Fessing up: I messed up the formulas for how you act on irreps

It's correct that $s_k v_T = \begin{cases} v_T & \boxed{k}\boxed{k+1} \\ -v_T & \begin{smallmatrix}\boxed{k}\\ \boxed{k+1}\end{smallmatrix} \end{cases}$

But in the other case, somewhere between Vershik-Okounkov, my notes, and the board, some factors got dropped. It's correct that $v_{s_kT} = s_k v_T - \frac{1}{a_{k+1}-a_k} v_T$ is a simultaneous eigenvector w/ correct eigenvalue. However:

$$s_k v_{s_kT} = v_T - \frac{1}{a_{k+1}-a_k} s_k v_T = \left(1 - \frac{1}{(a_{k+1}-a_k)^2}\right) v_T - \frac{1}{a_{k+1}-a_k} v_{s_kT}$$

Not symmetric! Of course it isn't! v_T and v_{s_kT} must be orthogonal, so

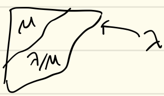
$$\langle v_T, v_T\rangle = \langle s_k v_T, s_k v_T\rangle = \frac{1}{(a_{k+1}-a_k)^2}\langle v_T, v_T\rangle + \langle v_{s_kT}, v_{s_kT}\rangle$$

Thus $\langle v_{s_kT}, v_{s_kT}\rangle = 1 - \frac{1}{(a_{k+1}-a_k)^2}$

2 solutions: break symmetry: $s_k v_T = \begin{cases} v_T & \boxed{k}\boxed{k+1} \\ -v_T & \begin{smallmatrix}\boxed{k}\\ \boxed{k+1}\end{smallmatrix} \\ v_{s_kT} + \frac{1}{a_{k+1}-a_k} v_T & \begin{smallmatrix} & \boxed{k} \\ \boxed{k+1} & \end{smallmatrix} \\ \left(1 - \frac{1}{(a_{k+1}-a_k)^2}\right) v_{s_kT} + \frac{1}{a_{k+1}-a_k} v_T & \begin{smallmatrix} & \boxed{k+1} \\ \boxed{k} & \end{smallmatrix} \end{cases}$

normalize $\tilde{v}_{s_kT} = v_{s_kT} \cdot \left(\sqrt{1 - \frac{1}{(a_{k+1}-a_k)^2}}\right)^{-1}$: $s_k v_T = \begin{cases} v_T & \boxed{k}\boxed{k+1} \\ -v_T & \begin{smallmatrix}\boxed{k}\\ \boxed{k+1}\end{smallmatrix} \\ \sqrt{1 - \frac{1}{(a_{k+1}-a_k)^2}}\,(v_{s_kT}) + \frac{1}{a_{k+1}-a_k} v_T & \begin{smallmatrix} & \boxed{k+1} \\ \boxed{k} & \end{smallmatrix} \text{ or } \begin{smallmatrix} & \boxed{k} \\ \boxed{k+1} & \end{smallmatrix} \end{cases}$

Note: these formulas define an S_n action on the formal span of tableau on a skew diagram: complement of diagram μ inside of λ. Denoted λ/μ. Let $V_{\lambda/\mu}$ be formal span of tableaux w/ this shape, w/ the induced S_n-action.



Young subgroups A Young (or parabolic) subgroup of S_n is the subgroup that preserves the partition of $[1,n]$ into sub sets $Y_1, \dots, Y_m$. Of course, this subgroup is isomorphic to $S_{\#Y_1} \times S_{\#Y_2} \times \cdots \times S_{\#Y_m}$, and up to inner automorphism, this only depends on the sizes $\#Y_i$ up to permutation. We can thus assume that $\lambda_i = \#Y_i$ give a partition. The resulting subgroup is often denoted $S_\lambda \subset S_n$. If we don't assume λ_i are ordered, these give a composition.

Thm The restriction of V_λ to $S_m \times S_{n-m}$ is of the form

$$\mathrm{Res}^{S_n}_{S_m \times S_{n-m}} V_\lambda \cong \bigoplus V_\mu \boxtimes V_{\lambda/\mu} \quad \text{w/ } \mu \text{ ranging over } \mu \vdash m$$

If $\mu \not\subset \lambda$, then $V_{\lambda/\mu} = 0$ by convention.

Pf Let $V_\lambda^{(\mu)}$ be the span all v_S where the first m entries have shape μ. This is isomorphic to $V_\mu \boxtimes V_{\lambda/\mu}$ as a $S_m \times S_{n-m}$-module as the formulas show.

More generally, $\mathrm{Res}^{S_n}_{S_\nu} V_\lambda \cong \bigoplus V_{\mu_1} \boxtimes V_{\mu_2/\mu_1} \boxtimes V_{\mu_3/\mu_2} \boxtimes \cdots \boxtimes V_{\lambda/\mu_{k-1}}$ where we sum over nested diagrams $\mu_1 \subset \mu_2 \subset \cdots \subset \mu_{k-1} \subset \lambda$, with μ_i/μ_{i-1} having ν_i boxes.

Lemma If λ/μ is disconnected, $\lambda/\mu = \nu_1 \cup \nu_2$, then $V_{\lambda/\mu} = \mathrm{Ind}^{S_n}_{S_{\#\nu_1} \times S_{\#\nu_2}} V_{\nu_1} \boxtimes V_{\nu_2}$.

Pf. The span of tableaux w/ $\{1, \dots, \#\nu_1\}$ in ν_1 and all other entries in ν_2 is isomorphic to a copy of $V_{\nu_1} \boxtimes V_{\nu_2}$. This induces a map $\mathrm{Ind} \to V_{\lambda/\mu}$, which is surjective, since these guys generate. This is an iso since dimensions are the same: $\binom{\# \lambda/\mu}{\# \nu_1} \cdot \dim V_{\nu_1} \cdot \dim V_{\nu_2}$.