On the Asymptotics of Connected Chord Diagrams

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Abstract. We pursue a combinatorial interpretation for expressions that appear in the asymptotic expansion of \( \mathcal{C}_n \), the number of connected chord diagrams on \( n \) chords. The main outcome presented here is a new combinatorial interpretation for entry A088221 of the OEIS. We will show that A088221 surprisingly counts pairs of connected chord diagrams (allowing empty diagrams). This question arose from a more applied context, namely, from quantum field theory where connected chord diagrams are used in describing solutions to the Dyson-Schwinger equations. The problem considered here come as a small outgrowth of a larger ongoing program aiming to replace the sometimes ill-defined analytic understanding of quantum field theory with a discrete combinatorial understanding that represents itself in a way that is more elementary, yet more robust.

1. Preliminaries

1.1. Overview. Connected chord diagrams stand as a rich structure that becomes handy and informative in a variety of contexts, including bioinformatics [11], quantum field theory [18, 3, 2], and data structures [8]. Our interest in chord diagrams comes from the context of quantum field theory, in particular, from the Dyson-Schwinger equations. The solutions to Dyson-Schwinger were recently shown to be described as series indexed by connected chord diagrams with extra conditions on the placing of terminal chords [14]. The work proposed here has been motivated as part of such combinatorial treatments on quantum field theory (see [18, 3, 12] for further readings), nevertheless is purely combinatorial. Throughout this paper \( C(x) \) will denote the generating function for \((\text{rooted})\) connected chord diagrams (Definition 1.4). In [3], M. Borinsky studied the asymptotic behaviour of \( C_n \), the number of connected chord diagrams on \( n \) chords, as an instance of the work on factorially divergent power series. Namely, if we start with an asymptotic expansion that consists of a sum where terms are scalar multiples of (modified) gamma functions, we can then associate the sequence of scalar coefficients to an ordinary power series instead of the gamma functions and study the algebraic advantage we get from this process. We discuss this in detail in Section 1.2. So, roughly speaking, it is shown in [3] that, after factoring out the factorial divergencies from the coefficients of the generating function \( C(x) \) of connected chord diagrams, we obtain the following expression in \( C(x) \):

\[
\frac{x}{C(x)} \exp \left( -\frac{1}{2x}(2C(x) + C(x)^2) \right).
\]

The generating series for connected chord diagrams \( C(x) \) is known to satisfy the following differential equation (see [9] for example)

\[
2x C''(x) C(x) = C(x) - x + C(x)^2,
\]

The author is greatly indebted to Karen Yeats, who originally introduced the problem, and whose notes and discussions on the subject were the most enlightening.
by which we can rewrite the expression inside the exponential as $-1$ times
\[
1 + \frac{1}{2x} C(x)(4x \frac{d}{dx} - 1)C(x).
\]
Ignoring the 1 and the $1/2$, this turns out to be the generating function for rooted chord diagrams with at most two connected components, counted by one less than the number of chords. The coefficients start as 3, 10, 63, 558, 6226, 82836, . . . , which coincide with those of the sequence A088221 of the OEIS: 1, 2, 3, 10, 63, 558, 6226, 82836, . . . . The only definition available for the latter is that (in abuse of notation!) $[x^n](A088221)^n = [x^{n+1}]A000698$, (https://oeis.org/A088221). Interestingly, A000698 is the series for indecomposable chord diagrams, so there has to be some bridge between chord diagrams with at most two connected components and indecomposable chord diagrams (Definition 1.5) as we shall here prove.

The problem of finding a better combinatorial interpretation for A088221 lies as a piece in a more general context. The ultimate goal is to interpret, on combinatorial basis, the process described by the map $A^3 \alpha \beta : \mathbb{R}[\!\{x\}\!]^\alpha \rightarrow \mathbb{R}[\!\{x\}\!]$, acting over the subring of $(\alpha, \beta)$-factorially divergent power series, as defined in [3] and discussed below. This map is also referred to as an alien derivative in the field of resurgence theory [15]. Throughout the paper we shall stick to the following notation:

1. $\mathcal{D}$ is the class of chord diagrams,
2. $\mathcal{C}$ is the class of connected chord diagrams,
3. $\mathcal{C}^*$ is the class of connected chord diagrams excluding the one chord diagram,
4. $\mathcal{C}_{\leq 2}$ is the class of chord diagrams with at most two connected components,
5. $\mathcal{I}$ is the class indecomposable chord diagrams, and finally
6. $\mathcal{E}$ will stand for indecomposable chord diagrams with exactly two components.

