Project Description

Diagrammatic and geometric techniques in representation theory

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Reader's guide. In the introduction, I want to lay out 2 basic goals (1, 2) that motivate my current work. While these are very broad, and far beyond what is achievable in a year, they serve to motivate and organize the more specific projects I suggest. Then I will address the broader impacts of this proposal.

Then, in the main body of the statement, I'll discuss 3 topics from my research in greater detail:

- the structure of categories $\mathcal{O}$ for symplectic singularities, with an emphasis on the case of Coulomb branches,
- problems relating to homological knot invariants,
- the S-duality conjecture of the author and collaborators.

Introduction. My research centers around the representation theory of deformation quantizations. While this might sound like an unfamiliar subject, in fact, it generalizes the study of universal enveloping algebras, which can be viewed as deformation quantizations of $\mathfrak{g}^*$. As with the representation theory of Lie algebras, these algebras have complex connections to topology, geometry and combinatorics.

Viewed from largest scale, my long-term goal is to give these algebras an investigation as thorough as that for $U(\mathfrak{g})$.

Goal 1. *Extend all known results about the representation theory of $U(\mathfrak{g})$ to a quantization A of an arbitrary conical symplectic singularity (defined below).*

I believe this is an important goal, because working out the specific cases (such as the "hypertoric enveloping algebra" considered by Musson, van der Bergh and others) has yielded very interesting mathematics, including some interesting breakdown of results from $U(\mathfrak{g})$ (for example, there can be non-semisimple finite dimensional representations), while many other structures seem to survive. The theory of rational Cherednik algebras also fits in this rubric, and gives a promising hint that this is an interesting approach.

The greatest motivation for us, though, has been to investigate a conjectured duality between symplectic singularities, which we call S-duality. At the moment, we have no general definition of this duality, but a very powerful method of constructing examples was recently given by Braverman, Finkelberg and Nakajima [Nak, BFNb]. One reflection of this duality is a Koszul duality between certain categories of representations of the algebras quantizing dual pairs of singularities.

This goal can be approached along two routes:

- proving general results when they are within our reach.
- gaining a better understanding of specific cases by exploiting special features of those cases.

The techniques in the first route are of geometric representation theory and algebraic geometry; in the second, diagrammatic techniques have shown themselves to be very powerful for computations of Hom-spaces between different representations, especially for the examples arising as Higgs and Coulomb branches in gauge theories.

In the specific case of quiver gauge theories, these diagrammatic techniques have already been developed by several authors in the theory of categorification of quantum groups; roughly, this theory investigates one (non-obvious) notion of an action of a Lie algebra on a category. This brings me to a second goal:

Goal 2. *Using the geometry of quiver varieties and affine Grassmannians (either directly or as inspiration), give a categorification of all structures in the theory of quantum groups in terms of categories with Lie algebra actions.*

Again, the importance of this may not be immediately apparent; however, in the work done thus far, difficult-to-motivate constructions from the theory of quantum groups such as R-matrix, quantum Weyl group, and even the coproduct appear completely canonically from geometry. This provides powerful evidence both of the naturality of the theory of quantum groups and of this particular notion of categorical Lie algebra action.

The Holy Grail of this line of research is the construction of a 4-dimensional extended TQFT providing "another level" of Chern-Simons theory, and yielding Chern-Simons when applied to $X \times S^1$. Previous work of mine gives bigraded-vector-space valued knot invariants whose graded Euler characteristics are Reshetikhin-Turaev knot invariants; this is evidence for the existence of such a TQFT, but much remains to be clarified.

Broader impacts. If accepted, this grant will allow the PI to distribute the results of his research by travelling to conferences and research visits, as well as funding visits to UVA. Most of the visitors the PI has invited to UVA since moving there have been graduate students and postdocs at the time of their visit (including Alex Weekes, Justin Hilburn, Qi You, Iordan Ganev, Ina Petkova, Steven Sam and others), so having travel funds has enhanced his ability to mentor young mathematicians. He is currently working with two graduate students at the start of their graduate careers, and has also helped to supervise students of his colleague W. Wang, several of whom are working on areas close to the PI's research (including Huanchen Bao, an ongoing collaborator of the PI).

The PI also has a track-record of expository talks and articles aimed at levels from high school to graduate, and plans to continue this; he is the scheduled workshop leader for WARTHOG, a Talbot-style workshop held in Eugene, OR for 2017. Similarly, he has been active in conference organizing, co-organizing 2 NSF funded workshops in Charlottesville in 2016, and a conference at Luminy accepted for 2018 (and 3 other conferences since 2013). He will submit a proposal for Fields Institute special semester to be held in 2019. He has also served as a moderator and board member of `mathoverflow.net` since 2010, and on the AMS Committee on Publications (2015-8) and Web Editorial Group (2014-6).

Contents

Research program. My plans for future research include:

- studying the construction of Coulomb branches for 3-dimensional gauge theories recently proposed by Braverman, Finkelberg and Nakajima. This construction gives a natural class of commutative algebras with deformation quantizations, one for each representation V of a reductive group G. This class includes universal enveloping algebras of type A, rational Cherednik algebras for the complex reflection groups $G(r, 1, \ell)$, truncated shifted Yangians of type ADE as defined in [KWWY14] and hypertoric enveloping algebras as studied in [MV98, BLPW12], as well as many algebras which have not been considered in great detail.
- Exploration of the symplectic duality conjecture stated by the author and collaborators [BLPW16, §10]. This conjecture suggests that symplectic singularities come in dual pairs, with the Coulomb branch mentioned above being dual to the Higgs branch of the same theory, which is a hyperkähler quotient $T^*V/\!\!/\!\!/\!\!/G$.

  The author has made significant progress in proving this result in the gauge theory situation discussed above, proving a weak form of the conjecture. One future goal is to complete this proof to give the conjecture in its full strength, and understand cases outside the gauge theory context.
- Developing the theory of categorifications of quantum symmetric pairs. One important special case has been developed by the author and Bao, Shan and Wang in [BSWW] using the geometry of type B flag varieties. These categorifications should play an important role in the representation theory of classical groups/algebras of types BCD.
- Continuing to study homological knot invariants defined in [Webd], and their connection to geometry. For each representation of a simple Lie algebra, these provide a bigraded vector space-valued knot invariant whose graded Euler characteristic is the quantum invariant for that representation, defined in [Webe]. Several aspects of these invariants are essentially completely unexplored at the moment:
  - constructing 3-manifold invariants and extended TQFTs from these invariants
  - connections to the knot invariants defined in terms of field theory by Witten [Wit]
  - effective/computer-assisted computation
  - functoriality
  - spectral sequences/Rasmussen-type concordance invariants

  Several of these questions have fruitful answers for Khovanov homology, so it is natural to look for analogues for these invariants which generalize Khovanov homology.

