Lecture 1: Representations of the symmetric group I

This class is on combinatorics and representation theory. That's a slippery concept, since "combinatorics" can mean just about anything. So, you'll be getting a very prejudiced view of how to think about the connections between these fields.

Combinatorial representation theory is more defined by what it's not than what it is. It means we're not going to use techniques from algebraic geometry or from analysis, and that points us toward certain kinds of problems. But it also means we're going to evaluate problems w/ an eye toward how interesting they are for combinatorialists. For example the symmetric group is of great interest in combinatorics. Combinatorial problems will tend to have symmetric groups as symmetries, so vector spaces whose dimensions are interesting will tend to have S_n actions, so we can ask not just about dimensions, but also about the underlying S_n-rep. For example, let $\mathbb{C}[x_1, \ldots, x_n]$ be polys w/ obvious S_n-action. Let C be the quotient where we mod out by every positive degree S_n invariant polynomial ($x_1 + \cdots + x_n$, $x_1x_2 + x_1x_3 + \cdots$, $\sum_{i_1 < \cdots < i_m} x_{i_1} \cdots x_{i_m}$ is enough).

<u>Thm</u> This quotient is isomorphic to the regular representation as an S_n-rep.

<u>Q</u> How does each individual degree decompose over S_n?

In order to answer a question like this, we need to first understand S_n-reps. That will be our first topic in this course: the connection between the representation theory of symmetric groups and Young diagrams.

<u>Thm</u> (Frobenius-Young) The irreducible representations of $\mathbb{C}S_n$ are in bijection w/ partitions of n (Young diagrams w/ n-boxes.

Of course, this theorem can be upgraded in many different ways to answer questions about the representations: What are their dimensions? Characters. How are they related under induction and restriction? What if we replace $\mathbb{C}$ by another field? So, our goal for the first part of this course is to prove this theorem; we'll do this in a way that facilitates answering all the other questions I mentioned. Then we'll consider how the same techniques carry over to other similar algebras, like Hecke algebras, and KLR.

Of course, these answers are interesting for representation theorists, but they're also interesting for combinatorialists. Often times representation theory points the way to interesting things to count, or shows a not-obviously-positive (or integer) quantity is the dimension of a vector space, or gives us two ways of writing quantities that are hard to match combinatorially. So, as often happens, when two areas of math talk to each other, it's not just a one way street.

I'm going to attack the symmetric group using a relatively modern perspective, developed by Vershik and Okounkov. The key to making it all run is the observation that $\mathbb{Z}S_n$ has a surprisingly large commutative subalgebra:

<u>Def</u> Let $X_k = \sum_{i<k} (ik) \in \mathbb{Z}S_n$ for $k \leq n$. Note that this def is compatible w/ $S_n \hookrightarrow S_{n+1}$, so we can leave n out of the notation.

Lemma X_k commutes w/ $\mathbb{C}S_{k-1}$, permutations fixing $\{k,\dots,n\}$.

Proof Assume $\pi \in S_{k-1}$. Then $\pi X_k \pi^{-1} = \sum_{i<k} (\pi(i),k) = X_k$.

Lemma The subalgebra generated by X_k $k=1,\dots,n$ in $\mathbb{Z}S_n$ is commutative

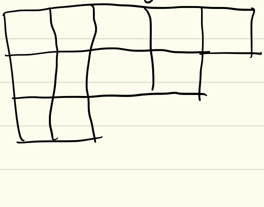
Pf By previous lemma, if $k<\ell$, then X_k and X_ℓ commute.

Whenever you have a commutative subalgebra in a bigger noncommutative algebra, you gain a new perspective on the representation theory of the algebra. On any $\mathbb{C}S_n$ representation, we have that X_k can be simultaneously be made upper-triangular (actually, diagonalized). So, we can try to understand $\mathbb{C}S_n$ reps by what eigenvalues of $X_1,\dots,X_n$ appear together and w/ what multiplicities.

For example, on the trivial representation, X_k acts by $k-1$. On the sign, it acts by $-k+1$. On the permutation rep $\{(x_1,\dots,x_n) \mid \sum x_i = 0\}$, we have an eigenbasis given by $v_\ell = (\overbrace{1,\dots,1}^{\ell},-\ell)$ for $\ell=1,\dots,n-1$. $X_k \cdot v_\ell = \begin{cases} (k-1)\cdot v_\ell & \ell \geq k \\ -v_\ell & \ell+1=k \\ k-2 & k>\ell+1 \end{cases}$. Thus, you can tell apart these 3 reps.

In the general case, we say that $(a_1,\dots,a_n)$ lies in the J-M spectrum of a representation if there is a vector v such that $X_k \cdot v = a_k v$ for all k.

Recall that a <u>partition</u> of n is a sequence $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_\ell \geq 0$ s.t. $\sum \lambda_i = n$, and that the Young diagram is a digram w/ λ_1 boxes in one row, λ_2 in next, etc. Whether "next" should read from top to bottom or vice versa is "English" v.s. "French" notation. I'll use English. Thus $(5,4,2)$ has Y.D.



A Young tableau (often called standard tableau) is a filling of boxes w/ $\{1,\dots,n\}$ which increases as we get further from top left corner.

We give boxes coordinates as w/ entries of matrices: (i,j) is the jth box from left in ith row, counting from top. The "diagram" is often taken to mean pairs (i,j) w/ $i,j \in \mathbb{Z}_{>0}$, and $j \leq \lambda_i$.

A tableau S is thus a bijection between this diagram and $\{1, \dots, n\}$ such that $S(i+1,j) > S(i,j)$ and $S(i,j+1) > S(i,j)$.

1	2	4	5	7
3	6	10	11	
8	9			

a tableau

0	1	2	3	4
-1	0	1	2	
-2	-1			

The contents of boxes.

The content of a box is just $j-i$.

The content vector of a tableau is the contents of the boxes, listed in order given by the entries of the tableau. In the example above, this is $(0,1,-1,2,3,0,4,-2,-1,1,2)$.

Thm (VO) The bijection of representations and Young diagrams is uniquely characterized by the fact that V_λ for $\lambda_1 \geq \dots \geq \lambda_\ell$ has spectrum which is precisely the content vectors of Young tableaux of shape λ. Each of the corresponding simultaneous (generalized) eigenspaces is dimension 1.

Note that given a content vector, you can reconstruct the tableau one box at a time. We often think of a tableau as a series of Young diagrams, given by the shape given by boxes w/ entry $\leq k$. The tableau conditions are exactly that this is always a Young diagram. Given the shape of entries $\leq k$, and content of $k+1$st box, there's only one place it can go. Thus, given a rep, you can reconstruct the YD from these eigenvalues. For perm, we have $(0, \dots, k-1, -1, k, \dots, n-2)$, which gives tableaux

1	2	--	k	k+2	---	n
k+1						