Project Description

Representation theory of symplectic singularities

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Research program. My research centers around the representation theory of deformation quantizations. While this might sound like an unfamiliar subject, in fact, it generalizes the study of universal enveloping algebras, which can be viewed as deformation quantizations of $\mathfrak{g}^*$. As with the representation theory of Lie algebras, these algebras have complex connections to topology, geometry and combinatorics.

My main concentration at the moment is generalizing the rich theory surrounding category $\mathcal{O}$ and Harish-Chandra bimodules. While these questions are interesting for their own sake, applications of this work include the definition of knot homology categorifying quantum knot invariants for all Lie groups and a new perspective on the canonical basis theory of Lusztig and others.

My plans for future research include:

- Investigating these categories for quiver varieties and slices between Schubert varieties in the affine Grassmannian. These are conjecturally equivalent to algebraically defined categories $\mathfrak{V}$ from [Weba], which categorify tensor products of representations of simple Lie algebras. Some progress toward this conjecture was made in that paper, but much is left to be resolved.

  In particular, even if there one can prove an equivalence to the algebraic category, there are a number of functors between these categories which are important for the construction of knot invariants (the action of Lie algebra elements, the R-matrix for quantum groups, quantum trace, etc.). I also aim to find a geometric description of these.

  These categories include as a special case, for quiver varieties of extended Dynkin diagrams, category $\mathcal{O}$ or the category of finite dimensional representations for a symplectic reflection algebra, so it may be that some of the beautiful results which have appeared in that context are more generally applicable.
- Continuing to study homological knot invariants defined using the aforementioned categories $\mathfrak{V}$, and their connection to geometry. For each representation of a simple Lie algebra, these provide a bigraded vector space-valued knot invariant whose graded Euler characteristic is the quantum invariant for that representation, defined in [Webb]. Several aspects of these invariants are essentially completely unexplored at the moment:
  - constructing 3-manifold invariants and extended TQFTs from these invariants
  - connections to the knot invariants defined in terms of field theory by Witten [Wit]
  - effective/computer-assisted computation
  - functoriality
  - spectral sequences/Rasmussen-type concordance invariants

Several of these questions have fruitful answers for Khovanov homology, so it is natural to look for analogues for these invariants which generalize Khovanov homology.

- More exploration of general category O's. The hypertoric case studied by myself, Braden, Licata and Proudfoot [BLPW10, BLPWa] reveals deep analogies between the deformation quantizations of hypertoric varieties and universal enveloping algebras: both the original and hypertoric category O's have Koszul graded lifts, versions of the Beilinson-Bernstein localization theorem, Verma modules which play a special intermediary role between projectives and simples, natural actions of complements to complex hyperplane arrangements, etc. However, in each case these are proven separately "by hand." It seems likely that these properties have more uniform proof which would apply to other interesting cases (such as quiver varieties).

  Also, it seems that the Koszul dual of the category O for one symplectic singularity is often the category O for a second, seemingly unrelated singularity. This has led us to conjecture that this fact reflects some underlying duality between singularities. The same duality appears in the physics of 3 dimensional field theories, a coincidence which at the moment remains largely unexplained.

CONTENTS

1. CATEGORIES O AND SYMPLECTIC RESOLUTIONS

1.1. **Symplectic resolutions.** The geometric objects which we will deform are symplectic singularities. These have attracted great interest in recent years as their theory has been developed by Kaledin and others (see the survey [Kal]).

Definition 1. *A* **conical symplectic singularity** *is a pair (X, ω) consisting of*

- *a normal affine algebraic variety X,*
- *a holomorphic 2-form $\omega \in \Omega^2(X)$, non-degenerate on the smooth locus, such that on some resolution of singularities $Y \to X$, ω extends to a closed 2-form on Y and*
- *an action of $\mathbb{S} \cong \mathbb{C}^*$ on X such that ω is a weight vector for the $\mathbb{S}$ action of positive weight n, and which contracts X is a fixed point o.*

Note that it is enough for some resolution Y of X to be itself symplectic, but that this is stronger than being a symplectic singularity. Examples of such singularities include:

- The nilpotent cone $X = \mathcal{N}_\mathfrak{g}$ for a semi-simple Lie group G has a resolution given by $Y = T^*(G/B)$ for any Borel B.
- For any finite subgroup $\Gamma \subset SL(2)$, the affine quotient surface $\mathbb{C}^2/\Gamma$ has a unique symplectic resolution $\widetilde{\mathbb{C}^2/\Gamma}$.

- More generally, the symmetric power $X = \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ has a resolution given by the Hilbert scheme $Y = \mathsf{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma})$.
- There are slices (much like Slodowy slices) transverse to one $G(\mathbb{C}[[t]])$-orbit inside another in the affine Grassmannian $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ of a complex algebraic group G.
- For any action of a compact group K on a complex vector space V, there is a hyperkähler structure on T^*V, and we can consider the hyperkähler quotient $\mathfrak{M}_{\boldsymbol{\alpha}} = T^*V////_{\boldsymbol{\alpha}}K$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ are a triple of moment map values. This quotient may be singular, but it always carries a hyperkähler structure on its generic locus. In fact, if $\mathfrak{M}_{\boldsymbol{\alpha}}$ is smooth, it is a symplectic resolution of $\mathfrak{M}_{\boldsymbol{\alpha}'}$ where $\boldsymbol{\alpha}' = (0, \alpha_2, \alpha_3)$. This allows us to construct a large number of examples of symplectic resolutions, including
  * the quiver varieties of Nakajima [Nak94, Nak01] and
  * the hypertoric varieties studied by Bielawski-Dancer and others [BD00].