1. CATEGORIES $\mathcal{O}$ AND SYMPLECTIC RESOLUTIONS

1.1. **Symplectic resolutions.** The geometric objects which we will deform are symplectic singularities. These have attracted great interest in recent years as their theory has been developed by Kaledin and others (see the survey [Kal09]).

Definition 1. *A* **conical symplectic singularity** *is a pair* (X, ω) *consisting of*

- *a normal affine algebraic variety* X*,*
- *a holomorphic 2-form* $\omega \in \Omega^2(X)$*, non-degenerate on the smooth locus, such that on some resolution of singularities* $Y \to X$*,* ω *extends to a closed 2-form on* Y *and*
- *an action of* $\mathbb{S} \cong \mathbb{C}^*$ *on* X *such that* ω *is a weight vector for the* $\mathbb{S}$ *action of positive weight* n*, and which contracts* X *is a fixed point* o*.*

Note that it is enough for some resolution Y of X to be itself symplectic, but that this is stronger than being a symplectic singularity. Examples of such singularities include:

- The nilpotent cone $X = \mathcal{N}_\mathfrak{g}$ for a semi-simple Lie group G has a resolution given by $Y = T^*(G/B)$ for any Borel B.
- For any finite subgroup $\Gamma \subset SL(2)$, the affine quotient surface $\mathbb{C}^2/\Gamma$ has a unique symplectic resolution $\widetilde{\mathbb{C}^2/\Gamma}$.
- More generally, the symmetric power $X = \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ has a resolution given by the Hilbert scheme $Y = \mathsf{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma})$.
- There are slices (much like Slodowy slices) transverse to one $G(\mathbb{C}[[t]])$-orbit inside another in the affine Grassmannian $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ of a complex algebraic group G.
- For any action of a compact group K on a complex vector space V, there is a hyperkähler structure on T^*V, and we can consider the hyperkähler quotient $\mathfrak{M}_{\boldsymbol{\alpha}} = T^*V////_{\boldsymbol{\alpha}}K$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ are a triple of moment map values. This is the **Higgs branch** of the corresponding gauge theory. This quotient may be singular, but it always carries a hyperkähler structure on its generic locus. In fact, if $\mathfrak{M}_{\boldsymbol{\alpha}}$ is smooth, it is a symplectic resolution of $\mathfrak{M}_{\boldsymbol{\alpha}'}$ where $\boldsymbol{\alpha}' = (0, \alpha_2, \alpha_3)$. This allows us to construct a large number of examples of symplectic resolutions, including
  - the quiver varieties of Nakajima [Nak94, Nak01] and
  - the hypertoric varieties studied by Bielawski-Dancer and others [BD00].
- There is also a general construction of a **Coulomb branch** depending on a choice of G and representation V by Braverman-Finkelberg-Nakajima [BFNb]. This construction defines the Coulomb branch by directly constructing its coordinate ring using convolution in homology of a bundle on the affine Grassmannian of G. Examples include hypertoric varieties, $\mathsf{Hilb}^n(\mathbb{C}^2)$ and the affine Grassmannian slices.

1.2. **The definition of $\mathcal{O}_X^\xi$.** If X is a conical symplectic singularity, then by [BPW16, Los], we have that:

Theorem 2. *There is a universal family* A *of deformation quantizations of the coordinate ring* $\mathbb{C}[X]$ *parametrized by a finite dimensional vector space* H *(if* X *possesses a symplectic resolution* Y*, then these are parameterized by* $H^2(Y; \mathbb{C})$ *mod a Coxeter group). We denote the ring corresponding to* $\eta \in H$ *by* A_η*.*

- In the case of $X = \mathcal{N}_\mathfrak{g}$, this deformation is $U(\mathfrak{g})$, and the algebras A_η are central quotients of $U(\mathfrak{g})$.
- In the case there X is a Slodowy slice, A is a finite W-algebra.
- In the case of $X = \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$, this deformation is the spherical symplectic reflection algebra for Γ and η corresponds to the parameter $\mathbf{c}$.
- If X is a hyperkähler quotient by a reductive group, then A_η is a noncommutative Hamiltonian reduction of $\mathcal{D}_V$.

- If X is a Coulomb branch, then [BFNb] also provides a quantization of this algebra by considering convolution in equivariant homology for the $\mathbb{C}^*$ acting by loop rotation.

Given a Hamiltonian $\mathbb{C}^*$-action $\xi : \mathbb{T} \cong \mathbb{C}^* \to \mathsf{Aut}(X)$, we can construct (by naturality of the deformation) a corresponding $\mathbb{T}$-action on A_η whose derivative is given by $[\hat{\xi}, -]$ for some $\hat{\xi} \in A_\eta$ quantizing the comoment map. Let A_η^+ be the subalgebra of non-negative weight under this $\mathbb{T}$-action.

Definition 3. *Let $\mathcal{O}_X^\xi$ be the category of A_η-modules M such that:*
- *M is finitely generated over A_η.*
- *The action of A_η^+ on M is locally finite.*

Several of these categories have already appeared in the literature and are of interest to representation theorists. For example:
- If $X = \mathcal{N}_\mathfrak{g}$, this is equivalent to a block of the original BGG category $\mathcal{O}$. More generally, we can arrive at any block of parabolic category $\mathcal{O}$ for any parabolic by considering the intersection of Slodowy slices with nilpotent orbits [Web11].
- If $Y = \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$, we obtain category $\mathcal{O}$ for the rational Cherednik algebra of the wreath product $S_n \wr \Gamma$ by [EGGO07].
- In the case where X is a hypertoric variety, these categories are equivalent to certain explicit algebras based on the combinatorics of hyperplane arrangements defined by myself, Braden, Licata, and Proudfoot [BLPW10, BLPW12].