This first section, Preliminaries, will provide an overview for factorially divergent series, chord diagrams, and Lagrange’s inversion. In the second section we prove some bijections describing connected chord diagrams and provide a reversible algorithm to move between the class of two lists of indecomposable chord diagrams (allowing empty lists) and the class of rooted trees, in which vertices are of special type, and where a $\mathcal{C}_{\leq 2}$-structure is set over the children of every vertex.

By the results in [4], it might be nice to pass this problem into the context of maps on oriented surfaces, but we shall not discuss this here.

1.2. Factorially Divergent Power Series. This section aims to provide the necessary background for factorially divergent series, as introduced in Chapter 4 in [3]. For this section we will need to use the usual big and small o-notation for asymptotic analysis: Given a sequence $a_n$, $O(a_n)$ will denote the class of sequences $b_n$ satisfying $\limsup_{n \to \infty} \frac{|b_n|}{a_n} < \infty$; whereas $o(a_n)$ shall denote the sequences $b_n$ such that $\lim_{n \to \infty} \frac{b_n}{a_n} = 0$. Moreover, $a_n = b_n + O(c_n)$ should mean that $a_n - b_n \in O(c_n)$. Following [3], we adopt the notation $\Gamma^\alpha_\beta(n) := \alpha^{n+\beta} \Gamma(n + \beta)$, where $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$ for $\text{Re}(z) > 0$ is the gamma function.

**Definition 1.1** (Factorially Divergent Power Series). For real numbers $\alpha$ and $\beta$, with $\alpha > 0$, the subset $\mathbb{R}[\!\{x\}\!]^\alpha_\beta$ of $\mathbb{R}[\!\{x\}\!]$ will denote the set of all formal power series $f$ for which there exists a sequence $(c_k^R)_{k \in \mathbb{N}}$ of real numbers such that

\[
f_n = \sum_{k=0}^{R-1} c_k^R (n-k) + O(\Gamma^\alpha_\beta(n-R)), \quad \text{for all } R \in \mathbb{N}_0.
\]
Proposition 1.1 ([3], ch.4). Given $\alpha, \beta \in \mathbb{R}$, with $\alpha > 0$, the set $\mathbb{R}[[x]]_\beta^\alpha$ is a subring of $\mathbb{R}[[x]]$. Moreover, the sequence $(c_k^f)_{k\in\mathbb{N}}$ is unique for every $f \in \mathbb{R}[[x]]_\beta^\alpha$; actually $c_N^f = \lim_{n \to \infty} \frac{\sum_{n=0}^{N-1} c_k^f (n-k)}{\binom{n}{N-k}} = 0$ for $N \in \mathbb{N}_0$. 

Note that the identity in (1.1) stands for an asymptotic expansion with asymptotic scale $\alpha^{n+\beta} \Gamma(n+\beta)$ (refer to [6] for a detailed literature). The ring $\mathbb{R}[[x]]_\beta^\alpha$ is referred to as a ring of factorially divergent power series. Now, given $f \in \mathbb{R}[[x]]_\beta^\alpha$, we can associate the coefficients $(c_k^f)_{k\in\mathbb{N}}$ of the asymptotic expansion with a new ordinary power series:

Definition 1.2 ([3]). For $\alpha, \beta \in \mathbb{R}$, with $\alpha > 0$, let $A_\beta^\alpha : \mathbb{R}[[x]]_\beta^\alpha \to \mathbb{R}[[x]]_\beta^\alpha$ has the following action for every $f \in \mathbb{R}[[x]]_\beta^\alpha$:

$$(A_\beta^\alpha f)(x) = \sum_{k=0}^{\infty} c_k^f x^k.$$ 

In [3] M. Borinsky provides an extensive analysis for the map $A_\beta^\alpha$; in particular, it is shown that $A_\beta^\alpha$ is actually a derivation over the ring $\mathbb{R}[[x]]_\beta^\alpha$ ([3], Prop. 4.3.1.). More interestingly, the next theorem serves as a powerful tool for our purposes in this paper.

For notation, we set $\text{Diff}(\mathbb{R}, 0) = (\{g \in \mathbb{R}[[x]] : g_0 = 0, g_1 = 1\}, \circ)$, the group of formal diffeomorphisms tangent to the identity, under composition of maps. Similarly, we set $\text{Diff}(\mathbb{R}, 0)_\beta^\alpha = (\{g \in \mathbb{R}[[x]]_\beta^\alpha : g_0 = 0, g_1 = 1\}, \circ)$.