1.2. **The definition of $\mathcal{O}_X^\xi$.** If X is a conical symplectic singularity, then applying work of Bezrukavnikov and Kaledin [BK04], we prove the following theorem [BLPWb]:

Theorem 2. *There is a canonical family A of deformation quantizations of the coordinate ring $\mathbb{C}[X]$ parametrized by a finite dimensional vector space H (if X possesses a symplectic resolution Y, then these are parameterized by $H^2(Y;\mathbb{C})$ mod a Coxeter group). We denote the ring corresponding to $\eta \in H$ by A_η.*

- In the case of $X = \mathcal{N}_\mathfrak{g}$, this deformation is $U(\mathfrak{g})$, and the algebras A_η are central quotients of $U(\mathfrak{g})$.
- In the case there X is a Slodowy slice, A is a finite W-algebra.
- In the case of $X = \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$, this deformation is the spherical symplectic reflection algebra for Γ and η corresponds to the parameter $\mathbf{c}$.
- If X is a hyperkähler quotient by a reductive group, A is the the algebra $\mathcal{D}_V^G$ of G-invariant differential operators on V and A_η is a noncommutative Hamiltonian reduction of $\mathcal{D}_V$.

Given a Hamiltonian $\mathbb{C}^*$-action $\xi : \mathbb{T} \cong \mathbb{C}^* \to \mathsf{Aut}(X)$, we can construct (by naturality of the deformation) a corresponding $\mathbb{T}$-action on A_η whose derivative is given by $[\hat{\xi}, -]$ for some $\hat{\xi} \in A_\eta$ quantizing the comoment map. Let A_η^+ be the subalgebra of non-negative weight under this $\mathbb{T}$-action.

Definition 3. *Let $\mathcal{O}_X^\xi$ be the category of A_η-modules M such that:*
- *M is finitely generated over A_η.*
- *The action of A_η^+ on M is locally finite.*

This category carries a natural action of the monoidal category of **Harish-Chandra bimodules** over A_η, that is, the category of bimodules whose associated graded is supported on the diagonal in $X \times X$ and which satisfy a "regularity" condition.

Several of these categories have already appeared in the literature and are of interest to representation theorists. For example:

- If $X = \mathcal{N}_\mathfrak{g}$, this is equivalent to a block of the original BGG category $\mathcal{O}$. More generally, we can arrive at any block of parabolic category $\mathcal{O}$ for any parabolic by considering the intersection of Slodowy slices with nilpotent orbits [Webc].

- If $Y = \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$, we obtain category $\mathcal{O}$ for the rational Cherednik algebra of the wreath product $S_n \wr \Gamma$ by [EGGO07].
- In the case where X is a hypertoric variety, these categories are equivalent to certain explicit algebras based on the combinatorics of hyperplane arrangements defined by myself, Braden, Licata, and Proudfoot [BLPW10, BLPWa].

One very interesting case which is less fully understood is when Y is a Nakajima quiver variety; Nakajima has already shown that a tensor product

$$V_{\underline{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell}$$

of simple $\mathfrak{g}$-modules for $\mathfrak{g}$ a simple, simply-laced Lie algebra have a corresponding quiver variety with associated $\mathbb{T}$-action. This variety has an associated category $\mathcal{O}_X^{\underline{\lambda}}$.

Conjecture 4. *There is a derived equivalence between $\mathcal{O}_X^{\underline{\lambda}}$ and $T^{\underline{\lambda}}-\mathsf{mod}$ for a finite dimensional algebra $T^{\underline{\lambda}}$ defined in terms of an explicit presentation in* [Weba].

The representation category $T^{\underline{\lambda}}-\mathsf{mod}$ carries a **categorical $\mathfrak{g}$-action** in the sense of Rouquier or Khovanov-Lauda and one can recover the original tensor product representation by taking Grothendieck group.

Conjecture 5. *Under the equivalence above, the auto-functors $\mathcal{E}_i$ and $\mathcal{F}_i$ on $T^{\underline{\lambda}}-\mathsf{mod}$ categorifying the action of the Chevalley generators are sent to tensor product with explicit Harish-Chandra bimodules.*

Both of these conjectures hold in type A, by the results of [Weba, §5]. In the special case when the associated quiver is an affine Dynkin diagram and λ is the highest weight Λ_0 of the basic representation, this would imply a conjecture of Etingof:

Conjecture 6. *The K-group of the space of finite dimensional representations of the symplectic reflection algebra for $S_n \wr \Gamma$ at a generic integral set of parameters is the $\Lambda_0 - n\delta$ weight space of the basic representation; in a certain cone in the space of parameters, the simple modules correspond to Lusztig's canonical basis.*

When the quiver is a single cycle with k vertices (i.e. $\mathfrak{g}$ is affine type A), then Nakajima's actions do not have isolated fixed points in any interesting examples, but there is a larger torus acting on the quiver variety which does have isolated fixed points. Thus we expect that the category $\mathcal{O}$ for this torus action to be an enlargement of $T^{\underline{\lambda}}-\mathsf{mod}$ which includes more objects.

If the weights $\underline{\lambda}$ are fundamental, then based on the representation of $\mathfrak{g}$ on cohomology of quiver varieties, we expect it to be a categorification of a higher level Fock space. Such an algebra is already known, the **cyclotomic q-Schur algebra $\mathscr{H}^{\underline{\lambda}}$**.