1.3. **A duality for singularities.** One of our principle motivations for studying category $\mathcal{O}$ is understanding a notion of duality between symplectic singularities. While we're still investigating the best set of hypotheses for the conjecture below, I will state an optimistic version here.

Conjecture 4 (Braden, Licata, Proudfoot, Webster)**.** *For each conical symplectic singularity X, there is a dual singularity $X^\vee$ such that*
1. *there is an order reversing bijection between strata of X and $X^\vee$ (by* [Kal09]*, there is a canonical Whitney stratification given by symplectic leaves).*
2. *there is a bijection between symplectic $\mathbb{C}^*$ actions on X, up to conjugacy, and pairs consisting of a crepant partial resolution of $X^\vee$, and a choice of ample line bundle on that resolution. Thus, to each partial resolution $Y \to X$ with $\mathbb{C}^*$-action ξ, we have a dual $Y^\vee \to X^\vee$ with action $\xi^\vee$.*
3. *Furthermore, the categories $\widetilde{\mathcal{O}}_Y^\xi$ and $\widetilde{\mathcal{O}}_{Y^\vee}^{\xi^\vee}$ are Koszul dual, and, in particular, derived equivalent. This duality switches the action of two pairs of natural functors on category $\mathcal{O}$ which arise from changing the choice of $\mathbb{C}^*$-action (**shuffling functors**) and from the choice of period (**twisting functors**).*

For example:
- If $Y = T^*G/B$ then $Y^\vee = T^{*L}G/{}^LB$, and duality claim is equivalent to the Koszul duality theorem for categories $\mathcal{O}$ due to Beilinson, Ginzburg, and Soergel [BGS96].
- If X is a hypertoric singularity associated to a hyperplane arrangement H, then $X^\vee$ is another hypertoric singularity, associated to the Gale dual $H^\vee$. Braden, Licata, Proudfoot, and I have proven the conjecture in this case [BLPW10, BLPW12].

- Conjecturally, if Y is the space of G-instantons on the algebraic surface $\widetilde{\mathbb{C}^2/\Gamma}$, then $Y^\vee$ is the G' instantons on $\widetilde{\mathbb{C}^2/\Gamma'}$ where G and Γ' (resp. G' and Γ) are matched by the MacKay correspondence. If Γ and Γ' are both cyclic groups, then both of these spaces are affine type A Nakajima quiver varieties, and the desired duality is confirmed in [Web17]
- Conjecturally, if Y is the Nakajima quiver variety for weights λ and μ in type ADE, then $Y^\vee$ is the slice to Gr_μ in $\overline{\mathrm{Gr}_\lambda}$ where Gr_μ denotes an orbit of polynomial loops in the affine Grassmannian of the Langlands dual group. Denote this slice $\mathfrak{W}^\lambda_\mu$.

  Conjecturally, this perspective will relate the homological knot invariants defined in terms of quiver varieties (including Khovanov-Rozansky homology) to those defined using the affine Grassmannian by work of Seidel-Smith [SS06], Manolescu [Man07], and Cautis-Kamnitzer [CK08].

In the first three cases considered above, we can confirm the desired Koszul duality "by hand" using explicit computations. In the fourth, I'll present an approach to doing so below.

After formulating an early version of this conjecture, we discovered that physicists had found many of the same pairs of examples as the Higgs and Coulomb branches of 3-dimensional $\mathcal{N} = 4$ supersymmetric field theories. This was established in the physics literature for T^*G/B [GW], and for hypertoric singularities [KS99], and both finite and affine type A quiver varieties [dBHOO97]. Since a precise mathematical construction of Coulomb branches for certain of these theories (those with a "Lagrangian description") was discovered in [Nak, BFNb], we can now systematically explain the examples above, but this doesn't seem to give the most general construction of dual pairs.

Goal 3. *Construct more general dual pairs of singularities, and direct method of constructing $X^!$ directly from X.*

2. Coulomb branches

The "freshest" set of varieties discussed here are the Coulomb branches constructed by Braverman, Finkelberg and Nakajima. While it's quite hard to give a presentation of quantized Coulomb branches in general, this algebra can be realized as the endomorphisms of an object in a larger category, as described in [Webf]. The representation V has weights $\varphi_1, \dots, \varphi_d$, which we can think of linear functions of a Cartan subalgebra $\mathfrak{t}$ of the Lie algebra of G; we let $\mathfrak{t}_\mathbb{Z}$ be the group of cocharacters in the corresponding torus T in T, and $\mathfrak{t}_R \cong R \otimes \mathfrak{t}_\mathbb{Z}$. The objects of the category $\mathscr{B}$ are the chambers in the complement in $\mathfrak{t}_\mathbb{R}$ of the hyperplanes defined by $\varphi_i(x) \in \mathbb{Z} + 1/2$. The morphisms are generated by morphisms of 4 types:

- each chamber has a copy of the coordinate ring $S := \Bbbk[\mathfrak{t}_\Bbbk]$ acting on it.
- there is a morphism $r(P) \cong C \to D$ for each path P between chambers C and D
- there is a morphism $w \colon C \to wC$ for each element w of the extended affine Weyl group $\tilde{W}$.
- if a reflection s_α in the extended affine Weyl group preserves a chamber $s_\alpha \cdot C = C$ then we formally adjoin the action of the Demazure operator $\psi_\alpha = \frac{s_\alpha - 1}{\alpha}$.

The relations between these operators are straightforward: polynomials and paths commute, conjugation by elements of the extended Weyl group act in the natural action on paths, and on polynomials by the level 1 affine action on $\mathfrak{t}_{\Bbbk}$. Any two paths of minimal length between two chambers are equal. There is only other needed relation which also the only which involves the choice of V; this relation also depends on a set of parameters c_i for $i = 1, \dots, d$ which we can think of as numerical or formal. When a path crosses a hyperplane H in the arrangement defined above by $\varphi_i(x) = n$ twice, then we can remove both crossings at the cost of multiplying by $\varphi_i + \hbar(c_i - n + 1/2)$.

Let C_0 be the chamber containing the origin, and let $e = \frac{1}{\#W}\sum_{w\in W} w$ be the sum of the elements of the finite Weyl group, thought of as an idempotent morphism $e\colon C_0 \to C_0$. We can formally adjoin the "image" eC_0 of this morphism; this is an object in the Karoubi envelope of $\mathscr{B}$.

