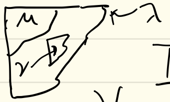
One other basic property: let λ/μ is a skew shape, and it contains a translate of the diagram of a partition ν.



Thm Every irrep of S_n that is a sub rep of $V_{\lambda/\mu}$ is of the form $V_{\nu'}$ w/ $\nu' \supset \nu$.

Pf. $\exists$ a tableau on λ/ν s.t. ν is filled by $\{k+1, \ldots, k+\#\nu\}$.
Pick any tableau on the diagram which consists of boxes either left or above top left corner of ν. Let k be the number of boxes in this subdiagram. We can then fill ν w/ $\{k+1, \ldots, k+\#\nu\}$, and then continue. Consider the span of v_T where T agrees w/ this tableau off ν. As a rep of $S_{(\overbrace{1,\ldots,1}^{k}, \#\nu, 1, \ldots 1)}$, this is just V_ν. Thus, we have a surjective map $\mathrm{Ind}_{S_{(1,\ldots,1,\#\nu,1,\ldots)}}^{S_n} V_\nu \twoheadrightarrow V_{\lambda/\nu}$.

Frobenius reciprocal of formula from last time:
$$\mathrm{Ind}_{S_k}^{S_n} V_\nu \cong \bigoplus_{\nu' \supset \nu} V_{\nu'} \boxtimes V_{\nu'/\nu}^{*} \quad \text{(where we ignore } S_{n-k} \text{ action on latter)}$$

important part for us is just when it is non-zero.

Lemma $(V_{\lambda/\mu})^{S_n} = \begin{cases} \mathbb{C} & \text{if } \lambda/\mu = \square\!\square \;\; \square\!\square\!\square \;\; \square\!\square \\ 0 & \text{if } \lambda/\mu \supset \begin{smallmatrix}\square\\\square\end{smallmatrix}. \end{cases}$ In CST "totally disconnected"

Pf If λ/μ is totally disconnected, then $V_{\lambda/\mu} \cong \mathrm{Ind}_{S_x}^{S_n}(\mathrm{triv})$
x is composition given by lengths of rows. This has
$$(V_{\lambda/\mu})^{S_n} = \mathrm{Hom}_{S_n}(V_{\lambda/\mu}, \mathrm{triv})^* \cong \mathrm{Hom}_{S_x}(\mathrm{triv}, \mathrm{triv})^* \cong \mathbb{C}.$$

Otherwise $\begin{smallmatrix}\square\\\square\end{smallmatrix} \subset \lambda/\mu$, so every simple appearing has at least two rows in its Young diagram, so $(V_{\lambda/\mu})^{S_n} = 0$.