Now, we wish to show that Spec(n) is also a maximal set w/ equivalence relation satisfying 1,2,3.

It's clear that 1 holds: this is just the fact that $X_1 = 0$.

OK, so how do we check properties 2 and 3? We have to isolate how X_k and X_{k+1} behave. The trick here is to think of them as independent variables (and then impose relations later). Let $s_i = (i, i+1)$. Then we have $s_i X_k s_i = X_k$ if $k \neq i, i+1$, and $s_k X_k s_k = \sum_{i<k} (i, k+1) = X_{k+1} - s_k$. Alternately, $s_k X_{k+1} - X_k s_k = 1$.

<u>Def</u> The degenerate affine Hecke algebra H_n is the algebra generated by $\mathbb{Z}S_n$ and $\mathbb{Z}[X_1, \ldots, X_n]$ modulo the relations:

$$s_m X_k s_m = \begin{cases} X_k & m \neq k, k+1 \\ X_{k+1} - s_k & m = k \\ X_{k-1} + s_k & m = k+1 \end{cases}$$

We have a surjective map $H_n \to \mathbb{Z}S_n$ sending $s_m \mapsto s_m$, $X_k \mapsto X_k$. The kernel of this map is the 2-sided ideal generated by X_1. (Since modulo this ideal, we have $X_2 = s_1 X_1 s_1 + s_1 \equiv s_1$, $X_3 = s_2 X_2 s_2 + s_2 \equiv (13) + (23)$. In general,

$X_k = s_k s_{k-1} \cdots s_1 X_1 s_1 \cdots s_k + s_{k-1} + s_{k-1} s_{k-2} s_{k-1} + s_{k-1} s_{k-2} s_{k-3} s_{k-2} s_{k-1} + \cdots$

Perhaps more importantly, X_k, X_{k+1}, S_k satisfy relations of H_2, and X_k, X_{k+1}, X_{k+2}, and s_k, s_{k+1} the relations of H_3.

We call a representation of H_n unitary if it has an inner-product such that X_i, S_i are self adjoint. Note that any rep of $\mathbb{C}S_n$ is unitary because any inner product invariant under S_n sends X_k to self-adjoint operators. Note, this implies that X_k is diagonalizable, and so all X_k's are simultaneously diagonalizable.

Now, assume that V is an S_n-rep, v is a common eigenvector of the X_k's and $(a_1, \ldots, a_n)$ is the weight. Then $\{v, s_k v\}$ span a subrep under the action of H_2.

Let $a = a_k$, $b = a_{k+1}$ be the eigenvalues of X_k, X_{k+1}.

This is a quotient of the tensor product $H_2 \otimes_{\mathbb{C}[X_k, X_{k+1}]} \mathbb{C}\cdot w$, where $X_k \cdot w = aw$, $X_{k+1} \cdot w = bw$

This is 2-dimensional, spanned by $w, s_k w$. If this is unitary w/ $\langle w, w\rangle = 1$, then $\langle s_k w, s_k w\rangle = 1$, and since $X_k \cdot s_k w = s_k X_{k+1} w - w = b \cdot s_k w - w$, we must have

$$a\langle w, s_k w\rangle = \langle X_k w, s_k w\rangle = \langle w, X_k s_k w\rangle = b\langle w, s_k w\rangle - 1 \quad \text{so} \quad (a-b)\langle w, s_k w\rangle = -1.$$

$\begin{pmatrix} 1 & \frac{1}{b-a} \\ \frac{1}{b-a} & 1 \end{pmatrix}$ defines an inner product iff $a \neq b, b \pm 1$.

Furthermore, any subrep is spanned by $w \pm s_k w$ (by s_k-invariance). We have

$X_k \cdot (w + s_k w) = (a-1)w + b s_k w.$ $\qquad X_{k+1} \cdot (w + s_k w) = (b+1)w + a\, s_k w$

$X_k \cdot (w - s_k w) = (a+1)w - b s_k w$ $\qquad X_{k+1} \cdot (w - s_k w) = (b-1)w - a\, s_k w.$

Thus $w \pm s_k w$ spans a sub iff $a = b \pm 1$. Otherwise we have a 2-d irrep.

This implies 2: if $a_k - a_{k+1} \notin \{1, 0, -1\}$, then $v, s_k v$ must be linearly independent, and $s_k v - \frac{1}{a_{k+1} - a_k} v$ is of weight $(a_1, \ldots, a_{k+1}, a_k, a_{k+2}, \ldots)$. It also gives us some of 3: there is no nonzero unitarizable rep w/ $a_k = a_{k+1}$; you must have the full 2-d rep, and that's not unitary. Note also that if $a_k = a_{k+1} \pm 1$, then unitarity implies that v and $s_k v$ are proportional. Thus, the span of $\{s_k v \mid k = 1, \ldots, n-1\}$ and $\{v\}$ is spanned by elems of weight gotten by an admissible move. Inductively, this shows V is spanned by such vectors.

One last thing to check: that $(\ldots, a, a\pm 1, a, \ldots)$ is impossible. By our calcs, a vector w/ this weight spans a line invariant under s_k and s_{k+1}. Since S_3 only has two 1-d irreps, they must both act trivially or by -1. But our calculations before show that this requires $(\ldots, a, a+1, a+2, \ldots)$ or $(\ldots, a, a-1, a-2, \ldots)$, so the pattern above is impossible.

This shows 1, 2, 3, so the weights of an irrep correspond to content vectors of a Young diagram.

Furthermore, this shows that the YD determines a unique $\mathbb{C}S_n$ module: it has a basis v_T for T the different tableaux, w/

$$s_k v_T = \begin{cases} v_{s_k T} + \frac{1}{a_{k+1} - a_k} v_T & s_k T \text{ still a tableau.} \\ v_T & s_k T \text{ breaks row condition} \\ -v_T & s_k T \text{ breaks column condition.} \end{cases}$$

Every irrep must be of this form. We know that the number of irreps is the number of partitions, so we must have an irrep for each YD.

This completes the proof of the theorem.