Theorem 5. *The algebra* $A_{\mathbf{c}} := \operatorname{Hom}(eC_0, eC_0) \cong e\operatorname{Hom}(C_0, C_0)e$ *is isomorphic to the quantized Coulomb branch defined in* [BFNb].

In particular, $\operatorname{gr} A_{\mathbf{c}}$ is the coordinate ring of a conic symplectic variety $\mathfrak{M}$.

This perspective is a powerful tool for understanding the representation theory of quantum Coulomb branches. We call a module over $A_{\mathbf{c}}$ (for numerical parameters) **integral** if it is the sum of the generalized eigenspaces for S with eigenvalues in $\mathfrak{t}^*_{\mathbb{Z}}$, and each of these eigenspaces is finite dimensional (but we allow the whole module to be infinite dimensional). We can naturally extend this definition to representations of the category $\mathscr{B}$ (i.e. functors $\mathscr{B} \to \Bbbk$-vect), requiring that the copy of S acting on each object have this property.

Note that if we consider any $\xi \in \mathfrak{t}$, and we adjust the parameters by $c_i \mapsto c_i + \langle \varphi_i, \xi\rangle$, then we obtain an isomorphic algebra, but a changed definition of integral, so this notion can actually capture weight modules with weights in any fixed coset of $\mathfrak{t}^*_{\mathbb{Z}}$ (and every indecomposable weight module has weights in a single coset).

2.1. **Characteristic 0.** Assume that $\Bbbk$ is characteristic 0. We can give a description of the category of weight modules in terms of a finite dimensional algebra: we consider the hyperplane arrangement defined by the hyperplanes $\varphi_i(x) = c_i - 1/2$ for $c_i \in \mathbb{Z}$. Let $W_{\mathbf{c}}$ be the subgroup of the Weyl group consisting elements w such that exists a permutation $\tilde{w} \in S_d$ such that $c_i = c_{\tilde{w}\cdot i}$ and $w \cdot \varphi_i = \varphi_{\tilde{w}\cdot i}$. We define a category $\mathscr{X}$ whose objects are the chambers of this new hyperplane arrangement, with morphisms given

- each chamber has a copy of the completed coordinate ring $\hat{S} := \Bbbk[[\mathfrak{t}_{\Bbbk}]]$ acting on it.
- there is a morphism $\pi(P) \cong C \to D$ for each path P between chambers C and D
- there is a morphism $w\colon C \to wC$ for each $W_{\mathbf{c}}$.
- if a reflection s_α in $W_{\mathbf{c}}$ preserves a chamber $s_\alpha \cdot C = C$ then we formally adjoin the action of the Demazure operator $\psi_\alpha = \frac{s_\alpha - 1}{\alpha}$.

The relations are unchanged, except that when crossing a wall, we multiply by φ_i without any shift. Let $\mathscr{X}'$ be the subcategory where we only consider objects corresponding to chambers containing integral points.

While this category may look slightly unusual, it is a natural generalization of the KLR algebra [KL09, Rou] (we can think of $\mathscr{X}$ as an algebra by taking the sum of all morphism spaces). The KLR algebra (and more generally, weighted KLR algebra [WebI]) appears if we consider the case where we fix a quiver Γ and dimension vector $\mathbf{v}$, and consider $V = \oplus_{i\to j}\operatorname{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j})$ and $G = \prod GL_{v_i}(\mathbb{C})$.

In the case where G is abelian, we again obtain a well-known algebra, which was studied by Musson and van der Bergh [MV98] in connection with the geometry of toric varieties.

Theorem 6. *The category of integral weight modules over $A_{\mathbf{c}}$ is equivalent to the finite-dimensional representations of the category $\mathscr{X}'$. The equivalence matches generalized weight spaces for $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$ of the module N to the image in $\Bbbk$-vect of the chamber C containing λ.*

This functor is induced by a equivalence of integral weight modules for $\mathscr{B}$ to finite-dimensional representations of $\mathscr{X}$.

While this theorem is relatively straightforward, a major line of research moving forward will be unpacking its consequences, since outside the quiver and abelian cases, both sides of this equivalences are relatively unexplored. In particular, this result can be used in specific cases, like rational Cherednik algebras of $G(r, 1, n)$ to determine the character of simple modules. It seems likely this can be extended more generally, but at the moment, we cannot even easily index the set of simple modules:

Goal 4. *Use the theorem above to determine the set of and structure of simple weight modules over $A_{\mathbf{c}}$.*

On example of how this may be possible is the "quantum Hikita conjecture":

Conjecture 7. *The simple modules in category $\mathcal{O}$ for a symplectic singularity X are in bijection with the spectrum of $H^{G^! \times \mathbb{C}^*}(\tilde{X}^!)$ specialized at $h = 1$ for $G^!$ the group of Hamiltonian endomorphisms of $\tilde{X}^!$ commuting the conic $\mathbb{C}^*$-action, and h the equivariant parameter of the conic action.*

It seems likely that Theorem 6 can be applied to give an explicit proof of this theorem when X is a Coulomb branch. It would be quite interesting (though considerably more challenging) to prove this when X is a Higgs branch instead.

Based on examples, this theorem should have a powerful application to localization. Every choice of $\mathbf{c}$ has an associated partial resolution of the Coulomb branch $\tilde{\mathfrak{M}}_{\mathbf{c}}$. There is a natural functor from $A_{\mathbf{c}}$-modules to the category of modules over a quantization of the structure sheaf on $\tilde{\mathfrak{M}}_{\mathbf{d}}$:

Conjecture 8. *Localization for integral weight modules to the partial resolution $\tilde{\mathfrak{M}}_{\mathbf{d}}$ holds for $A_{\mathbf{c}}$ if the chambers of the arrangment defined by $\varphi_i(x) = d_i$ for $c_i \in \mathbb{Z}$ match those of the arrangement $\varphi_i(x) = c_i + 1/2$ which contain integral points (where we match two chambers if they have the same sign vector describing which side of each hyperplane they lie on). Localization holds for all modules if and only if it holds for integral weight modules for the parameters $c_i + \langle \varphi_i, \xi \rangle$ for all $\xi \in \mathfrak{t}$.*

One interesting possibility is to consider how this algebraic construction can be modified to apply in situations where the geometric construction of [BFNb] does not. For example, if we have a symplectic action of a group on a vector space which does not admit an invariant Lagrangian subspace, then there is still an attached Higgs branch and gauge theory, but the construction of [BFNb] does not make sense. Similarly, there are Yangians defined for non-simply laced Lie algebras, but with no associated representation V as defined above.