Conjecture 7. *We have a derived equivalence $D(\mathscr{H}^{\underline{\lambda}}) \cong D(\mathcal{O}_X^{\xi})$ for ξ a generic cocharacter factoring through the enlarged torus.*

These categories include in them as a special case the category $\mathcal{O}$ of spherical Cherednik algebras of $G(n,1,k)$ when the dimension vector is $(n,\dots,n)$, and this result is quite interesting and new from this perspective. One special case is

Conjecture 8. *Every block of the representation theory of q-Schur algebra with q a primitive k-th root of unity is Koszul dual to category $\mathcal{O}$ for the Cherednik algebra of $G(n,1,k)$ with a particular choice of integral parameters, where the choice of n depends on the block (and all n appear).*

Shan, Vasserot and Varagnolo have recently shown a Koszul duality exists between different blocks of parabolic categories $\mathcal{O}$ for affine Lie algebras which are conjecturally equivalent to the categories mentioned above. Thus, the algebras attached to affine quiver varieties provide a fascinating generalization of the theory of rational Cherednik algebras, which has proved a remarkable and rich theory over recent years.

I plan to attack the conjectures above using techniques similar to those used in [Weba, §4] and [SWb]; one can explicitly compute Ext algebras of semi-simple perverse sheaves on the space one plans to do hyperkahler reduction on, and one can show that taking hyperkähler reduction corresponds to a recognizable quotient of this algebra. This allows computation of the Ext algebra of simples.

Koszul dually, one can also geometrically approach the computation of Hom between projectives, using microlocal techniques. Computations along these lines by Braden were a major inspiration for our work in the case of hypertoric varieties, and Kevin McGerty [McG] has done some analogous work for affine type A quiver varieties, which I hope to utilize.

1.3. **Localization.** One of the most interesting themes of geometric representation theory is the interplay between sheaves of algebras such as the structure sheaf or differential operators, and their sections, most famously captured by the localization theorem of Beilinson-Bernstein [BB81].

We assume in this section that X possesses a symplectic resolution Y. In fact, there is a natural sheaf $\mathcal{A}_\eta$ on Y which "localizes" A_η: $\mathcal{A}_\eta$ is a deformation of the structure sheaf of Y, and A_η can be constructed from the sections of $\mathcal{A}_\eta$ in a straightforward way.

We have a natural functor $\Gamma : \mathcal{A}_\eta - \mathsf{mod} \to A_\eta - \mathsf{mod}$, essentially given by taking sections, and an adjoint to this, the localization functor $\mathsf{Loc} : A_\eta - \mathsf{mod} \to \mathcal{A}_\eta - \mathsf{mod}$. The localization functor sends category $\mathcal{O}$ to a category of sheaves supported (set-theoretically) on the subvariety

$$Y(\xi) = \left\{ y \in Y \middle| \lim_{t\to\infty} \xi(t) \cdot y \text{ exists} \right\}$$

which satisfy a "regularity" condition.

Conjecture 9. *There are finitely many effective classes in $x_i \in H_2(Y;\mathbb{Z})$ and rational numbers a_i such that the functor $\mathsf{Loc} : A_\eta - \mathsf{mod} \to \mathcal{A}_\eta - \mathsf{mod}$ is an equivalence if and only if $\langle \eta, x_i \rangle - a_i \notin \mathbb{Z}_{\geq 0}$. The derived functor $\mathbb{L}\mathsf{Loc}$ is an equivalence if and only if $\langle \eta, x_i \rangle \neq a_i$ for all i.*

- For the BGG category $\mathcal{O}$, this is precisely the localization theorem [BB81].
- For the case of a Slodowy slice, a version of this theorem has been proven by Ginzburg [Gin].
- For hypertoric varieties, this is a theorem of Bellamy and Kuwabara [BK].

What we can prove is a weaker version of this:

Theorem 10. *There are finitely many effective classes in $x_i \in H_2(Y;\mathbb{Z})$ such that the functor $\mathsf{Loc} : A_\eta - \mathsf{mod} \to \mathcal{A}_\eta - \mathsf{mod}$ is an equivalence for all η with $\langle \eta, x_i \rangle$ sufficiently large.*

There is a similar asymptotic statement for derived localization.

This allows us to understand category $\mathcal{O}$ as a category of sheaves on Y, and use techniques similar to those applied to $\mathcal{D}$-modules in more classical geometric representation theory. For example, we expect the techniques of mixed Hodge theory allow us to construct a graded version $\widetilde{O}^{\xi}_{X}$ of O^{ξ}_{X}. A graded lift can be constructed "by hand" in most important cases.

It also gives a geometric interpretation of the Grothendieck group $K^0(O^{\xi}_{X})$, or more naturally, the graded Grothendieck group $K^0_q(\widetilde{O}^{\xi}_{X})$ (which has base ring $\mathbb{Z}[q,q^{-1}]$, where q is grading shift). As with $\mathcal{D}$-modules, each $\mathcal{A}_\eta$ module has a characteristic cycle, defined by the sum of the classes of the components of its support variety, with multiplicity given by the graded dimension of the residue at that point. Since these cycles are not compact, we should interpret them as elements of Borel-Moore homology $H^{BM}_{*}(Y)$. Identifying $H^*_{\mathbb{C}^*}(pt) \cong \mathbb{Z}[q]$, we ultimately obtain a characteristic cycle map

$$CC : K^0_q(\widetilde{O}^{\xi}_{Y}) \to H^{BM,\mathbb{C}^*}_{*}(Y)[q^{-1}] \cong H^*_{\mathbb{C}^*}(Y)[q^{-1}].$$

We could instead consider the quantum cohomology $QH^*(Y)$, but the quantum cohomology of symplectic resolutions is not sufficiently well understood to know whether this is a more natural target for the characteristic cycle map. Moving forward, it would be very interesting to investigate these quantum cohomologies for their own interest, and to understand connections to our categories.

