Hypertoric mirror symmetry

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Recall: a **hyperkähler manifold** is a manifold M equipped with a metric, and 3 complex structures I, J, K which generate a copy of $\mathbb{H}$, each of which make M Kähler.

In the complex structure I, the $\mathbb{C}$-valued symplectic form $\omega_J + i\omega_K$ is holomorphic. Often it's easier to show that varieties are holomorphic symplectic without the additional metric structure.

Hyperkähler manifolds naturally arise in physics, for example in the moduli of vacua of a 3d $\mathcal{N} = 4$ QFT. If you're a physicist, maybe you think those are intrinsically interesting, but if you're a mathematician, I argue you should care because they spin off interesting mathematics.

Braverman-Finkelberg-Nakajima give a mathematical description of part of this space: the **Coulomb branch**. They define as Spec of the homology of a space ("the BFN space") related to affine Grassmannian under convolution.

In this talk, I'll discuss homological mirror symmetry for some of these varieties. People often tell me that hyperkähler varieties in one Kähler structure will be dual to a rotation (one of the other Kähler structures); probably this needs a lot of caveats, but I'll present one piece of evidence for this.

Most basic example: let $\mathbb{H}_\circ = T^*\mathbb{C} \setminus \{zw = -1\}$ with the holomorphic symplectic form $\Omega = \omega_J + i\omega_K = \frac{dx \wedge dy}{xy-1}$ (ω_I is very messy).

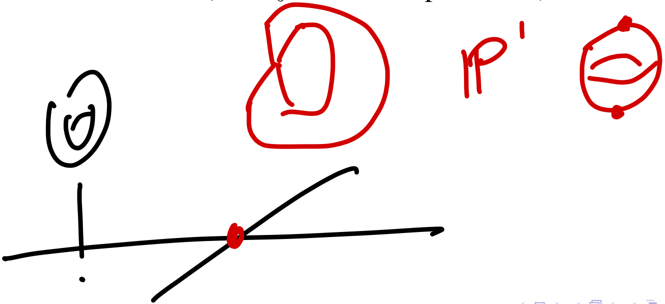
This is a standard example in mirror symmetry, since it is $\mathbb{P}^1$ minus a canonical divisor. I want to think of it in its own right.

This is self-dual: the map $(z, w) \mapsto (\log(|zw - 1|), |z|^2 - |w|^2)$ has special Lagrangian, self-dual fibers (except at $(0, 0)$). Can be proven rigorously using Abouzaid-Auroux-Katzarkov.

Our philosophy is a bit different: we use that $\mathbb{H}_\circ$ has a Lagrangian skeleton, given by

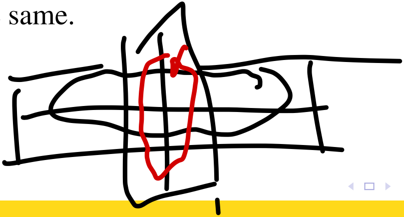
$$\mathcal{S} = \{(\eta\sqrt{1-\zeta}, \eta^{-1}\sqrt{1-\zeta}) | \eta, \zeta \in U(1)\}.$$

This is a nodal torus (with $\zeta = 1$ a collapsed fiber).



Using microlocal philosophy of Ganatra-Pardon-Shende, we can show that constructing an object in the Fukaya category is equivalent choosing a local system on $\mathcal{S} - \{(0,0)\}$, together with gluing data at the node.

The gluing data at the node looks like choosing a sheaf on $\mathbb{C}$ with singular support in $\mathbb{C} \cup T_0^*\mathbb{C}$ such that local system on both punctured components is the same.



- The fiber of the local system gives a vector space V with invertible monodromy endomorphism $m\colon V \to V$.
- The variation and canonical maps for vanishing cycles give maps $x, y\colon V \to V$.
- These maps are related by $xy = yx = 1 + m$.

Thus, we find that V has an action of $\mathbb{C}[x, y]$ such that $(xy - 1)^{-1}$ exists.

It's a coherent sheaf on $\mathbb{H}_\circ$! Congrats, you understand mirror symmetry now!

We can construct examples by **hyperhamiltonian reduction**.

Consider a hyperkähler manifold M (for example $\mathbb{H}^n \cong T^*\mathbb{C}^n$), and an action of a compact group G. This is hyperhamiltonian if we have a triple of moment maps $\vec{\mu}\colon M \to (\mathbb{R} \oplus \mathbb{C}) \otimes \mathfrak{g}^*$ for the three different symplectic structures.

An $\mathbb{H}$-linear action on $T^*\mathbb{C}^n$ is hyperhamiltonian with moment map

$$\langle \vec{\mu}(m), X \rangle = (\omega_I(m, Xm), \omega_J(m, Xm) + i\omega_K(m, Xm)).$$

The simplest case is when G is abelian. In this case, we can choose bases so that $G \subset D = U(1)^N$ is contained in the unitary diagonal matrices.