Theorem 1.2 ([3], Th. 4.4.2.). Let $\alpha, \beta \in \mathbb{R}$, with $\alpha > 0$. Then $\text{Diff}(\mathbb{R}, 0)_\beta^\alpha$ is a subgroup of $\text{Diff}(\mathbb{R}, 0)_\beta^\alpha$; moreover, for any $f \in \mathbb{R}[[x]]_\beta^\alpha$ and $g \in \text{Diff}(\mathbb{R}, 0)_\beta^\alpha$ the following holds:

1. $f \circ g$ and $g^{-1}$ are again elements in $\mathbb{R}[[x]]_\beta^\alpha$.
2. The derivation $A_\beta^\alpha$ satisfies a chain rule, namely

$$(A_\beta^\alpha (f \circ g))(x) = f'(g(x))(A_\beta^\alpha g)(x) + \left(\frac{x}{g(x)}\right)^\beta e^{g(x)-x} (A_\beta^\alpha f)(g(x)), \quad x,$$

$$(A_\beta^\alpha g^{-1})(x) = -(g^{-1})'(x) \left(\frac{x}{g^{-1}(x)}\right)^\beta e^{g^{-1}(x)-x} (A_\beta^\alpha g)(g^{-1}(x)).$$

1.3. Chord Diagrams. A chord diagram on $n$ chords is simply a circle with $2n$ nodes that are matched into disjoint pairs through chords (no labels). A rooted chord diagram is a chord diagram with a selected node called the root. Thus, a rooted chord diagram can be represented in a linear order, by numbering the nodes in counterclockwise order, starting from the root which receives the label ‘1’. The chord carrying the root is called the root chord. An interval in the rooted chord diagram is the space to the right of one of the nodes in the linear representation (for example, this includes the space to the right of the last node in the linear order). Notice that, through this linear order, there is a clear bijection between rooted chord diagrams and matchings on the set $\{1, 2, \ldots, 2n\}$. A chord in the diagram may be referred to as $c = \{a < b\}$, where $a$ and $b$ are the nodes in the linear order.

Unless otherwise stated, all chord diagrams in this paper are assumed to be rooted. If we denote the generating function for rooted chord diagrams by $D(x)$, then it is easily seen that

$$D(x) = \sum_{n=0}^{\infty} (2n-1)!! x^n.$$ 

Definition 1.3 (Oriented Intersection Graph). Given a (rooted) chord diagram on $n$ chords, consider the following graph: the chords of the diagram will serve as vertices for the new graph, and there is an edge between the two nodes $c_1 = \{v_1 < v_2\}$ and $c_2 = \{w_1 < w_2\}$ if $v_1 < w_1 < v_2 < w_2$, i.e. if the chords cross each other. The graph so constructed is called the oriented intersection graph of the given chord diagram.
Definition 1.4 (Connected Chord Diagrams). A (rooted) chord diagram is said to be *connected* if its oriented intersection graph is connected.

**Example:** The diagram $D$ below is a connected chord diagram in linear representation, where the root node is drawn in black.

![Diagram](image)

The generating function for connected chord diagrams (in the number of chords) is denoted by $C(x)$. Thus $C(x) = \sum_{n=0}^{\infty} C_n x^n$, where $C_n$ is the number of connected chord diagrams on $n$ chords. The first terms of $C(x)$ are found to be

$$C(x) = x + x^2 + 4x^3 + 27x^4 + 248x^5 + \cdots;$$

the reader may refer to OEIS sequence A000699 for more coefficients. The next lemma lists some classic decompositions for chord diagrams (see [9] for example). The first equation operates by extracting the connected component that contains the root chord (known as the root component).

**Lemma 1.3.** If $D(x)$ is the generating series of chord diagrams then

(i) $D(x) = 1 + C(xD(x)^2),$
(ii) $D(x) = 1 + xD(x) + 2x^2D'(x)$, and
(iii) $2xC(x)C'(x) = C(x)(1 + C(x)) - x.$

**Proof.** We sketch the underlying decompositions as follows:

1. The ‘one’ term is for the empty chord diagram. Now, given a nonempty chord diagram, we see that for every chord in the root component there live two chord diagrams to the right of its two ends. This gives the desired decomposition.

2. There are three situations for a root chord: it is either non-existent (empty diagram); or it is concatenated with a following diagram; or the root chord has its right end landing in one of the intervals of a diagram. These situations correspond respectively with the terms in (ii).

3. Follows from (i) and (ii).

We end this section with the definition of an *indecomposable* chord diagram, these diagrams will become a key ingredient later in this paper.

**Definition 1.5.** A chord diagram is said to be *indecomposable* if, when represented linearly, it is not the concatenation of separate chord diagrams.