Question 9. *Can the construction of $\mathscr{B}$ be modified to give quantizations of the Coulomb branch for general symplectic representations and to recover the Yangians for non-simply-laced Lie algebras?*

Is the category of representations of a non-simply-laced Yangian related to the KLR algebra of this type, or to some new class of algebras?

We can naturally describe the category $\mathcal{O}$ introduced above as a subcategory of weight modules: it is the category of representations of $\mathscr{X}$ modulo 2-sided ideal generated by the identity morphism of any chamber on which ξ (thought of as a function on $\mathfrak{t}_{\mathbb{R}}$) does not achieve a maximum. Thus, these category $\mathcal{O}$'s have a relatively simple presentation.

We can apply this to truncated shifted Yangians Y^λ_μ in types ADE. These algebras that appear as quantizations of slices within the affine Grassmannian $\mathfrak{W}^\lambda_\mu$; they are also examples of quantized Coulomb branches associated with $V = (\oplus_{i\to j}\operatorname{Hom}(\mathbb{C}^{v_i},\mathbb{C}^{v_j}))\oplus(\oplus_i \operatorname{Hom}(\mathbb{C}^{v_i},\mathbb{C}^{w_i})$ and $G=\prod GL_{v_i}(\mathbb{C})$. The corresponding Higgs branch is a Nakajima quiver variety, and the algebra $\mathscr{X}$ corresponds to an algebra $\tilde{T}^{\mathbf{c}}(\mu)$ constructed in [Webd, §4] as part of an approach to categorifying tensor products.

Conjecture 10. *The integral representations in category $\mathcal{O}$ of a truncated shifted Yangian for each choice of* $\mathbf{c}$ *are equivalent to the representations of the algebra $eT^{\mathbf{c}}(\mu)e$ for an idempotent e depending on* $\mathbf{c}$.

In particular, the highest weights of category $\mathcal{O}$ of Y^λ_μ are in bijection with elements of the monomial crystal $\mathscr{B}(\mathbf{c})$ of weight μ and given by the conjecture of [KTW+].

One of the most significant features of $eT^{\mathbf{c}}e$ is a categorical action of the underlying Lie algebra $\mathfrak{g}$ of type ADE. It seems natural that these could be constructed geometrically in terms of Yangians.

Conjecture 11. *The action of the restriction functor $\mathcal{E}\cong\oplus\mathcal{E}_i$ on modules over $eT^{\mathbf{c}}e$ is intertwined with the functor of taking Whittaker vectors for a morphism $r(*,*-x)$ for a particular choice of $x\in\mathfrak{t}$. The adjoint $\mathcal{F}\cong\oplus\mathcal{F}_i$ is given by taking the obvious left adjoint to Whittaker vectors, and then the universal weight module receiving a map from the result.*

2.2. **Characteristic** p**.** Now let $\Bbbk$ be a field of characteristic p, and let $\tilde{\mathfrak{M}}=\tilde{\mathfrak{M}}_{\mathbf{d}}$ for any $\mathbf{d}$ generic. This approach to Coulomb branches is especially interesting in this case. The algebra $A_{\mathbf{c}}$ now has a large center:

Conjecture 12. *The center of $A_{\mathbf{c}}$ is isomorphic to $\Bbbk[\mathfrak{M}^{(1)}]$, the functions on the Frobenius twist of the Coulomb branch.*

There is an Azumaya sheaf $\mathscr{A}$ of algebras on $\tilde{\mathfrak{M}}^{(1)}$ such that $A_{\mathbf{c}}\cong\Gamma(\tilde{\mathfrak{M}}^{(1)};\mathscr{A})$ identifying the center of $A_{\mathbf{c}}$ with the sections of $\mathscr{O}_{\mathfrak{M}}\subset\mathscr{A}$.

This Azumaya sheaf cannot be split (since $A_{\mathbf{c}}$ is an integral domain), but work of Stadnik [Sta13] suggests how it can be split on an étale cover. Consider the subalgebra in $A_{\mathbf{c}}$ generated by the center and S^W, the invariants of the finite Weyl group W acting on S. Since S is a commutative algebra, the resulting algebra is commutative, and its spectrum $\mathfrak{P}$ has a finite map $\mathfrak{P}\to\mathfrak{M}^{(1)}$. The intersection of S with the center is the W-invariant polynomials in the elements X^p-X with $X\in\mathfrak{t}_{\Bbbk}$, so this map is a base change by the Artin-Schreier map AS: $\operatorname{Spec}S^W\to\operatorname{Spec}S^W$.

Conjecture 13. *The map $\mathfrak{P}\to\mathfrak{M}^{(1)}$ is étale and base change by it splits the Azumaya algebra $\mathscr{A}$. The splitting bundle on a formal neighborhood of the fiber over a maximal ideal $\mathfrak{n}$ in* $\operatorname{Spec}S^W$ *is given by*

$$\mathcal{S}=\oplus_{\mathfrak{m}\in\mathrm{AS}^{-1}(\mathfrak{n})}\varinjlim\mathscr{A}/\mathscr{A}\mathfrak{m}^n$$

These completions $\varinjlim\mathscr{A}/\mathscr{A}\mathfrak{m}^n$ have correspond to the modules over $A_{\mathbf{c}}$ that represent the functor of taking the summand of a module where $\mathfrak{m}$ acts nilpotently. This is effectively a weight space as discussed in earlier sections.

In this case, the weights of an integral module are a subset of $\mathfrak{t}_{\mathbb{Z}}/p\cdot\mathfrak{t}_{\mathbb{Z}}$; we can view these as lying in the torus $\mathfrak{t}_{\mathbb{R}}/p\cdot\mathfrak{t}_{\mathbb{Z}}$. Consider the toric arrangement in $\mathfrak{t}_{\mathbb{R}}/p\cdot\mathfrak{t}_{\mathbb{Z}}$ defined by the hyperplanes $\varphi_i(x) = c_i + 1/2$ for $c_i \in \mathbb{F}_p$. Define the category $\mathscr{X}_p$ to have objects given by the chambers of this arrangment in $\mathfrak{t}_{\mathbb{R}}/p\cdot\mathfrak{t}_{\mathbb{Z}}$, and all generators and relations otherwise the same as $\mathscr{X}$. Despite the similarity, this category is quite different. In particular, the closed paths give a much larger endomorphism algebra for each chamber.

