Note that since transpose switches moving boxes SW and NE, we have that $\lambda \trianglerighteq \mu$ iff $\lambda^T \trianglelefteq \mu^T$.

Cor V_λ is the unique S_n-irrep such that $V^{S_\lambda} \neq \{0\}$ and $V^{S_{\lambda^T}, sgn} \neq \{0\}$.

In fact, both spaces are 1-d.

Pf Obviously $\lambda \leq \lambda$ and $\lambda^T \leq \lambda^T$. In both cases, $\exists$ SSYT of desired type, the "super-standard tableau" $\begin{matrix} 1 & \cdots & 1 \\ 2 & \cdots & 2 \\ 3 & \cdots & 3 \\ \vdots \end{matrix}$
On the other hand, if $\lambda \triangleleft \nu$, then $\nu^T \triangleleft \lambda^T$, so if $V_\nu^{S_\lambda} \neq \{0\}$, then $V_\nu^{S_{\lambda^T}, sgn} = \{0\}$ and vice versa, if $\lambda \neq \nu$.

Let $M^\lambda = \mathrm{Ind}_{S_\lambda}^{S_n}(\mathrm{triv})$ and $\widetilde{M}^\lambda = \mathrm{Ind}_{S_\lambda}^{S_n}(\mathrm{sgn})$. Note that these are given by skew diagrams w/ disconnected rows or columns respectively.

You'll sometimes see V_λ constructed using this principle or the related one:
Cor V_λ is the unique simple submodule appearing in both $\mathrm{Ind}_{S_\lambda}^{S_n}(\mathrm{triv})$ and $\mathrm{Ind}_{S_{\lambda^T}}^{S_n}(\mathrm{sgn})$, in both cases w/ mult 1. Thus any non-zero map $\mathrm{Ind}_{S_\lambda}^{S_n}(\mathrm{triv}) \to \mathrm{Ind}_{S_{\lambda^T}}^{S_n}(\mathrm{sgn})$ has image V_λ.

The Pieri rule shows $\dim \mathrm{Hom}(M^\lambda, \widetilde{M}^{\lambda^T}) = 1$, so unique up to scalar.
One way of doing this is to embed $S_\lambda \hookrightarrow S_n \hookleftarrow S_{\lambda^T}$ as the subgroups of permutations of boxes preserving rows and columns.
In this case, $\mathrm{Ind}_{S_\lambda}^{S_n}(\mathrm{triv}) \cong \mathbb{C}S_n \cdot a_\lambda$ where $a_\lambda = \sum_{\pi \in S_\lambda} \pi$.
$\mathrm{Ind}_{S_{\lambda^T}}^{S_n}(\mathrm{sgn}) \cong \mathbb{C}S_n \cdot b_\lambda$ where $b_\lambda = \sum_{\pi \in S_{\lambda^T}} (-1)^{\ell(\pi)} \pi$. Thus $\mathbb{C}S_n a_\lambda b_\lambda \cong V_\lambda$ is the image of such a map. $a_\lambda b_\lambda$ is what's called the Young symmetrizer.

This strategy of analyzing S_n-reps via restriction to Young subgroups can also be applied to computing characters.

Strategy: Fix a partition μ. We only need to find character of
$\pi = (1, \dots, \mu_1)(\mu_1+1, \dots, \mu_1+\mu_2)\cdot(\mu_1+\mu_2+1, \dots, \mu_1+\mu_2+\mu_3)\cdots$.
Restrict to $S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_n}$. $V_\lambda \cong \bigoplus_{\substack{\nu_1 \vdash \mu_1 \\ \nu_2 \vdash \mu_2 \\ \vdots}} (V_{\nu_1} \otimes V_{\nu_2} \otimes \cdots \otimes V_{\nu_\ell})^{\oplus m_{\underline{\nu}}}$.

$$\chi_{V_\lambda}(\pi) = \sum_{\underline{\nu}} m_{\underline{\nu}} \cdot \chi_{V_{\nu_1}}(1 \cdots \mu_1) \cdot \chi_{V_{\nu_2}}(1 \cdots \mu_2) \cdots \chi_{V_{\nu_\ell}}(1 \cdots \mu_\ell).$$

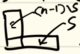
So we "only" need to calculate $m_{\underline{\nu}}$, and the character of a single n-cycle on reps of S_n.

The latter is actually pretty easy:

<u>Lemma</u> $X_n \cdots X_2 = C_n$.

<u>Pf</u>. Let's induct. $C_2 \in \mathbb{C}S_2 = (12) = X_2$. Consider $C_{n-1} \in \mathbb{C}S_{n-1} = \sum (1, i_1, \dots, i_{n-2}) \cdot \sum (i, n)$. We have $(1, i_1, \dots, i_{n-2}) \cdot (i_k, n) = (1, i_1, \dots, i_{k-1}, n, i_k, \dots, i_{n-2})$. That is, sum is over all ways of inserting n.

<u>Thm</u>
Character of $(1, \dots, n)$ on $V_\lambda \circlearrowleft \mathbb{C}S_n$ is $\begin{cases} 0 & \lambda \text{ not a hook.} \\ (-1)^{k+1} & \lambda = (k, 1^{n-k}) \end{cases}$

<u>Pf</u> C_n acts by the scalar given by product of all contents, excluding the bottom corner. This is 0 if $(2,2)$ is in diagram, i.e. not a hook. If it is a hook $(k, 1^{n-k})$ then we get $(-1)^{k+1}(k-1)! \cdot (n-k)!$. There are $(n-1)!$ n-cycles so its trace is $(-1)^{k+1} \frac{1}{\binom{n-1}{n-k}} \cdot \dim V_{(1^k, n-k)}$.
This latter dimension is given by tableaux on this hook. 