Theorem 11. *If ξ has isolated fixed points, CC is an isomorphism for generic η.*

Conjecture 12. *This map intertwines the Euler product on the Grothendieck group with the equivariant intersection product (generalizing Ginzburg's index theorem* [Gin86], *which implies the corresponding result for cotangent bundles).*

Thus, at least in the case with isolated fixed points, the images of the simple modules would give a basis for $H^*_{\mathbb{C}^*}(Y)[q^{-1}]$. This basis will satisfy the familiar properties of a canonical basis (i.e. self-duality, semi-orthogonality with standard classes).

Conjecture 13. *When Y is a quiver variety for a simply-laced simple Lie algebra $\mathfrak{g}$, this canonical basis coincides with Lusztig's canonical basis for a tensor product for certain choices of parameter.*

It's worth noting that for this would be a consequence of proving Conjecture 4, since the projective modules over $T^{\underline{\lambda}}$ are already known to correspond to the canonical basis by [Webb, 1.15].

1.4. **Twisting functors.** One interesting question to be studied is the relationship between categories O for different $\mathbb{C}^*$-actions on X. The categories O^{ξ}_{X} and $O^{\xi'}_{X}$ are both subcategories of $A_\eta-\mathsf{mod}$. One can show the existence of projection functors $\pi_\xi : D^b(A_\eta-\mathsf{mod}) \to D^b(O^{\xi}_{X})$, left adjoint to the inclusion ι_ξ. This allows one to define a canonical functor

$$\Phi_{\xi,\xi'} = \pi_{\xi'} \circ \iota_\xi : D^b(\widetilde{O}^{\xi}_{X}) \to D^b(\widetilde{O}^{\xi'}_{X}).$$

Let G be a maximal reductive subgroup of the symplectic automorphisms of X which commute with $\mathbb{S}$. This is a finite dimensional reductive group, and we let S be a maximal torus of this group, with $W = N_G(S)/S$ its Weyl group. We let $\Phi_w \colon O^{\xi}_{X} \to O^{w\cdot\xi}_{X}$ be the functor induced by lifting the action of a fixed representation of w in G to the algebra A. This is obviously an equivalence of categories.

Let $\mathfrak{s}$ be the Lie algebra of S, and $\check{\mathfrak{s}} \subset \mathfrak{s}$ the set of elements with the same fixed points as all of S. This is the set of elements where no weight that appears in the tangent space at a fixed point vanishes, and thus the complement of a hyperplane arrangement. We call a $\mathbb{T}$ action which factors through a map $\xi' \colon \mathbb{T} \to S$ **generic** if its fixed points coincide with those of S, that is, if its associated vector in $\mathfrak{s}$ lies in $\check{\mathfrak{s}}$.

Conjecture 14. *If ξ, ξ' are both generic, then the functor $\Phi_{\xi,\xi'}$ is an equivalence.*

Interestingly, this equivalence is not canonical. For example, $\Phi_{\xi,\xi'} \circ \Phi_{\xi',\xi} \ncong \mathrm{Id}$. Instead, these functors generate an interesting groupoid.

Definition 15. *We call a functor of $D(\widetilde{O}^{\xi}_Y) \to D(\widetilde{O}^{\xi'}_Y)$ a* **pure twisting functor** *if it is of the form $\Phi_{\xi_n,\xi'} \circ \Phi_{\xi_{n-1},\xi_n} \circ \cdots \circ \Phi_{\xi,\xi_1}$ for some sequence of generic cocharacters $\{\xi_i\}$. We call such a functor a* **twisting functor** *if it is a composition of pure twisting functors and the functors Φ_w.*

Conjecture 16. *There is a natural map $\pi_1(\check{\mathfrak{s}}/W) \to \mathsf{Aut}(D(\widetilde{O}^{\xi}_Y))$ whose image is the twisting functors.*

In the usual BGG category O, G is the adjoint group of the chosen Lie algebra, so $\mathfrak{s}$ is its Cartan, $\check{\mathfrak{s}}$ the regular elements of the Cartan, and W its Weyl group; thus $\pi_1(\check{\mathfrak{s}}/W)$ is the usual Artin braid group of $\mathfrak{g}$ and these specialize to Arkhipov's twisting functors. This conjecture is proven for hypertoric varieties by Braden, Licata, Proudfoot, and myself in [BLPWa].

This conjecture is particularly interesting in the case of a quiver variety which we mentioned earlier, since a braid groupoid action (of which Conjecture 16 only concerns the pure part) on the categories $D(T^{\underline{\lambda}}\text{–}\mathsf{mod})$ has been constructed in [Webb, §1], which categorifies the braiding in the category of quantum groups (i.e. the action of R-matrices times flips).

Conjecture 17. *The conjectured equivalence $D(\widetilde{O}^{\xi}_X) \cong D(T^{\underline{\lambda}}\text{–}\mathsf{mod})$ where X is a quiver variety intertwines this braid groupoid action with one by twisting functors.*

As I mentioned in the introduction, this action of a fundamental group at the moment lacks a good geometric explanation (instead, all proven cases of Conjecture 14 pass through a combinatorial description of $\pi_1(\check{\mathfrak{s}}/W)$ specific to those cases).

In certain cases of particular interest, we recover actions which were already known

- for the nilpotent cone, we recover the braid group acting on the Hecke algebra by the left multiplication via the usual quotient map.
- for a quiver variety (conjecturally), we recover the braiding in the tensor category of quantum group representations
- for an affine Grassmannian slice (conjecturally), we recover the quantum Weyl group action on a representation of a quantum group.