If $G = D$ is the diagonal matrices, then this map is given by

$$\vec{\mu}(\mathbf{z}, \mathbf{w}) = ((|z_1|^2 - |w_1|^2, \ldots, |z_n|^2 - |w_n|^2), (z_1 w_1, \ldots, z_n w_n))$$

For $G \subset D$, obtained by composing with projection

$$(\mathbb{R} \oplus \mathbb{C}) \otimes \mathfrak{d}^* \to (\mathbb{R} \oplus \mathbb{C}) \otimes \mathfrak{g}^*.$$

Definition

The hyperhamiltonian reduction at (δ, β) is $\mathfrak{M}^{(+)} = \vec{\mu}^{-1}(\delta, \beta)/G$.

When we apply this to the action of a (connected) torus $G \subset D$, the result is a **(additive) hypertoric variety** $\mathfrak{M}^{(+)}$ (also called "toric hyperkähler".

In physics, this makes the hypertoric variety $\mathfrak{M}^{(+)}$ the **Higgs branch** of a 3d $\mathcal{N} = 4$ gauge theory with gauge group G and matter $\mathbb{H}^n$.

It's also Coulomb branch à la BFN of the theory with matter $\mathbb{H}^n$ and gauge group $G^! = {}^L(D/G) \subset {}^LD = D^!$.

The space I want to talk about is related but stranger: we replace everywhere

$$\mathbb{H} \mapsto \mathbb{H}_\circ = T^*\mathbb{C} \setminus \{zw = -1\} \qquad (\mathbb{R} \oplus \mathbb{C}) \otimes \mathfrak{g}^* \mapsto \mathfrak{g}^* \times {}^L G_\mathbb{C}$$

For the obvious action of D on $\mathbb{H}_\circ^n$, we have partly group-valued moment map $\vec{\mu}^{(\times)} \colon \mathbb{H}_\circ^n \to \mathbb{R}^n \times (\mathbb{C}^*)^n = \mathfrak{d}^* \times {}^L D_\mathbb{C}$

$$\vec{\mu}^{(\times)}(\mathbf{z}, \mathbf{w}) = ((|z_1|^2 - |w_1|^2, \dots, |z_n|^2 - |w_n|^2), (z_1 w_1 + 1, \dots, z_n w_n + 1)).$$

For G, just apply canonical map $\mathfrak{d}^* \times {}^L D_\mathbb{C} \to \mathfrak{g}^* \times {}^L G_\mathbb{C}$.

Definition

The **multiplicative hypertoric variety** is the quasi-hyperhamiltonian reduction $\mathfrak{M}^{(\times)}_{\beta,\delta} = \vec{\mu}^{\times}(\delta, \beta)/G$.

By Kirwan-Ness, we can write either kind of hypertoric variety as a GIT quotient:

$$\mathfrak{M}_{\beta,\delta}^{(+/\times)} = (\mu_{\mathbb{C}}^{(+/\times)})^{-1}(\beta) /\!\!/_{\delta} G_{\mathbb{C}}$$

of a fixed level for the second and third components of the additive/multiplicative moment map:

$$\mu_{\mathbb{C}}^{(+)} : \mathbb{H}^n \to \mathfrak{g}_{\mathbb{C}}^* \qquad \mu_{\mathbb{C}}^{(\times)} : \mathbb{H}_{\circ}^n \to {}^L G_{\mathbb{C}}.$$

There's also a way of thinking of $\mathfrak{M}_{1,\delta}^{(\times)}$ on the Coulomb side.

Theorem

$\mathbb{C}[\mathfrak{M}_{1,\delta}^{(\times)}]$ *is the K-theory of the BFN space for the gauge theory with gauge G and matter* $\mathbb{H}^n$.

In physics, this comes from a related $4d$ $\mathcal{N} = 2$ gauge theory; this is the algebra of line operators in the Coulomb phase wrapped around S^1 in the product $\mathbb{R}^3 \times S^1$.

I'll exploit this fact in the proof, but its significance in physics seems a bit unclear; physicists usually tell me considering (2d) mirror symmetry for this sort of space is a weird thing to do.

To make parameters more symmetric, think of ${}^LG_{\mathbb{C}} \cong \mathfrak{g}^* \times {}^LG$, and instead of δ, choose $\alpha = (\delta, \delta') \in {}^LG_{\mathbb{C}}$.

The other component δ' looks like it should be some kind of "B-field" but we have not yet made this precise in most cases.

Ansatz

The hypertoric varieties $(\mathfrak{M}^{(\times)}_{\alpha,\beta}, \mathfrak{M}^{(\times)}_{\beta,\alpha})$ are a mirror pair.

In Teleman's framework for mirror symmetry:

- a $G_{\mathbb{C}}$-action should correspond to a map to ${}^LG_{\mathbb{C}}$ on the other side,
- Hamiltonian reduction should correspond to taking a fiber.

