

4. Proof of Theorem 1.2

Let $\varphi \in \mathcal{E}$ and define F_φ as in (1.7). We shall construct recursively an increasing sequence of primes $p_1, p_2, \dots$ with the property that for each positive integer r the set of positive integers generated by $\{p_1,\dots,p_r\}$ satisfies (1.8). The result follows by taking T to be $\{p_1, p_2, \dots\}$.

Certainly the set of positive integers generated by $\{p_1\}$ satisfies (1.8) whenever p_1 is an odd prime. We shall take $p_1 = 23$. Suppose that we have determined primes $p_1 < p_2 < \cdots < p_r$ such that the set of integers, all of whose prime factors are from $\{p_1,\dots,p_r\}$, satisfies (1.8). We shall prove that there is a prime p_{r+1} with

$p_{r+1} > p_r$ such that (1.8) holds for the integers generated by $\{p_1, \dots, p_{r+1}\}$, and the result will then follow.

Accordingly let t be a real number with $p_r < t/2$ and let p be a prime with $t/2 < p \le t$ for which (1.8) does not hold for $\{p_1, \dots, p_r, p\}$. If $1 = n_1 < n_2 < \cdots$ is the sequence of positive integers all of whose prime factors are from $\{p_1, \dots, p_r, p\}$, then for some integer i we have

$$n_{i+1} - n_i \le n_i / F_\varphi(n_i). \tag{4.1}$$

Put $n_{i+1} = b$ and $n_i = a$ with

$$a = p_1^{\alpha_1} \cdots p_r^{\alpha_r} p^{\alpha} \quad \text{and} \quad b = p_1^{\beta_1} \cdots p_r^{\beta_r} p^{\beta},$$

where $\alpha_1, \dots, \alpha_r, \alpha, \beta_1, \dots, \beta_r, \beta$ are non-negative integers. Since φ is non-decreasing, so is F_φ, and we may choose i minimal in (4.1) and thus ensure that a and b are coprime. Thus at least one of α_i and β_i is zero for $i = 1, \dots, r$. By our inductive hypothesis p divides ab, and since a and b are coprime, p divides exactly one of a and b. Further

$$\max(\alpha_i, \beta_i) \le (\log b)/\log p_i \le (\log 2a)/\log 23 \le \log a \tag{4.2}$$

for $i = 1, \dots, r$.

We shall now bound $\log a$ from above. Notice that a is not 1 and so a is divisible by a prime of size at least $p_1 = 23$; hence $\log a$ exceeds 3. It follows from (4.2) and Lemma 2.1 with $n = r + 1$ and

$$\Lambda = \log(b/a) = (\beta_1 - \alpha_1)\log p_1 + \cdots + (\beta_r - \alpha_r)\log p_r + (\beta - \alpha)\log p$$

that

$$\log(b/a) > \exp(-c_0^{r+1} \log p_1 \cdots \log p_r \log t \log\log a). \tag{4.3}$$

On the other hand, by (4.1),

$$0 < \log(b/a) < F_\varphi(a)^{-1}. \tag{4.4}$$

Since $a > e^e$ by (1.7), (4.3) and (4.4),

$$\log \varphi(\log a \log\log a) < c_0^{r+1} \log p_1 \cdots \log p_r \log t.$$

Put $c = c_0^{r+1} \log p_1 \cdots \log p_r$. Since φ^{-1} is defined and increasing on the positive real numbers,

$$\log a \log\log a < \varphi^{-1}(t^c). \tag{4.5}$$

Since $\max(\alpha, \beta)$ is at most $\log a$, it follows from (4.2) and (4.5) that there are at most

$$\varphi^{-1}(t^c)^{r+1} \tag{4.6}$$

possible $2r+2$-tuples $(\alpha_1, \dots, \alpha_r, \alpha, \beta_1, \dots, \beta_r, \beta)$ for which (4.1) holds for the integers generated by $\{p_1, \dots, p_r, p\}$ with $t/2 < p \le t$. Each such $2r+2$-tuple determines an interval in which p must lie. In particular, if we put $A = p_1^{\alpha_1 - \beta_1} \cdots p_r^{\alpha_r - \beta_r}$ we find from (4.1) that

$$A < p^{\beta - \alpha} \le A(1 + F_\varphi(a)^{-1}).$$

If $\beta > \alpha$, then

$$A^{1/(\beta-\alpha)} < p \le A^{1/(\beta-\alpha)}(1 + F_\varphi(a)^{-1}),$$

whereas if $\alpha > \beta$, then

$$(1 + F_\varphi(a)^{-1})^{1/(\beta-\alpha)} A^{1/(\beta-\alpha)} \le p < A^{1/(\beta-\alpha)};$$

and in both cases p lies in an interval of length at most $A^{1/(\beta-\alpha)}/F_\varphi(a)$.

By choosing t, hence also p and a, sufficiently large we can ensure that $F_\varphi(a)$ exceeds 2, and so $A^{1/(\beta-\alpha)}$ is at most $2t$. Therefore each $2r+2$-tuple determines an interval in which p must lie of length at most $2t/F_\varphi(a)$. Since $2a \geq b \geq p \geq t/2$ and since F_φ is non-decreasing, we see that the number of primes determined by each $2r+2$-tuple is at most

$$1 + (2t/F_\varphi(t/4)).$$

But $F_\varphi(t/4) > F_\varphi(t)/4$ and t exceeds $F_\varphi(t)$ by (1.6) and (1.7) for t sufficiently large. Thus at most $9t/F_\varphi(t)$ primes are determined by each $2r+2$-tuple. Therefore, by (4.6), there are at most

$$(9t/F_\varphi(t))(\varphi^{-1}(t^c))^{r+1} \tag{4.7}$$

primes from the interval $(t/2, t]$ which will extend $\{p_1, \ldots, p_r\}$ to give a sequence for which (1.8) does not hold. But for t sufficiently large there are at least $t/2\log t$ primes in $(t/2, t]$, and so to be sure that there is a prime p for which (1.8) holds with $\{p_1, \ldots, p_r, p\}$ it suffices to check that $t/2\log t$ exceeds (4.7). But $18\log t < \varphi^{-1}(t^c)$ for t sufficiently large, and so it suffices to check that

$$(\varphi^{-1}(t^c))^{r+2} < F_\varphi(t).$$

Since $r+2$ is less than $\log\log t$ for t sufficiently large, we need only check that

$$\varphi^{-1}(t^c) < \varphi(\log t \log\log t)$$

or, equivalently,

$$t^c < \varphi(\varphi(\log t \log\log t)) = t^{\log\log t}.$$

But this plainly holds for t sufficiently large and the result follows.

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