We can deepen our understanding of what's going on by remembering there isn't just one symmetric group. There's a whole tower

$$\begin{array}{ccccccccc} S_0 & \hookrightarrow & S_1 & \hookrightarrow & S_2 & \hookrightarrow & S_3 & \hookrightarrow & S_4 \hookrightarrow \cdots \\ \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel & & \\ \{e\} & & \{e\} & & C_2 & & D_6 & & \end{array}$$

Thus, we have functors of induction and restriction

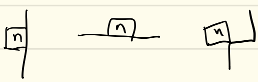
$$\mathrm{Res}: \mathbb{C}S_n\text{-mod} \to \mathbb{C}S_{n-1}\text{-mod} \qquad \mathrm{Ind}: \mathbb{C}S_{n-1}\text{-mod} \to \mathbb{C}S_n\text{-mod}.$$

These functors are biadjoint.

The J-M elements allow us to refine these functors: if V is an $\mathbb{C}S_n$-module, then the action of X_n commutes w/ $\mathbb{C}S_{n-1}$. That is, it's an endomorphism of $\mathrm{Res}(V)$. Thus, the eigenspaces of X_n are S_{n-1}-subreps.

<u>Thm</u> The c-eigenspace of X_n on $\mathrm{Res}(V_\lambda)$ is an irrep of S_{n-1} corresponding to λ w/ a box of content c removed, if that results in a Young diagram and is 0 otherwise.

<u>Pf</u> The c-eigenspace of X_n on $\mathrm{Res}(V_\lambda)$ is spanned by the vectors v_T w/ the box filled by n having content c. To understand the action of S_{n-1}, just need to consider simultaneous eigenvectors for $X_1, \ldots, X_{n-1}$. These are given by the content vectors of the tableau that result when the box w/ content c and label n is removed. Note that if c is fixed, the label n can only be in the last box of content c $(k, k+c)$ for some k. A tableau w/ this box having the label n exists if and only if it is <u>removable</u>: $(k+1, k+c)$ and $(k, k+c+1)$ are not in the diagram.

 removable  not removable.

Removing the box of label n gives a bijection
$\{\text{tableau w/ } (k, k+c) \text{ labelled } n\} \longleftrightarrow \{\text{tableau of shape } \lambda' = (\lambda_1, \dots, \lambda_k - 1, \lambda_{k+1}, \dots, \lambda_\ell)\}$

The effect on content vectors is just removing the final c. Thus as an irrep of S_{n-1}, this eigenspace is $V_{\lambda'}$.

<u>Cor.</u> $\mathrm{Res}^{S_n}_{S_{n-1}} V_\lambda = \bigoplus_{\substack{\mu \subset \lambda \\ |\mu| = |\lambda| - 1}} V_\mu$. $\mathrm{Ind}^{S_n}_{S_{n-1}} V_\mu = \bigoplus_{\substack{\mu \subset \lambda \\ |\mu| = |\lambda| - 1}} V_\lambda$.

Note: $\mathrm{Ind}^{S_n}_{S_{n-1}} V_\mu := \mathbb{C}S_n \otimes_{\mathbb{C}S_{n-1}} V_\mu$. This has endomorphism $a \otimes v \mapsto a x_n \otimes v$, and if λ is gotten from μ by adding a box of content c, then V_λ is the c-eigenspace of this endo. This gives another construction of V_λ.

This means that $(\mathbb{C}S_{n-1}, \mathbb{C}S_n)$ form a <u>Gelfand pair</u>: any $\mathbb{C}S_n$ irrep restricted to S_{n-1} is a sum of <u>distinct</u> irreps. All multiplicities 1.

There's also an abstract proof of this:
<u>Theorem</u> Let A be a semi-simple algebra over $\mathbb{C}$ and B a semi-simple subalgebra. Then (B, A) is a Gelfand pair if $C = \{a \in A \mid [a,b] = 0 \text{ for } b \in B\}$ is a commutative algebra.
<u>Pf</u> Since $A \cong \bigoplus_{V \text{ irrep}} \mathrm{End}_{\mathbb{C}}(V)$, we have $C \cong \bigoplus_{V \text{ irrep}} \mathrm{End}_B(V) \cong \bigoplus \mathrm{Mat}_{c_{ij}}(\mathbb{C})$ where c_{ij} is mult. of B-rep W_j in $\mathrm{Res}^A_B V_i$ for A-irreps V_i. Thus, commutative iff $c_{ij} \in \{0,1\}$ for all i,j.

<u>Thm</u> If $A = \mathbb{C}S_n$, $B = \mathbb{C}S_m$, then C is commutative.
<u>Pf.</u> C is algebra of sums $\sum_{g \in S_n} a_g g$ such that $a_g = a_{\pi g \pi^{-1}}$ for $\pi \in S_{n-1}$. Note that $\mathbb{C}S_n$ has an antiautomorphism $(\sum a_g g)^* = \sum a_g g^{-1}$, since $(gh)^* = h^* g^*$.
Note that not only is every permutation conjugate to its inverse via an elmt of S_{n-1}. Suffices to prove this for an n-cycle in S_n. This cycle is of form $(n, a_1, \dots, a_{n-1})$. This is conjugate to its inverse via $(a_1, a_{n-1}) \cdot (a_2, a_{n-2}) \cdot (a_3, a_{n-3}) \cdots$.
Thus if $\sum a_g g \in C$, then $a_g = a_{g^{-1}}$. Thus $x^* = x$ for $x \in C$. This shows $ab = (ab)^* = b^* a^* = ba$