**Example:** Consider the following two diagrams:

$$D_1 = \text{[Diagram]} \quad \text{and} \quad D_2 = \text{[Diagram]}.$$

Then $D_1$ is indecomposable, whereas $D_2$ is not since it is the concatenation of three (indecomposable chord diagrams). Notice that an indecomposable chord diagram is not necessarily connected, but the converse is clearly true, namely, any connected diagram is indecomposable. Sequence A000698 of the OEIS counts indecomposable chord diagrams, the first terms start as

$$I(x) = 1 + x + 2x^2 + 10x^3 + 74x^4 + 706x^5 + \cdots,$$

where $I(x)$ is the generating power series for indecomposable chord diagrams.
1.4. Asymptotic Analysis for Connected Chord Diagrams. Here we follow [3] in applying the results discussed in Section 1.2 to chord diagrams. The full study of this approach, as well as more combinatorial applications, can be found in [3]. We have $(2n-1)!! = \frac{2n+1}{\sqrt{2\pi}} \Gamma(n + \frac{1}{2}) = \frac{1}{\sqrt{2\pi}} \Gamma^2_\frac{1}{2}(n)$, and so $D(x) \in \mathbb{R}[\!\!x\!\!]^{\frac{3}{2}}_2$. According to Definition 1.2 we have $(A^2_\frac{1}{2}D(x)) = 1/\sqrt{2\pi}$. From the recursion (i) in Lemma 1.3 we see that $C(xD(x)^2) \in \mathbb{R}[\!\!x\!\!]^{\frac{3}{2}}_2$. Also, $xD(x)^2 \in \mathbb{R}[\!\!x\!\!]^{\frac{1}{2}}_2$ since $\mathbb{R}[\!\!x\!\!]^{\frac{3}{2}}_2$ is a ring by Proposition 1.1. Now, applying Theorem 1.2 to dissolve the composition, we get that $C(x) \in \mathbb{R}[\!\!x\!\!]^{\frac{3}{2}}_2$.

From the other way round, we can apply the chain rule of $A^2_\frac{1}{2}$ (Theorem 1.2), doing so we obtain the equation

$$
\frac{1}{\sqrt{2\pi}} = (A^2_\frac{1}{2}D)(x) = \left( A^2_\frac{1}{2} \left( 1 + C(xD(x)^2) \right) \right)(x) = \left( A^2_\frac{1}{2} C(xD(x)^2) \right)(x) = 2xD(x)C'(xD(x)^2)(A^2_\frac{1}{2}D)(x) + \left( \frac{x}{xD(x)^2} \right)^2 \frac{e^{xD(x)^2-x}}{2\pi} \left( A^2_\frac{1}{2} C \right)(xD(x)^2)
$$

This tells us that

$$
\left( A^2_\frac{1}{2} C \right)(xD(x)^2) = \frac{-D(x) - 2xD(x)^2C'(xD(x)^2)}{\sqrt{2\pi}} e^{-\frac{x}{2\pi(xD(x)^2)}}.
$$

Now notice that $xD(x)^2$ has a compositional inverse, $h(x)$, say. That is, $h(x)D(h(x))^2 = x$, and hence, if we replace $x$ by $h(x)$ in the last equation we get

$$
\left( A^2_\frac{1}{2} C \right)(x) = \frac{h(x)}{\sqrt{2\pi}} e^{-\frac{x}{2\pi(h(x))^2}}.
$$

But $D(h(x)) = 1 + C(x)$ by (i) in Lemma 1.3, so we get

$$
\left( A^2_\frac{1}{2} C \right)(x) = \frac{1 + C(x) - 2xC''(x)}{\sqrt{2\pi}} e^{-\frac{x}{2\pi(2C(x)+C(x)^2)}}
$$

(†)

where the second equality is achieved by appealing to (iii) in Lemma 1.3. Obtaining such a computable formula for $A^2_\frac{1}{2}C$ means that we have all the coefficients for the asymptotic expansion of $C(x)$, in the sense of Definition 1.1. As provided in [3], the first coefficients are

$$
(1.4) \quad \left( A^2_\frac{1}{2} C \right)(x) = \frac{1}{e^{\frac{x}{\sqrt{2\pi}}}} \left( 1 - \frac{5}{2} x - \frac{43}{8} x^2 - \frac{579}{16} x^3 - \frac{4477}{128} x^4 - \frac{5326191}{1280} x^5 \ldots \right).
$$

Accordingly, by Definition 1.1, and since $(2n-1)!! = \frac{1}{\sqrt{2\pi}} \Gamma^2_\frac{1}{2}(n)$, we have for all $R \in \mathbb{N}_0$:

$$
C_n = \sum_{k=0}^{R-1} \Gamma^2_\frac{1}{2}(n-k) \left[ x^k \right] \left( A^2_\frac{1}{2} C \right)(x) + O \left( \Gamma^2_\frac{1}{2}(n-R) \right)
$$

$$
= \sqrt{2\pi} \sum_{k=0}^{R-1} (2(n-k) - 1)!! \left[ x^k \right] \left( A^2_\frac{1}{2} C \right)(x) + O \left( (2(n-R) - 1)!! \right).
$$