All of these actions have something in common; they arise from the monodromy of a flat connection of the form

$$\nabla - \log q \sum_{\alpha} \frac{df_i}{f_i}\Phi_i$$

where $f_i : H^2(Y, \mathbb{C}) \to \mathbb{C}$ is a collection of linear functionals and $\Phi_i : H^*(Y, \mathbb{C}) \to H^*(Y, \mathbb{C})$ a collection of linear operators *independent of q*. These are

- the Coxeter-KZ connection of Cherednik
- the Knizhnik-Zamolodchikov connection

- the Casimir connection of De Concini, Milson, and Toledano Laredo

respectively.

Furthermore each of these connections arises as a rational degeneration of the connection that arises in quantum cohomology of a symplectic manifold: CKZ for a nilcone, KZ for an affine Grassmannian slice and Casimir for a quiver variety. Note that these are **not** the varieties we attached them to before: KZ and Casimir have switched places! This is one manifestation of a more general duality phenomenon we will discuss below.

Conjecture 18. *The action of* $\pi_1(\check{\mathfrak{s}}/W)$ *on* $K^0_q(\widetilde{O})$ *is isomorphic to the monodromy representation of a flat connection of the form*

$$\nabla - \log q \sum_{\alpha} \frac{df_i}{f_i}\Phi_i$$

where $f_i : H^2(Y, \mathbb{C}) \to \mathbb{C}$ *is a collection of linear functionals and* $\Phi_i : H^*(Y, \mathbb{C}) \to H^*(Y, \mathbb{C})$ *a collection of linear operators independent of* q*. This connection is a rational degeneration of the quantum connection of a symplectic resolution of singularities (which we later denote* $Y^\vee$*).*

We expect that the full quantum connection will appear when one passes to quantizations over a field of characteristic p.

2. Knot homology

One interesting direction in which the study of these categories leads is the construction of homological knot invariants.

As described in the introduction, the geometric considerations in the introduction inspired a purely algebraic definition of homological invariants given in [Weba, Webb]. These invariants are constructed by associating a functor between categorifications of tensor products attached to any labeled tangle. The conjectures in the previous section would show that this invariant has a geometric origin when the Lie algebra $\mathfrak{g}$ is simply laced, but there are also plenty of questions which remain to be resolved from an algebraic perspective.

For instance, these invariants are not the only known categorifications of the quantum invariants for the defining representation of $\mathfrak{sl}(n)$. Khovanov and Rozansky [KR08] have already defined such a categorification using the seemingly unrelated tool of matrix factorizations.

Conjecture 19. *The knot invariants of* [Webb] *for the defining representation of* $\mathfrak{sl}_n$ *coincide with those of Khovanov-Rozansky.*

When $n = 2, 3$, this conjecture follows from results of Mazorchuk and Stroppel [MS]. My approach to the general case is to relate matrix factorizations to singular category O via the realization of that category as partial Whittaker sheaves on the flag variety (discussed in [Webc]).

Also, Witten has suggested a construction of knot invariants based on the Morse homology of certain spaces of solutions to PDEs; while this perspective sounds quite distant from ours, there are in fact unexpected links. Witten's perspective should attach a category to a disk labeled with finitely many representations of a simple Lie group. The category which seems to arise is a version of A-branes on a convolution space for the affine Grassmannian; as we will discuss later, our quantization framework is a mathematical version of this category, and conjecturally gives rise to the categorifications of tensor products $T^{\underline{\lambda}}-\mathsf{mod}$ discussed earlier. Thus, we conjecture

Conjecture 20. *Witten's construction can be made precise using quantizations of affine Grassmannians and naturally arising functors between them.*

There are two other structures which appeared in the original Khovanov homology (which is associated to the defining representation of $\mathfrak{sl}_2$) whose extensions to this more general theory could be quite interesting.

Khovanov homology was shown to be functorial (up to sign) by Jacobsson [Jac04] and this functoriality was interpreted in the representation-theoretic context by Stroppel using the adjunction of twisting functors [Str05, Str]. When the weights λ_i are all miniscule, then this prescription can be extended to the invariants from [Webb] to give a possible functoriality map, but this map is constructed using a handle decomposition of the cobordism, and it is unclear if the map is independent of cobordism.

Conjecture 21. *For each finite dimensional semi-simple Lie algebra $\mathfrak{g}$, there is a functorial invariant of links labeled with miniscule representations of $\mathfrak{g}$, valued in bigraded vector spaces, whose Euler characteristic is the Reshetikhin-Turaev invariant of this knot.*

An important barrier to the understanding functoriality and to general computation of these invariants is that their construction uses in a quite essential way the functor of tensor product with a module, which while finitely presented, has no known explicit projective resolution. Thus, at several important points, one uses the fact that any two projective resolutions of a module are homotopic; while this is sufficient for checking that the underlying vector space is a knot invariant, in order to show functoriality, one needs some understanding of the underlying homotopies.

Question 22. *Is there an explicit projective resolution of the functor attached to an arbitrary tangle projection, together with explicit and coherent homotopies between different projections of isotopic tangles?*

While not easy, this is not quite as daunting as it sounds; if one constructs projective resolutions of the functors for a single crossing, a single cup and a single cap, then by naturality this gives a projective resolution for any tangle projection. Similarly by usual topological arguments, one need only give homotopies corresponding to Reidemeister moves, and check a finite number of "movie moves" to check coherence.

Such a description could also be extremely useful for computation, which at the moment is the weak point of this theory.