Conjecture 14. *The endomorphism algebra of the splitting bundle $\mathcal{S}$ is Morita equivalent to the sum of the morphism spaces in $\mathscr{X}_p$.*

General results of Kaledin guarantee that the splitting bundle on the fiber of the cone point of $\mathfrak{M}^{(1)}$ can be extended over $\tilde{\mathfrak{M}}^{(1)}$ and give a tilting generator (for p sufficiently large and $\mathbf{c}$ generic). This tilting generator can be lifted to characteristic 0. On the other hand, we can define a category $\mathscr{X}_p(\mathbb{k})$ for $\mathbb{k}$ of any characteristic; we let $\underline{\mathscr{X}}_p(\mathbb{k})$ be this same category, but with an action of an uncompleted polynomial ring on each chamber, instead of the completion.

Recall that we call a coherent sheaf $\mathscr{F}$ on a variety Y a **tilting generator** if $\mathrm{Ext}^i(\mathscr{F},\mathscr{F})$ vanishes for all $i > 0$, and $\mathscr{F}$ generates the category of coherent sheaves. Since the structure sheaf of an affine variety is a tilting generator, this provides a generalization of the notion of an affine variety. In particular, we get an equivalence of derived categories $D^b(\mathrm{End}(\mathscr{F})) \cong D^b(\mathsf{Coh}(Y))$.

Conjecture 15. *For any field $\mathbb{k}$, there is a tilting generator on $\tilde{\mathfrak{M}}$ with endomorphism ring given by the sum of morphisms in $\underline{\mathscr{X}}_p(\mathbb{k})$.*

For the case of quiver gauge theories discussed earlier (which includes the cases of affine Grassmannian slices in types ADE, type A Slodowy slices and affine type A quiver varieties), the resulting algebra is a cylindrical KLR algebra, an algebra given by string diagrams drawn on a cylinder which satisfy the same local relations as the KLR algebra.

In fact, there are many such tilting generators, corresponding to different choices of $\mathbf{c}$ and obtained by tensoring a given tilting generator with a line bundle. Each of these gives a different t-structure on coherent sheaves. Conversely, the same algebras appear as endomorphisms of tilting generators on the different partial resolutions $\tilde{\mathfrak{M}}_{\mathbf{d}}$ for different d. Assuming Conjecture 15, these isomorphisms give equivalences of derived categories

$$D^b(\mathsf{Coh}(\tilde{\mathfrak{M}}_{\mathbf{d}})) \cong D^b(\mathsf{Coh}(\tilde{\mathfrak{M}}_{\mathbf{d}'})).$$

Crucially, these equivalences depend on the choice of quantization parameter $\mathbf{c}$, and thus the compositions of these equivalences give non-trivial automorphisms of these derived categories.

Bezrukavnikov, Maulik and Okounkov [MO] have suggested a remarkable program relating these different t-structures to the quantum connection for $\tilde{\mathfrak{M}}$; in particular, they conjecture that there is an action of π_1 of the complement of a subtorus arrangement on the derived category that categorifies the quantum connection of $\tilde{\mathfrak{M}}$.

Goal 5. *Use the results of Conjectures 12–15 to verify this program in the Coulomb case.*

2.3. **Cherednik algebras.** One Coulomb branch of particular interest is that attached to the group $G = GL(n)$ and $V = \mathfrak{gl}(n) \oplus (\mathbb{C}^n)^{\oplus \ell}$.

Theorem 16 (Kodera-Nakajima). *The commutative Coulomb branch in this case is* $\mathsf{Hilb}^n(\mathbb{C}^2/\widetilde{(\mathbb{Z}/\ell\mathbb{Z})})$ *and the noncommutative Coulomb branch is given by the spherical rational Cherednik algebra for the complex reflection group* $G(\ell,1,n)$ *(the group of permutation matrices with non-zero entries in the* ℓ*th roots of unity).*

Based on this observation, the author has given a new presentation of the Cherednik algebra of $G(\ell,1,n)$, which makes this isomorphism more natural, and allows an algebraic construction of the restriction, induction and KZ functors. This presentation also allows for a new proof of the relationship of this Cherednik algebra to weighted KLR algebras, analogous to Theorem 6.

Question 17. *How can these results be extended to other complex reflection groups, especially* $G(\ell e,e,n)$*? What is the correct generalization of Rouquier's description of decomposition numbers for the Cherednik category* $\mathcal{O}$ *of* $G(\ell,1,n)$ *in terms of canonical bases? Can the category of weight modules over the Cherednik algebra be given a Koszul grading?*

A solution to this question would also lead to a grading on the Hecke algebras of these groups (and this characteristic p group algebras) analogous to the grading found for $G(\ell,1,n)$ by Brundan and Kleshchev [BK09].

Having an algebraic presentation of the category of weight modules over Cherednik algebra has another interesting consequence: we can consider its reduction modulo p (note that this is quite different from the modules of the Cherednik algebra itself over characteristic p), and obtain a quasi-hereditary cover of the representations of any complex reflection group in characteristic p analogous to the Schur algebra.

2.4. **Higgs branches.** In the Higgs cases, we also have powerful techniques for studying the category $\mathcal{O}$'s that appear. For the Higgs case, we can study D-modules on V which are equivariant for K. We can construct one of these for every lift $\tilde{\xi}$ of ξ to a linear action of $\mathbb{C}^*$ on V, by letting V_ξ be the positive weight spaces for this lift, and $\mathfrak{Y}_\xi$ be the D-module pushforward of the structure sheaf on $(K\times V_\xi)/P_\xi$ (where $P_\xi\subset K$ is a parabolic preserving V_ξ).

This is a sum of shifts of semi-simple D-modules by the Decomposition Theorem. The Hamiltonian reduction $\mathfrak{r}(\mathfrak{Y}_\xi)$ of this D-module lies in category $\mathcal{O}$ as shown in [Web17].