Consequently, for large $n$ we have

$$
C_n = e^{-1} \left( (2n-1)!! - \frac{5}{2}(2n-3)!! - \frac{43}{8}(2n-5)!! - \frac{579}{16}(2n-7)!! \ldots \right).
$$
This provides a full generalization for the computations in the work of Kleitman [13], Stein and Everett [16] and Bender and Richmond [1], where only the first term in the expansion has been known. Finally, this also tells us that the probability for a diagram on \( n \) chords to be connected is 
\[
e^{-\frac{1}{4n}} + \mathcal{O}(1/n^2).
\]

1.5. Lagrange’s Inversion. The material provided in this section can be found in many references on algebraic combinatorics, e.g. the reader can refer to [10].

**Theorem 1.4 (Lagrange’s Implicit Function Theorem (LIFT)).** Let \( G(x) \in \mathbb{C}[[x]] \) be invertible, then there is a unique solution in \( \mathbb{C}[[x]] \) to the functional equation
\[
R(x) = xG(R(x)).
\]
Moreover, if \( F \in \mathbb{C}[[x]] \), then, for \( n \geq 1 \),
\[
[x^n]F(R(x)) = \frac{1}{n}[t^{n-1}]F'(t)G(t)^n.
\]

**Corollary 1.5.** In the same context of LIFT, if \( a_n = [x^n]H(x)G^n(x) \) where \( H \in \mathbb{C}[[x]] \), then
\[
\sum_{n \geq 0} a_n x^n = \frac{H(R(x))}{1 - x G'(R(x))}.
\]

**Proof.** Differentiating the defining equation of \( R \) we get
\[
R' = G(R) + xG'(R)R'.
\]
Now, given that \( G \) is invertible we can write \( H = F'G \) for some \( F \in \mathbb{C}[[x]] \). Thus we have
\[
\frac{F'(R)G(R)}{1 - x G'(R)} = \frac{dF(R)}{dx} = \sum_{n \geq 1} x^{n-1}[t^{n-1}]F'(t)G(t)^n = \sum_{n \geq 0} x^n[t^n]F'(t)G(t)^{n+1},
\]
and the result follows. Note that the second equality above is through LIFT. □

2. Main Results

First we recall the decomposition of a chord diagram by means of extracting the root component (Lemma 1.3-(i)) :
\[
D(x) = 1 + C(xD(x)^2).
\]
This decomposition will be of great help in the construction presented here, hence it may be wise to accompany it with a suitable notation.

**Notation:** The two diagrams that correspond to each chord in the root component will be referred to as the right dangling and the left dangling diagrams. Given a chord diagram \( D \), the root component will be denoted as \( C_\bullet(D) \), while the dangling diagrams will then be \( d_r \) and \( d_l \). The symbols \( d_r, d_l \) and \( C_\bullet \) are to be often used as operators.

**Example:** Consider the following chord diagram.

Yet, this can also be seen as
which clarifies the decomposition of the lemma. Note that the thick red diagram is the root component $C_\bullet$ of the original diagram. Also, notice that, for example, $d_l(c_1) = \emptyset$. The reason for the names is now hopefully justified.

**Lemma 2.1.** There is a bijection $\Psi$ between the class $C^*$ of rooted connected chord diagrams excluding the one chord diagram, and the class $C$ of indecomposable chord diagrams with exactly two components. Thus, in terms of generating functions $C(x) = C(x) - x$.

**Proof.** The bijection $\Psi$ here works almost the same as what is known as the root share composition: Let $C$ be a rooted connected chord diagram. Removing the root chord shall leave us with a list of rooted connected components ordered in terms of intersections with the original root, the first of these components is denoted as $C_2$, while $C_1$ is obtained by removing $C_2$ from the original $C$. Then $(C_1,C_2)$ is the root share decomposition of $C$ (root share decomposition is defined in [14] by Karen Yeats and N. Marie). $\Psi$ then places the whole $C_1$ at the interval of $C_2$ where the last end of the original root of $C$ would be if the non-root chords of $C_1$ are removed. As seen, the image $\Psi(C)$ in this case is an indecomposable chord diagram with exactly two components. This definition is reversible. Indeed, given an indecomposable chord diagram with exactly two connected components, $C_2$ will be the outer component and $C_1$ should be the inner one, and $C$ is obtained by just pulling out the first end of the root of $C_1$ to the leftmost position. $\square$

**Example:**

In the next theorem, given a finite set $S$ and a class $G$ of combinatorial objects, the term $G$-structure on $S$ will simply mean an arrangement of the elements of $S$ into an object from $G$. The operation $*$ stands for the usual ordered product for combinatorial classes. For example, if $T$ is the class of trees and $K_{\text{or}}$ is the class of oriented complete graphs, then an element from the class $T \ast K_{\text{or}}$ will be an ordered pair $(T,K)$ where $T \in T$ and $K \in K_{\text{or}}$. Notice that, for such a product structure to be applied on a finite set there has to be a partition of the set.