One particularly interesting possibility is that this may allow us to construct some sort of homological version of Witten-Reshetikhin-Turaev invariants. In this case, the difficulties of defining WRT invariants for q not a root of unity could be explained by the fact that infinite dimensional vector spaces often do not have well defined graded dimensions.

Another important structure on Khovanov homology is the existence of the Lee spectral sequence which converges from Khovanov homology to a link invariant which only counts the number of link components, described in [Ras]. This spectral sequence and the functoriality of Khovanov homology was used by Rasmussen to construct a lower bound on slice genus which provides, for example, the first combinatorial (i.e. not using gauge theory) proof that the slice genus of a (p,q)-torus knot is $(p-1)(q-1)/2$. Lobb used a similar spectral sequence on Khovanov and Rozansky's homology to construct an infinite sequence of such lower bounds.

Conjecture 23. *There are spectral sequences which converge from the knot invariants of* [Webb] *to knot homologies only depending on the number of link components. More generally, there are spectral sequences from the homology for any choice of labeling to the homology attached to any Levi subgroup by the restriction of the corresponding representations to that Levi.*

The induced filtration on the limit for a knot defines a lower bound on slice genus of that knot.

3. S-duality and category $\mathcal{O}$

3.1. **A duality for singularities.** I am also working on another direction of research, jointly with Braden, Licata, and Proudfoot, on understanding a notion of duality between symplectic singularities. While we're still investigating the best set of hypotheses for the conjecture below, I will state a maximally optimistic version here and a more concrete conjecture using quantum field theory in Conjecture 28.

Conjecture 24 (Braden, Licata, Proudfoot, Webster). *For each conical symplectic singularity X, there is a dual singularity $X^\vee$ such that*

(1) *there is an order reversing bijection between strata of X and $X^\vee$ (by* [Kal]*, there is a canonical Whitney stratification given by symplectic leaves).*
(2) *there is a bijection between symplectic $\mathbb{C}^*$ actions on X, up to conjugacy, and pairs consisting of a crepant partial resolution of $X^\vee$, and a choice of ample line bundle on that resolution. Thus, to each partial resolution $Y \to X$ with $\mathbb{C}^*$-action ξ, we have a dual $Y^\vee \to X^\vee$ with action $\xi^\vee$.*
(3) *Furthermore, the categories $\widetilde{\mathcal{O}}^{\xi}_{Y}$ and $\widetilde{\mathcal{O}}^{\xi^\vee}_{Y^\vee}$ are Koszul dual, and, in particular, derived equivalent. This duality switches the action of twisting functors with functors defined using the monodromy in a canonical deformation of $Y^\vee$.*

For example:

- If $Y = T^*G/B$ then $Y^\vee = T^{*L}G/{}^LB$, and this duality claim is equivalent to the Koszul duality theorem for categories $\mathcal{O}$ due to Beilinson, Ginzburg, and Soergel [BGS96].
- If X is a hypertoric singularity associated to a hyperplane arrangement H, then $X^\vee$ is another hypertoric singularity, associated to the Gale dual $H^\vee$. Braden, Licata, Proudfoot, and I have proven the conjecture in this case [BLPW10, BLPWa].
- Conjecturally, if Y is the space of G-instantons on the algebraic surface $\widetilde{\mathbb{C}^2/\Gamma}$, then $Y^\vee$ is the G' instantons on $\widetilde{\mathbb{C}^2/\Gamma'}$ where G and Γ' (resp. G' and Γ) are matched by the MacKay correspondence.
- Conjecturally, if Y is the Nakajima quiver variety for weights λ and μ in type ADE, then $Y^\vee$ is the slice to Gr_μ in $\overline{\mathrm{Gr}_\lambda}$ where Gr_μ denotes an orbit of polynomial loops in the affine Grassmannian of the Langlands dual group. Denote this slice $\mathfrak{W}^\lambda_\mu$.

  Conjecturally, this perspective will relate the homological knot invariants defined in terms of quiver varieties (including Khovanov-Rozansky homology) to those defined using the affine Grassmannian by work of Seidel-Smith [SS06], Manolescu [Man07], and Cautis-Kamnitzer [CK08].

In the first two cases, the Koszul duality results are well understood and should be regarded as evidence for Conjecture 24. The case of instanton spaces is less well understood. One special case is when Y is the Hilbert scheme of $\widetilde{\mathbb{C}^2/\Gamma}$ (which are moduli spaces of $U(1)$-instantons) and thus O_Y^ξ is category O as defined by [GGOR03] for a Cherednik algebra. In this case, our conjecture reduces to:

Conjecture 25. *Category O for the rational spherical Cherednik algebra of $S_n \wr C_\ell$ has a graded lift which is Koszul (for parameters away from a finite number of hyperplanes), and its Koszul dual is a category O for the space of $U(\ell)$-instantons on $\mathbb{C}^2$.*

The last case is particularly interesting from the perspective of knot homology since it suggests a different approach to the categorifications of tensor products arising from quiver varieties. At the moment, outside type A, the algebras A_η quantizing $\mathfrak{W}^\lambda_\mu$ are not well-understood; however, in type A, they have been described by Brundan and Kleshchev as quotients of shifted Yangians. The Poisson geometry of the affine Grassmannian suggests that this description will carry over to arbitrary case.

Conjecture 26. *The algebra A_η for $\mathfrak{W}^\lambda_\mu$ is a natural quotient of a shifted version of the Yangian of the corresponding Lie algebra ${}^L\mathfrak{g}$.*

I am currently working on this conjecture jointly with Joel Kamnitzer. Once these quantizations are well-understood, they should have a rather interesting representation theory.