Since the construction of $\mathfrak{M}^{(\times)}_{\beta,\alpha}$ involves taking a fiber and a Hamiltonian reduction, this construction is self-dual.

Theorem (Gammage-McBreen-W.)

In the I-Kähler structure, the wrapped Fukaya category of $\mathfrak{M}^{(\times)}_{\alpha,1}$ (which is independent of α) is quasi-equivalent to the derived category of coherent sheaves on $\mathfrak{M}^{(\times)}_{1,\alpha}$ (which seems to depend on α, but different choices give). equivalent

For $\alpha \in \mathfrak{g}^*$, these are hyperkähler rotations of each other.

So why do I keep that parameter α around? It doesn't change the underlying A_∞ categories up to quasi-equivalence, but the identification for different parameters isn't canonical.

We match the two categories by considering the situation where $\alpha \in {}^L G_{\mathbb{C}}$ is unitary.

- on the Fukaya side, this means that we have a Lagrangian skeleton where the components are holomorphic in the J-complex structure (i.e. they are (A, B, A)-branes). This induces a t-structure on the Fukaya category.
- on the coherent side, we have an associated **non-commutative** resolution of singularities for the singular variety $\mathfrak{M}_{(1,1)}$. This is a non-commutative algebra A such that $D^b(A\text{-mod}) \cong D^b(\mathsf{Coh}(\mathfrak{M}_{(1,\delta)}))$.

This Lagrangian skeleton gives us another way to think our variety topologically:

- given $\alpha \in {}^L G$, we can consider the preimage K_α of α in ${}^L D$.
- K_α is topologically a torus, but doesn't have a distinguished origin. The subtori where coordinates vanish divide K_α up into polytopic regions.
- each of these regions can be "puffed" up in to a toric variety, and we can plumb the cotangent bundles of these together along the toric varieties corresponding to intersections.

Lemma

The result embeds in $\mathfrak{M}_{\alpha,1}$ with the toric varieties giving the skeleton.

Skeletons in the closet

On the other side, we have to construct the desired non-commutative resolution of singularities.

Definition

A **tilting generator** on a scheme X is a vector bundle $\mathcal{T}$ on X such that $\mathrm{Ext}^{>0}(\mathcal{T},\mathcal{T}) = 0$ and $\langle \mathcal{T} \rangle = D^b(\mathsf{Coh}(X))$.

In this case, we have an equivalence of derived categories

$$\mathbb{R}\mathrm{Hom}(\mathcal{T}, -) \colon D^b(A\text{-mod}) \cong D^b(\mathsf{Coh}(X)) \colon \mathcal{T} \overset{L}{\otimes}_A -$$

A = End(T)

A theorem of Kaledin gives a construction of a tilting generator on a formal neighborhood of a fiber of any symplectic resolution, in fact one for each generic $\alpha \in {}^L G$. Unfortunately, Kaledin's usual trick (spread this out using a $\mathbb{C}^*$-action) doesn't work to construct one on $\mathfrak{M}_{1,\beta}$.

On the other hand, in this case, Kaledin's construction in this case gives a sum of line bundles, and all of them are the associated bundles for G-modules. So, we can just take the corresponding line bundles on the whole variety.

Theorem

The resulting vector bundle $\mathcal{T}_\alpha$ is a tilting generator on $\mathfrak{M}_{1,\beta}$, and the endomorphism ring A_α is a NCCR which only depends on α.

This is a special case of a general construction for K-theoretic Coulomb branches, where Kaledin's construction gives us a sum of vector bundles coming from surface (?) operators in the 4d theory.

NCCRs and tilting bundles

So, the proof of our theorem goes as follows:

- Using work of Ganatra-Pardon-Shende, we write the wrapped Fukaya category in terms of microlocal sheaves on the skeleton.
- Using standard calculations in microlocal sheaves on a coordinate cross, we compute the algebra A given by the natural transformations between the functors of microstalk on the different components of the skeleton.
- We then note A is exactly A_α the non-commutative resolution of singularities attached to α.
- We construct the equivalence $D^b(A_\alpha\text{-mod}) \cong D^b(\mathsf{Coh}(\mathfrak{M}_{(1,\delta)}))$.

Note that $D^b(A_\alpha\text{-mod}) \cong D^b(\mathsf{Coh}(\mathfrak{M}_{(1,\delta)}))$ for any α and δ. Even worse/better, the algebra A_α is unchanged if we tensor $\mathcal{T}$ by a line bundle.

Theorem

These equivalences generate an action of $\pi_1({}^L G^\circ_{\mathbb{C}})$*, where* ${}^L G^\circ_{\mathbb{C}}$ *is the complement of certain "bad" choices of* α*. This is dual to the action on* $\mathrm{Fuk}(\mathfrak{M}^{(\times)}_{\alpha,1})$ *by parallel transport.*

These can even be upgraded to an equivalence of Schobers (perverse sheaves of categories), but maybe that's a story for another time.

Thanks

Thanks for listening.