**Theorem 2.2.** Let $Z$ be the class of rooted trees where vertices are nonempty ordered sets (paths) and where there is a $C_{\leq 2}$-structure over the children of every vertex. Then there is a bijection $\Theta$ between $Z$ and the class $X \ast (D \ast D)$, where $D$ is the class of chord
diagrams. Consequently, if \( Z(x) \) is the generating series for \( Z \), then

\[
Z = x \left( \frac{1}{1 - I_0} \right)^2,
\]

where \( I_0 \) is the generating series for nonempty indecomposable chord diagrams.

**Proof.** Well, for simplicity, we will assume first that the objects are labelled. Let’s start with a labelled object \( P \) from the class \( \mathcal{X} \ast (D \ast D) \). In abuse of notation we shall write \( d_l(P) \) and \( d_r(P) \) for the two diagrams involved. Also the chord of \( P \) will mean the unique chord of which \( P \) starts (represented by \( \mathcal{X} \) in the decomposition). Then the corresponding \( Z \)-tree is obtained through the following algorithm:

**Algorithm 1: Make \( Z \)-Tree**

Input: \( P = \vec{s} \circ (d_l, d_r) \)
initially \( Q_1 = P \);
queue \( Q = (Q_1) \);
integer \( L = \text{length}(Q) \) (automatically modified by any alteration of \( Q \));
vertex \( v = \emptyset \);
label \( (v) = \text{label given to the chord of } Q_1 \); diagrams \( D_l = D_r = \emptyset \);
tree \( Z = v \);

Set \( v \) as the root vertex of \( Z \)
While \( Q \neq \emptyset \) {

1. Set \( D_l = d_l(Q_1) \) and \( D_r = d_r(Q_1) \);

2. If \( D_l = \emptyset = D_r \) then:
   - push \( Q_1 \) out of \( Q \) (i.e. for all \( 1 \leq k < L, Q_k \leftarrow Q_{k+1} \));
   - Go to step (1) again.

3. If \( D_l = \emptyset \) and \( D_r \neq \emptyset \) then:
   - Create \( |C_\bullet(D_r)| \) children attached to \( v \), and set their labels to be the same as the chords in \( C_\bullet(D_r) \). Namely, let \( \{w_1, \ldots, w_{|C_\bullet(D_r)|}\} \) be the children and set \( \text{label}(w_i) = \text{label}(i^{\text{th}} \text{ chord}) \) in the obvious meaning;
   - For each \( i \in \{1, \ldots, |C_\bullet(D_r)|\} \) add \( Q_{L+i} \) to the queue \( Q \), where \( Q_{L+i} := \vec{s} \circ (d_l(i^{\text{th}} \text{ chord}), d_r(i^{\text{th}} \text{ chord})) \), where the single chord is standing for the \( i^{\text{th}} \) chord in \( C_\bullet(D_r) \) and the \( d_l, d_r \) are the dangling diagrams of this chord in \( D_r \);
   - Set \( C_\bullet(D_r) \) as the \( C_{\leq 2} \)-structure over the children of \( v \);
   - Push \( Q_1 \) out of \( Q \);
   - Set \( v = \text{vertex for the chord of } Q_1 \) (where \( Q_1 \) has been updated);
   - Go to step (1);

4. If \( D_l \neq \emptyset \) and \( D_r \neq \emptyset \) then:
   - Create \( |C_\bullet(D_l)| + |C_\bullet(D_r)| \) children attached to \( v \), and set their labels to be the same as the corresponding chords, as before;
   - For each \( i \in \{1, \ldots, |C_\bullet(D_l)| + |C_\bullet(D_r)|\} \) add \( Q_{L+i} \) to the queue \( Q \), where \( Q_{L+i} := \vec{s} \circ (d_l(i^{\text{th}} \text{ chord}), d_r(i^{\text{th}} \text{ chord})) \), where the single chord is standing for the \( i^{\text{th}} \) chord in \( C_\bullet(D_l) \) if \( 1 \leq i \leq |C_\bullet(D_l)| \) and for the \( (i - |C_\bullet(D_l)|)^{\text{th}} \) chord in \( C_\bullet(D_r) \) otherwise;
   - Set the concatenation \( C_\bullet(D_l)C_\bullet(D_r) \) as the \( C_{\leq 2} \)-structure over the children of \( v \);
- Push $Q_1$ out of $Q$;  
- Set $v$ = vertex for the chord of $Q_1$;  
- Go to step (1);  