Conjecture 27. *The category O_X^ξ for $X = \mathfrak{W}^\lambda_\mu$ is equivalent to the category $T^{\underline{\lambda}}$–mod with $\lambda = \sum \lambda_i$ with $\underline{\lambda}$ depending on η.*

This equivalence sends the functors of the categorical $\mathfrak{g}$-action to generalizations of Zuckerman functors relating the quantizations of different slices, and the braid actions to shuffling functors.

As we mentioned earlier, this conjecture suggests that our construction of categorified tensor products is compatible with the extended TQFT in Witten's description.

3.2. **Connections to physics.** Remarkably, the same list of examples of "dual" varieties has been known to physicists for some time. They are the Higgs branches of mirror dual $N = 4$ supersymmetric quantum field theories. This was established in the physics literature for T^*G/B [GW], and for hypertoric singularities [KS99], and both finite and affine type A quiver varieties [dBHOO97]. Alternatively, this duality switches the Higgs branch and quantum Coulomb branches of these QFT's, the Higgs branches of dual QFT's can be described as the Higgs and quantum Coulomb branches of the same theory.

Conjecture 28. *If X is the Higgs branch of the moduli space of vacua for an $N = 4$ supersymmetric $d = 3$ field theory, then $X^\vee$ is the Higgs branch of the mirror dual field theory, that is, the quantum Coulomb branch of the original field theory.*

Some, though certainly not all, of our expectations about connections between X and $X^\vee$ can be explained in this picture. For example, the connection between resolutions of X and $\mathbb{C}^*$ actions on $X^\vee$ corresponds to the switching of mass and Fayet-Illiopolous parameters under mirror duality.

All quiver varieties are the Higgs branches of appropriate QFT's, so this conjecture does suggest what the dual singularity to a quiver variety should be. Unfortunately, the quantum Coulomb branch is a very mysterious object, and present techniques in physics do not allow for a description of them in this generality which would satisfy a mathematician. I've had discussions with several physicists on this subject, but it is still a point which requires more investigation.

3.3. **Localization duality.** There are several other conjectural connections between X and $X^\vee$, which we expect will all be consequences of Conjecture 24.

For instance, we consider two smooth varieties M and N with torus actions of S on M and T on N, both with isolated fixed points, and both equivariantly formal in the sense of Goresky-Kottwitz-MacPherson [GKM98].

Definition 29. *We call M and N* **localization dual** *if there is a perfect pairing $H^2_S(M)\times H^2_T(N) \to \mathbb{C}$ such that*

- *$H^2_S(pt)$ and $H^2_T(pt)$ are mutual annihilators.*
- *There is a bijection $\Phi : M^S \cong N^T$ such that $\ker i_m^* \subset H^2_S(M)$ and $\ker i^*_{\Phi(m)} \subset H^2_T(N)$ are mutual annihilators, where i_m is the inclusion map of a fixed point.*

For example, T^*G/B and $T^{*L}G/{}^LB$ are localization dual, with the pairing given by the identifications $H^2_T(Y) \cong \mathfrak{t} \oplus \mathfrak{t}$ and $H^2_{{}^LT}(T^{*L}G/{}^LB) \cong \mathfrak{t}^* \oplus \mathfrak{t}^*$ (the kernels of maps to points are the graphs of the elements of the Weyl group, and thus mutual annihilators since they are the graphs of dual linear maps). An easy calculation also shows that Gale dual hypertoric varieties are also localization dual.

Let (Y, ξ) and $(Y^\vee, \xi^\vee)$ be duals, in the sense of Conjecture 24. Let T and S be the maximal Hamiltonian tori acting on Y and $Y^\vee$ which contain the image of ξ and $\xi^\vee$.

Conjecture 30. *The spaces Y and $Y^\vee$ are localization dual.*

Work by Braden, Licata, Proudfoot, Phan, and myself [BLP+] shows that Conjecture 30 is closely related to Conjecture 24, by showing that an algebraic analogue holds for a large class of Koszul algebras A, relating A to its dual $A^!$.

3.4. **The cell filtration.** I assume throughout this section that the fixed points of ξ are isolated, to simplify notation and statements.

One of the more interesting structures of the Hecke algebra which is clearer after realizing it as the Grothendieck group of the BGG category $\mathcal{O}$ is that of the cells. For each nilpotent orbit, there is an ideal of the Hecke algebra which is spanned by the classes of simples whose characteristic varieties in T^*G/B are contained in the preimage of that orbit closure. This is a two-sided cellular ideal.

This notion generalizes to arbitrary symplectic resolutions of affine singularities. The singular variety X has a natural stratification $X = \sqcup_\alpha X_\alpha$ by symplectic leaves [Kal], which generalizes the orbit stratification on $\mathcal{N}_\mathfrak{g}$. This allows us to define a generalization of two sided cells.

Definition 31. *For each stratum X_α of X, let $\widetilde{\mathcal{O}}_\alpha \subset \widetilde{\mathcal{O}}^\xi_X$ be the subcategory of objects whose supports are contained in $\overline{X_\alpha}$. Let*

$$\widetilde{\mathcal{O}}^\perp_\alpha = \{M | \mathrm{Ext}^\bullet(M,N) = 0 \text{ for all } N \in \widetilde{\mathcal{O}}_\alpha\}$$

be the annihilator subcategory.

In particular, we have two dual filtrations of $K^0_q(\widetilde{\mathcal{O}}^\xi_X)$ by subspaces $K_\alpha = K^0_q(\widetilde{\mathcal{O}}_\alpha)$ and $B_\alpha = K^0_q(\widetilde{\mathcal{O}}^\perp_\alpha)$, which are exchanged by Koszul duality.