These are in bijection w/ $S \subset [2, n]$ w/ $\#S = n-k$, so, we get $(-1)^{k+1}$. $\square$

OK, so we just need to find $m_{\underline{\nu}}$ where all of these guys are hooks. We can calculate these somewhat recursively using skew diagrams.

Maybe better to think that

$$\chi_\lambda(\pi) = \sum_{\substack{\lambda = \lambda^{(\ell)} \supset \cdots \supset \lambda^{(0)} = \emptyset \\ \#\lambda^{(i)}/\lambda^{(i-1)}}} \prod_{i=1}^{\ell} \chi_{V_{\lambda^{(i)}/\lambda^{(i-1)}}}(1 \cdots \mu_i).$$

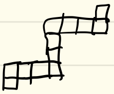
Lemma If $\lambda^{(i)}/\lambda^{(i-1)}$ is disconnected, then $\chi_{\lambda^{(i)}/\lambda^{(i-1)}}(1 \cdots \mu_i) = 0$.

If $\lambda^{(i)}/\lambda^{(i-1)}$ is disconnected then $V_{\lambda^{(i)}/\lambda^{(i-1)}} \cong \mathrm{Ind}_{S_{(k,\mu_i-k)}}^{S_{\mu_i}} (W \boxtimes W')$ for reps corresponding to the two pieces. Note, $(1 \cdots \mu_i)$ is not conjugate to an element of this subgroup, so its character is 0.

Lemma If $\boxplus \subset \lambda^{(i)}/\lambda^{(i-1)}$, then $\chi_{\lambda^{(i)}/\lambda^{(i-1)}}(1 \cdots \mu_i) = 0$.

Pf. Every summand of $V_{\lambda^{(i)}/\lambda^{(i-1)}}$ is V_γ w/ γ a partition which contains $\boxplus$, i.e., is not a hook.

If $\lambda^{(i)}/\lambda^{(i-1)}$ is connected and contains no $\boxplus$'s, then we call it a skew hook. Let $\langle \lambda^{(i)}/\lambda^{(i-1)} \rangle$, the height of the skew hook be the number of rows containing boxes -1. I.e. $\langle (n-k, 1^k) \rangle = k$.


is a skew hook of height 5.

Thm $\chi_{\lambda^{(i)}/\lambda^{(i-1)}}(1, \ldots, \mu_i) = \begin{cases} 0 & \lambda^{(i)}/\lambda^{(i-1)} \text{ not a skew hook} \\ (-1)^{\langle \lambda^{(i)}/\lambda^{(i-1)} \rangle} & \lambda^{(i)}/\lambda^{(i-1)} \text{ is a skew hook} \end{cases}$

Pf First case is covered by lemmata. For second, let λ = fill in skew hook:


Then $a = \mathrm{Hom}_{S_n}(V_\xi, V_{\lambda/\nu}) \cong \mathrm{Hom}_{S_{(m,n)}}(V_\nu \boxtimes V_\xi, \mathrm{Res}^{n+m}_{(m,n)} V_\lambda)$.
Here $\nu \vdash m$, $\lambda \vdash m+n$. Now, use conjugacy of $S_{(m,n)}$ and $S_{(n,m)}$. Thus, $a = \mathrm{Hom}_{S_{(n,m)}}(V_\xi \boxtimes V_\nu, \mathrm{Res}^{n+m}_{(n,m)} V_\lambda)$. Thus, if ξ is a hook, then $a = 0$, unless


$\langle \xi \rangle = \langle \lambda^{(i)}/\lambda^{(i-1)} \rangle$. In this case $\lambda/\xi = \nu$ so, $a = \dim \mathrm{Hom}(V_\nu, V_\nu) = 1$.

Thus $\chi_{\lambda^{(i)}/\lambda^{(i+1)}}(1 \cdots \mu_i) = \chi_\xi(1 \cdots \mu_i) = (-1)^{\langle \xi \rangle}$.

So, this tells us the Murnaghan-Nakayama rule: $\chi_\lambda(\pi) = \sum (-1)^{\langle S \rangle}$ where S ranges over chains $\lambda = \lambda^{(l)} \supset \lambda^{(l-1)} \supset \cdots \supset \lambda^{(0)} = \emptyset$ s.t. $\# \lambda^{(i)}/\lambda^{(i-1)} = \mu_i$, $\lambda^{(i)}/\lambda^{(i-1)}$ is a skew hook, and $\langle S \rangle = \sum \langle \lambda^{(i)}/\lambda^{(i-1)} \rangle$.

Probably better to think inductively: $\chi_\lambda(\pi) = \sum_{\nu \subset \lambda} (-1)^{\langle \lambda/\nu \rangle} \chi_\nu(\pi')$ where $\pi' = (1 \cdots \mu_1) \cdots (\mu_1 + \cdots + \mu_{\ell-2}+1, \ldots, \mu_1 + \cdots + \mu_{\ell-1})$, and $(-1)^{\langle \lambda/\nu \rangle} = 0$ if not a skew hook.