(5) If $D_L \neq \emptyset$ and $D_R = \emptyset$ then:  
   (a) In case $C_\bullet(D_l)$ is a single chord $c$ then:  
      - vertex $v$ absorbs another node with the label given to $c$. It is appropriate to think of a vertex here as some sort of stack comprising labelled nodes:  
        \[
        \begin{array}{c}
        \bullet \\
        \bullet \\
        \bullet \\
        \vdots \\
        \end{array}
        \]
      - Add $Q_{L+1}$ to $Q$, where $Q_{L+1}$ consists of $c$ and its dangling diagrams, i.e. $d_L(Q_{L+1}) = d_L(c)$ and $d_R(Q_{L+1}) = d_R(c)$;  
      - Push $Q_1$ out of $Q$;  
      - Set $v$ = vertex for the chord of $Q_1$;  
      - Go to step (1);  

   (b) Otherwise if $C_\bullet(D_l)$ is not a single chord then:  
      - Create $|C_\bullet(D_l)|$ children attached to $v$, and set their labels to be the same as the corresponding chords, as before;  
      - For each $i \in \{1, \ldots, |C_\bullet(D_l)|\}$ add $Q_{L+i}$ to the queue $Q$, where $Q_{L+i} = (d_L(i^{th\ chord}), d_R(i^{th\ chord}))$, where the single chord is standing for the $i^{th}$ chord in $C_\bullet(D_l)$;  
      - Set $\Psi(C_\bullet(D_l))$ as the $C_{\leq 2}$-structure over the children of $v$;  
      - Push $Q_1$ out of $Q$;  
      - Set $v = vertex$ for the chord of $Q_1$;  
      - Go to step (1);  

\]

Output: $\Theta(P) = Z$.

This algorithm uniquely generates the corresponding tree. Indeed, to see this it shall be enough to see that every branching from a vertex is uniquely translated into chord diagrams:

1. If the $C_{\leq 2}$-structure over the children is the concatenation of two connected components, then we know simply that there has been nonempty right and left dangling diagrams for the chord corresponding to the vertex. Further, the two connected components are, respectively, the root components of the dangling diagrams. The order of components in the (rooted) $C_{\leq 2}$-structure dictates which component is for the left or right dangling diagram.

2. If the $C_{\leq 2}$-structure is just a connected chord diagram, then for the chord corresponding to the vertex only the right dangling diagram existed (nonempty). In particular, this connected structure is the root component for the right dangling diagram. The next two cases are actually the most crucial.

3. If the $C_{\leq 2}$-structure is an indecomposable chord diagram with exactly two connected components, then we learn that for the corresponding chord only the left dangling diagram existed. The root component of which is determined by applying $\Psi^{-1}$ to the $C_{\leq 2}$-structure. This process is well-defined by virtue of $\Psi$ being a bijection.
(4) The only remaining case is when the vertex itself is a stack. This marks that, as in the previous case, only the left dangling diagram existed for the chord corresponding to the vertex, and that, further, the root component for this dangling diagram was a single chord with the label given next in the stack. Uniqueness in this case is clear as the information is encoded into the tree in a way that does not interfere with the previous cases, hence no ambiguity arises.

This outlines by which means the above algorithm is reversible, and hence establishes the desired bijection. To prove the second part of the theorem just notice that any rooted chord diagram can be viewed as a (possibly empty) list of nonempty indecomposable chord diagrams.

□

Example: Let $P \in X^* (D \ast D)$ be given by

$$P = \overset{2}{\circ \circ} (d_l(P), d_r(P)), \text{ where}$$

$$d_r(P) = \begin{array}{c}
3 \\
13 \\
16
\end{array} \quad \begin{array}{c}
15 \\
9 \\
19 \\
8 \\
12 \\
17 \\
10 \\
4 \\
18 \\
6
\end{array}$$

and

$$d_l(P) = \begin{array}{c}
11 \\
20 \\
1 \\
14 \\
5
\end{array}.$$
and then the vertex $v$ is set to be 11.

[3] In this iteration we find that $D_l \neq \emptyset$, whereas $D_r = \emptyset$, moreover, $C_\bullet(D_l)$ is the single chord labelled ‘20’. Thus, following the algorithm, one more node is appended to the vertex $v$ which, before this moment, only contained the node labelled 11. Thus, vertex $v$ is now given by $\left(\frac{11}{20}\right)$. Then we add the entry $Q_6 = \left(\frac{20}{14, 5}, \emptyset\right)$ to the queue $Q$ (from the end); update $Q$ by pushing out $Q_1$; and set the vertex $v$ to be at chord ‘1’, since it is the chord of the new $Q_1$.