Note that the set V_ξ will be the same for many different choices of ξ; we let B be a set of lifts that represents each of these different subspaces once. If we let $R=\operatorname{Ext}^\bullet(\oplus_{\xi\in B}\mathfrak{Y}_\xi)$, then:

Theorem 18. *The algebra* R *is isomorphic to the sum of morphisms in* $\mathscr{X}'$*. The abelian category generated by summands of* $\oplus_{\xi\in B}\mathfrak{Y}_\xi$ *is the Koszul dual of the category* $\mathscr{X}'$*.*

We can study $\mathcal{O}^\xi$ by considering the functor $\Theta\colon R\text{-dgmod}\to D^b(\mathcal{O}^\xi)$ induced by Hamiltonian reduction of D-modules.

Conjecture 19. *If* $\mathfrak{M}_\alpha = T^*V////_\alpha K$ *is smooth,* Θ *is a dg-quotient functor. In general,* Θ *is a dg-quotient functor onto its essential image, which is a block of* $\mathcal{O}^\xi$*. For general* ξ *and* η*, this kernel is generated by the simples corresponding to chambers on which* η *is not bounded.*

For the quiver and hypertoric cases, we have exploited the structure of the situation to confirm this result in [Web17].

We call the block given by the image of Θ the *integral block* of category $\mathcal{O}$ Note the similarity to the description of category $\mathcal{O}$ for a Coulomb branch given above. Thus, together these results should yield:

Conjecture 20. *The integral block of category $\mathcal{O}$ for a Higgs branch is the Koszul dual of a block for the corresponding Coulomb branch, with the $\mathbb{C}^*$-action ξ and period η switching roles.*

Theorem 18 is significant progress toward this goal, but certain technical issues embodied in the conjectures above remain to be resolved.

3. Categorified symmetric pairs

One very rich structure, which made an important appearance in the problems discussed above is the notion of a categorical Lie algebra action. These are given by functors $\mathcal{E}_i$ and $\mathcal{F}_i$ which categorify the Chevalley generators of a simple Lie algebra. We do not impose the Chevalley relations on these directly, however, but rather require the existence of natural transformations of these functors which force the Chevalley relations to hold.

Examples of categories with these actions include category $\mathcal{O}$'s for quiver varieties and truncated shifted Yangians as discussed above, including Cherednik categories $\mathcal{O}$ and classical category $\mathcal{O}$ in type A, and lot of generalizations of these categories. However, the categories $\mathcal{O}$ for other classical types BCD don't have such actions. Recent work of the author with Bao, Shan and Wang shows that these category $\mathcal{O}$'s instead have a natural action of a different 2-category, which categorifies a coideal subalgebra of $U_q(\mathfrak{sl}_{2n+1})$. This category was constructed using the geometry of type B flag varieties (much as that for $U_q(\mathfrak{sl}_n)$ can be constructed from type A flag varieties), but has a purely algebraic definition.

This construction naturally raises the question:

Question 21. *Is there a natural KLR-type categorification for any coideal subalgebra arising from a symmetric pair? That is, a 2-category whose Grothendieck group is the coideal subalgebra, equipped with a deformation to the KLR categorification of the ambient simple Lie algebra.*

The structure of deformation is suggested by the case explored above, and other situations where an inclusion of algebras is categorified.

Work of Yiqiang Li has suggested that symmetric pairs in simply laced Lie algebras have a corresponding generalization of Nakajima quiver varieties. Once these varieties are on a firm footing, we can expect to generalize the results of [Web15, Web17, Weba] to this context:

Conjecture 22. *The category of DQ-modules on $\imath$-quiver varieties carry a categorical action of the corresponding coideal subalgebra.*

The classes of simple DQ-modules subject to different support conditions give canonical bases in representations of the coideal subalgebra, and classes of indecomposable 1-morphisms in the categorified

This conjecture can be read in either direction: as a condition on a proposed definition of a categorical action of a coideal subalgebra, and as a guide to giving such a definition.

4. Knot homology

One interesting direction in which the study of these categories leads is the construction of homological knot invariants.

As described in the introduction, the geometric considerations in the introduction inspired a purely algebraic definition of homological invariants given in [Webd]. These invariants are constructed by associating a functor between categorifications of tensor products attached to any labeled tangle. The conjectures in the previous section would show that this invariant has a geometric origin when the Lie algebra $\mathfrak{g}$ is simply laced, but there are also plenty of questions which remain to be resolved from an algebraic perspective.

For instance, these invariants are not the only known categorifications of the quantum invariants for the fundamental representation of $\mathfrak{sl}(n)$. Khovanov and Rozansky [KR08] have already defined such a categorification using the seemingly unrelated tool of matrix factorizations, and Cautis and Kamnitzer defined one using the coherent sheaves on convolution varieties in the affine Grassmannian. In fact, these constructions all have a common origin: they are all given by taking the image under an action functor of a knot invariant constructed in a categorification of $\mathfrak{sl}(\infty)$. Recent work of the author and Mackaay [MW] shows that the construction based on categorified tensor products has this form, based on a categorified skew Howe duality (which is one version of the Koszul duality discussed in Section 1.3).

However, for other Lie algebras, the correct generalization of this statement is much harder to guess. In the cases of B_2/C_2 and G_2, the endomorphisms of tensor products of fundamental representations are described by Kuperberg in terms of spiders [Kup96]. Since [MW] proceeded by categorifying the webs in the $\mathfrak{sl}(n)$ spider into bimodules, it seems likely that a similar operation is possible here:

Conjecture 23. *The Kuperberg spiders B_2/C_2 and G_2 have categorifications which act on the categorified tensor products of fundamental representations over these algebras commuting with the categorical Lie algebra action. This construction generalizes to categorified Brauer algebra action on the tensor product of vector representations over $\mathfrak{sp}(2n)$.*

Also, Witten has suggested a construction of knot invariants based on the Morse homology of certain spaces of solutions to PDEs; while this perspective sounds quite distant from ours, there are in fact unexpected links. Witten's perspective should attach a category to a disk labeled with finitely many representations of a simple Lie group. The category which seems to arise is a version of A-branes on a convolution space for the affine Grassmannian; as we will discuss later, our quantization framework is a mathematical version of this category, and conjecturally gives rise to the categorifications of tensor products $T^{\underline{\lambda}}\text{-mod}$ discussed earlier. Thus, we conjecture:

Conjecture 24. *Witten's construction can be made precise using quantizations of affine Grassmannians and naturally arising functors between them.*

There are two other structures which appeared in the original Khovanov homology (which is associated to the defining representation of $\mathfrak{sl}_2$) whose extensions to this more general theory could be quite interesting.