We call the set of simples in $\widetilde{\mathcal{O}}_\alpha$ which do not lie in $\widetilde{\mathcal{O}}_\beta$ for $\beta < \alpha$ the **two-sided cell** *of α.*

Furthermore, we expect that there will be a finer decomposition corresponding to the left and right cells of the Hecke algebra. As in the BGG/Hecke algebra case, left cells are defined by primitive ideals of the algebra A_η (an ideal is called **primitive** if it annihilates a simple module). We let $\pi\colon Y \to X$ be a crepant $\mathbb{Q}$-factorial terminalization (if X possesses a symplectic resolution, then this implies that Y must be a symplectic resolution).

Definition 32. *Two simple modules lie in the same* **left cell** *if their annihilators coincide.*

Since all primitive ideals are two-sided, their associated varieties in X must be Poisson, and thus of the form $\overline{X_\alpha}$ for some α. The two-sided cell for α is a union of left cells corresponding to the primitive ideals whose associated varieties are $\overline{X_\alpha}$. We assume from now on that the local system on X_α given by top cohomology of the fiber of π is trivial. Note that this holds for hypertoric varieties and quiver varieties.

Conjecture 33. *There is a bijection between primitive ideals J associated to α (that is, left cells in the two-sided cell for α) and components of $\pi^{-1}(X_\alpha)$.*

This bijection should be given by considering the support of a localization of the corresponding simple module to a sheaf on Y.

Definition 34. *Two simple modules lie in the same* **right cell** *if each appears as a composition factor in the other tensored with a Harish-Chandra bimodule.*

Conjecture 35. *There is a bijection between right cells in the two-sided cell for α and MV cycles, the components of the locus in X_α where the limit $\lim_{t\to 0}\xi(t)\cdot x$ exists.*

We expect that this cell decomposition will behave well under duality. By Koszul duality, there is a bijection between simple modules of $\widetilde{\mathcal{O}}^{\xi}_{X}$ and $\widetilde{\mathcal{O}}^{\xi^\vee}_{X^\vee}$.

Conjecture 36. *This bijection preserves two-sided cells (compatibly with the bijection between strata of X_α and $X^\vee_\alpha$) and sends left cells to right cells.*

Conjectures 33, 35, and 36 are all true for type A flag varieties, and are all proven for hypertoric varieties in [BLPWa].

4. Results of prior support

From September 2003 through July 2007, I was supported by an NSF Graduate Research Fellowship. During this time I wrote 6 papers [Web08, WY07, Web07b, Web07a, PW07, Web06], on aspects of Lie theory, the theory of finite groups, and combinatorial algebraic geometry. Each of these papers has been published and distributed on the website `http://www.arxiv.org`.

From August 2007 though June 2011, I have been supported by an NSF Postdoctoral Research Fellowship with additional support from NSA grant Grant H98230-10-1-0199 from June 2010 through the present (anticipated ending in June 2012); during that time, I wrote eleven papers [WW08, WWa, WWb, BLPW10, BLP+, BLPWa, SWa, Webc, Weba, Webb], six of which have been accepted for publication.

- I coauthored a series of 3 papers [WW08, WWa, WWb] with G. Williamson exploring the connections between HOMFLYPT homology and the geometry of algebraic groups. This allowed us to give the first definition of a homology theory categorifying colored HOMFLYPT polynomial using spectral sequences attached to weight filtrations on Bott-Samelson type varieties, and to give a geometric construction of a Markov-type trace on Hecke algebras of all types. This method gave an elegant explanation of the expansion of this trace in terms of irreducible characters (coinciding with one given "by hand" by Gomi) using the geometry of character sheaves.

- I coauthored 2 papers [BLPW10, BLP+] with T. Braden, A. Licata, and N. Proudfoot related to the topics of this proposal, proving essentially the entire program described above in the case of hypertoric varieties. In these works, we defined the category O attached to a hypertoric variety and showed it was equivalent to a very simple, combinatorially presented algebra, which one can explicitly prove is standard Koszul; we also proved that twisting and shuffling functors give actions of the correct fundamental groups and are Koszul dual to each other.
- I also coauthored a paper [BLPWa] with T. Braden, A. Licata, C. Phan, and N. Proudfoot which connected the localization duality described above to Koszul duality. We showed that for standard Koszul dual algebras satisfying certain technical conditions, dual subspace arrangements like those from equivariant cohomology arise very naturally from canonical deformations.
- I authored a paper [Webc] which shows that the category $\mathcal{O}$, as we have defined it, attached to the intersection of a Richardson nilpotent orbit for a Lie algebra $\mathfrak{g}$ with the Slodowy slice to a Richardson nilpotent is equivalent to a singular block of parabolic category $\mathcal{O}$ (in the usual sense) for $\mathfrak{g}$. This gives a geometric explanation of Brundan's isomorphism of the centers of such blocks with cohomology of Spaltenstein varieties.
- I coauthored a paper [SWa] with C. Stroppel, describing connections between the geometry of 2-row Springer fibers and category $\mathcal{O}$.
- I wrote two papers, [Weba, Webb] discussed extensively in the proposal which define categorifications of tensor products of simple representations and categorified knot invariants based on this construction. This shows that there is a categorification of the quantum knot invariant attached to every representation of a simple Lie algebra.
- I coauthored a paper [SWb] with C. Stroppel showing the existence of a grading on cyclotomic q-Schur algebras, and relating decomposition numbers to canonical bases.

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