Following the algorithm to the end we generate the tree $\Theta(P)$ to be as below, where the right column indicates the $C_\leq 2$-structures pertinent to the children of each vertex (recall that vertices here are generally stacks of nodes). For clarity, structures are displayed level-wise:

\[
\Theta(P) =
\]

**Corollary 2.3.** Let $I_0(x)$ be the generating series for nonempty indecomposable chord diagrams as before, and set $B(x) = C_{\leq 2}(x) + x$, where $C_{\leq 2}(x)$ is the generating function for the class $C_{\leq 2}$. Then

\[
I_0(x) = \frac{x}{1 - xB'(Z)} ,
\]

where $Z$ is the generating series for the class $Z$ as before.

Before proving Corollary 2.3 we will need the following decomposition of indecomposable diagrams.

**Lemma 2.4.** The generating series $I_0$ for nonempty indecomposable chord diagrams satisfies the relation

\[
I_0(x) = x + \frac{2x^2 I_0'(x)}{1 - I_0(x)} .
\]

**Proof.** Given a nonempty indecomposable chord diagram we can argue as follows. If the diagram is not a single chord, then removing the root chord generally leaves us with a list of nonempty indecomposable chord diagrams. Moreover, the last diagram in
this list carries all the information about the removed root chord encoded as a marked interval that carried the right end of the root chord. Recall that the intervals are meant to be the spacings to the right of every end-node (this includes the last space to the right of the diagram), thus we have $2m$ intervals in a size-$m$ chord diagram. The relation in the lemma is exactly the translation of this decomposition into the world of generating series.

**Example:** In the following diagram, the diagram is decomposed into: the root, $D_1$, $D_2$, and $(D_3$, interval 4), where, among the 8 intervals in $D_3$, the root originally landed in interval 4 (marked by a red dashed line).

![Diagram](image)

**Proof of Corollary 2.3:** First of all, notice that by the definition of the class $Z$, the generating series $Z$ satisfies the recursion

$$Z(x) = xC_{c2}(Z) + xZ = xB(Z),$$

where $B(t) = C_{c2}(t) + t$. Thus $Z'(x) = B(Z) + xB'(Z)Z' = \frac{Z}{x} + xB'(Z)Z'$, and hence

$$Z = xZ'(1 - xB'(Z)).$$

By Theorem 2.2, we have that $Z = x\left(\frac{1}{1 - I_0}\right)^2$. Taking the logarithmic derivative of both sides and making use of the above identities we get

$$1 + 2x \frac{d}{dx} \log \left(\frac{1}{1 - I_0}\right) = x \frac{d}{dx} \log Z = x \frac{Z'}{Z} = \frac{1}{1 - xB'(Z)},$$

and hence

$$\frac{1}{1 - xB'(Z)} = 1 + \frac{2xI_0'}{(1 - I_0)}.$$}

Multiplying by $x$ we get that, by Lemma 2.4, $\frac{x}{1 - xB'(Z)} = x + \frac{2x^2I_0'}{(1 - I_0)} = I_0$, which completes the proof.

**Corollary 2.5.** Let $A(X)$ be the generating series for the sequence A088221. Then

$$A(x) = C_{c2}(x) + x.$$

**Proof.** From Corollary 1.5, we know that

$$[x^n] \frac{1}{1 - xB'(Z)} = [x^n]B^n(x).$$

Now, by the definition of the sequence A088221, we know that it is the sequence for which $[x^n]A^n(x) = [x^{n+1}]I_0(x)$, where $A(x)$ is assumed to be the generating series for the sequence A088221. This gives that $[x^n]A^n(x) = [x^{n+1}]I_0(x) = [x^{n+1}]\frac{x}{1 - xB'(Z)} = [x^n]B^n(x)$, and so $A(x) = B(x) = C_{c2}(x) + x$. □
Proposition 2.6. The \( n \)th entry in the sequence A088221 counts the number of pairs \((C_1, C_2)\) of connected chord diagrams (allowing empty diagrams) with total number of chords being \( n \).

Proof. Indeed, any chord diagram with at most two connected components falls into one of the following categories:

1. empty,
2. connected,
3. concatenation of two connected diagrams,
4. or is indecomposable with exactly two connected components.

By using Lemma 2.1 for the last case we get

\[
A(x) = C_{\leq 2}(x) + x = 1 + C(x) + C^2(x) + C(x) - x + x = (C(x) + 1)^2,
\]

and the result is established.

\(\square\)

References


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