Khovanov homology was shown to be functorial (up to sign) by Jacobsson [Jac04] and this functoriality was interpreted in the representation-theoretic context by Stroppel using the adjunction of twisting functors [Str05, Str]. When the weights λ_i are all minuscule, then this prescription can be extended to the invariants from [Webe] to give a possible functoriality map, but this map is constructed using a handle decomposition of the cobordism, and it is unclear if the map is independent of cobordism.

Conjecture 25. *For each finite dimensional semi-simple Lie algebra $\mathfrak{g}$, there is a functorial invariant of links labeled with minuscule representations of $\mathfrak{g}$, valued in bigraded vector spaces, whose Euler characteristic is the Reshetikhin-Turaev invariant of this knot.*

An important barrier to the understanding functoriality and to general computation of these invariants is that their construction uses in a quite essential way the functor of tensor product with a module, which while finitely presented, has no known explicit projective resolution. Thus, at several important points, one uses the fact that any two projective resolutions of a module are homotopic; while this is sufficient for checking that the underlying vector space is a knot invariant, in order to show functoriality, one needs some understanding of the underlying homotopies.

Question 26. *Is there an explicit projective resolution of the functor attached to an arbitrary tangle projection, together with explicit and coherent homotopies between different projections of isotopic tangles?*

While not easy, this is not quite as daunting as it sounds; if one constructs projective resolutions of the functors for a single crossing, a single cup and a single cap, then by naturality this gives a projective resolution for any tangle projection. Similarly by usual topological arguments, one need only give homotopies corresponding to Reidemeister moves, and check a finite number of "movie moves" to check coherence.

Such a description could also be extremely useful for computation, which at the moment is the weak point of this theory.

One particularly interesting possibility is that this may allow us to construct some sort of homological version of Witten-Reshetikhin-Turaev invariants. In this case, the difficulties of defining WRT invariants for q not a root of unity could be explained by the fact that infinite dimensional vector spaces often do not have well defined graded dimensions.

Another important structure on Khovanov homology is the existence of the Lee spectral sequence which converges from Khovanov homology to a link invariant which only counts the number of link components, described in [Ras10]. This spectral sequence and the functoriality of Khovanov homology was used by Rasmussen to construct a lower bound on slice genus which provides, for example, the first combinatorial (i.e. not using gauge theory) proof that the slice genus of a (p, q)-torus knot is $(p-1)(q-1)/2$. Lobb used a similar spectral sequence on Khovanov and Rozansky's homology to construct an infinite sequence of such lower bounds.

Conjecture 27. *There are spectral sequences which converge from the knot invariants of* [Webe] *to knot homologies only depending on the number of link components. More generally, there are spectral sequences from the homology for any choice of labeling to the homology attached to any Levi subgroup by the restriction of the corresponding representations to that Levi.*

The induced filtration on the limit for a knot defines a lower bound on slice genus of that knot.

5. Results of prior support

From June 2012 through June 2017, I have been supported by an NSF CAREER grant, with the title "CAREER: Representation theory of symplectic singularities" and numbers DMS-1151473 and DMS-1419500. During this time, I have completed 23 preprints, 14 of which are accepted for publication.

Each of these papers has been published and distributed on the website `http://www.arxiv.org`. A full list with citations is given in the bibliography. The lines of research covered by this work include:

- the categorification of tensor products and Fock spaces.
  - This research was motivated by my interest in the categorification of Reshetikhin-Turaev invariants, which was done in [Webd]. I continued investigation of these invariants in [Webj, MW], which showed that the invariants I defined for $\mathfrak{sl}(n)$ agree with a number of other proposed categorifications in type A.
  - Consideration of tensor products from an abstract perspective lead to an axiomatic characterization of categorified tensor products, given in [LW15]. This in turn led us to apply these results to category $\mathcal{O}$ of superalgebras, and give a new proof of the Brundan conjecture on the Kazhdan-Lusztig polynomials for $\mathfrak{sl}(m|n)$ in [BLW].
  - Higher level Fock spaces give a generalization of tensor products of simple modules in the affine type A case. The paper [SW] gave one geometric schema for categorifying these, which was further generalized in [Webl, Webg, Webi]. The latter paper gave a proof of Rouquier's conjecture relating the decomposition numbers of rational Cherednik algebras for the groups $G(r, 1, n)$ to the canonical basis of q-Fock space.
  - The papers [Web15, Webc] investigated the relationship between these categories and canonical bases.
  - The paper [BSWW] initiated the program of extending this notion of categorical representation theory to quantum symmetric pairs, treating a case of particular importance in the geometry of type BCD flag varieties.
  - The paper [Webk] studied deformations of KLR algebras, relating these algebras for different Cartan data, and providing a key step in the most general non-degeneracy result for categorical Lie algebra actions.
  - The papers [BHLW17, BHLW16, Webb] studied the relationship between current algebras, (co)centers of cyclotomic KLR algebras and the cohomology of quiver varieties.
- the representation theory of quantizations of conical singularities.
  - The paper [BPW16] considered the basic theory of these representations, including twisting functors and localization.
  - The paper [BLPW16] introduced a generalized category $\mathcal{O}$ attached to these varieties, developed its general properties, and used this category to state the symplectic duality conjecture on these varieties discussed in the proposal.
  - The papers [Weba, Web17] related this theory in the case of quiver varieties to the categorification of representation of Lie algebras. In particular, it shows that the categories $\mathcal{O}$ in these cases can be described using weighted KLR algebras.
  - The papers [KWWY14, KTW$^+$] and the appendix of [BFNa] initiated the study of truncated shifted Yangians for general Lie algebras, and related these algebras to slices in the affine Grassmannian.
  - The paper [Webh] gives a new presentation of the rational Cherednik algebras for the groups $G(r, 1, n)$, inspired by the Coulomb branch formalism discussed earlier.

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