S_1	1
(1)	1

S_2	$(1,1)$	(2)
(2)	1	1
$(1,1)$	1	-1

S_3	$(1,1,1)$	$(2,1)$	(3)
(3)	1	1	1
$(2,1)$	2	0	-1
$(1,1,1)$	1	-1	1

S_4	(1^4)	$(2,1^2)$	$(3,1)$	$(2,2)$	(4)
(4)	1	1	1	1	1
$(3,1)$	3	1	0	-1	-1
$(2,2)$	2	0	-1	2	0
$(2,1,1)$	3	-1	0	-1	1
(1^4)	1	-1	1	1	-1

S_5	(1^5)	$(2,1^3)$	$(3,1^2)$	$(4,1)$	$(2,2,1)$	$(3,2)$	(5)
$(3,2)$	5	1	-1	-1	1	1	0
$(3,1,1)$	6	0	0	0	-2	0	1
$(2,2,1)$	5	-1	-1	1	1	-1	0
$(4,1)$	4	2	1	0	0	-1	-1
(5)	1	1	1	1	1	1	1
(1^5)	1	-1	1	-1	1	-1	1
$(2,1,1,1)$	4	-2	1	0	0	1	-1

In addition to the usual character table, there's also the "Kostka character table" given by $\dim V_\lambda^{S_\mu}$.

S_5	(1^5)	$(2,1^3)$	$(3,1^2)$	$(4,1)$	$(2,2,1)$	$(3,2)$	(5)
$(3,2)$	5	3	1	0	1	1	0
$(3,1,1)$	6	3	1	0	1	0	0
$(2,2,1)$	5	2	0	0	1	0	0
$(4,1)$	4	3	2	1	2	1	0
(5)	1	1	1	1	1	1	1
(1^5)	1	0	0	0	0	0	0
$(2,1,1,1)$	4	1	0	0	0	0	0

$$(1^5) < (2,1,1,1) < (2,2,1) < (3,1,1) < (3,2) < (4,1) < (5)$$

This information can be packaged as symmetric functions.

Def We call a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ symmetric if it is invariant under the symmetric group S_n, i.e. $f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for any permutation π. Let $\mathrm{Sym}_n = \mathbb{C}[x_1, \ldots, x_n]^{S_n}$.

Thm The ring of symmetric polynomials is isomorphic to a polynomial ring in the elementary symmetric polynomials $e_k(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$.

$\mathbb{C}[x_1, x_2]^{S_2} \cong \mathbb{C}[x_1 + x_2, x_1 x_2]$. $\mathbb{C}[x_1, x_2, x_3] = \mathbb{C}[x_1 + x_2 + x_3,\ x_1x_2 + x_2x_3 + x_1x_3,\ x_1x_2x_3]$.

Specialization $x_k = 0$ induces a map $\mathrm{Sym}_k \to \mathrm{Sym}_k$ that sends $e_n(x_1, \ldots, x_k) \mapsto e_n(x_1, \ldots, x_{k-1})$. Note $e_n(x_1, \ldots, x_k) = 0$ if $n > k$.

Def The <u>symmetric functions</u> are the graded inverse limit

$$\varprojlim \mathrm{Sym}_n = \mathbb{C}[e_1, e_2, e_3, \ldots]$$

Def

The Frobenius <u>characteristic</u> of a rep V of S_n is the formal sum

$$F(V) = \sum_{\substack{(c_1, \ldots, c_k, \ldots) \\ \sum c_i = n}} \dim V^{S_c} \cdot x_1^{c_1} x_2^{c_2} \cdots$$

For example, $F(\square) = x_1 + x_2 + x_3 + \cdots = e_1$

$F\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) = x_1x_2 + x_1x_3 + x_2x_3 + \cdots = e_2$ $F(\square\square) = \begin{matrix} x_1^2 + x_2^2 + \cdots + \\ x_1x_2 + x_1x_3 + x_2x_3 + \cdots \end{matrix}$

Prop The FC's $F(V)$ of any rep is a symmetric function, and the $F(V_\lambda)$ for $\lambda \vdash n$ are a basis of degree n symmetric functions.

Def The Schur function of a partition λ is

$$s_\lambda = F(V_\lambda) = \sum_{\underline{c}} K(\lambda, \underline{c}) \cdot \underline{x}^{\underline{c}}$$