Representation theory and the Coulomb branch

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What are Coulomb branches? What are they good for?

- Coulomb branches are very nice non-commutative algebras, with a simple(ish) presentation.
- Lots of interesting algebras show up this way: cyclotomic Cherednik algebras, truncated shifted Yangians, hypertoric enveloping algebras.
- They connect to a lot of other interesting math: (weighted) KLR algebras, knot homology, and the geometry of symplectic resolutions.

The most basic example to understand are the representations of the Weyl algebra W generated by x, ∂ with the relations $[\partial, x] = 1$. If we let $H = x\partial$, then note that we have

$$\partial x = H + 1 \qquad xH = (H-1)x \qquad \partial H = (H+1)\partial$$

For $\mu \in \mathbb{C}$ and a rep V, let

$$V_\mu = \{v \in V \mid (H-\mu)^N v = 0 \text{ for } N \gg 0\}$$

be the generalized weight space for μ.

Definition

We call a W module **generalized weight** it is the sum of its generalized weight spaces, and these spaces are finitely generated.

Note that $x\colon V_{\mu-1} \to V_\mu$ and $\partial\colon V_\mu \to V_{\mu-1}$.

These maps are isomorphisms unless $\mu = 0$, since

$$H = x\partial\colon V_\mu \to V_\mu \qquad H + 1 = \partial x\colon V_{\mu-1} \to V_{\mu-1}$$

both only have the eigenvalue μ.

Theorem

The functor sending a generalized weight module with integral weights to the quiver representation

$$V_{-1} \underset{\partial}{\overset{x}{\rightleftarrows}} V_0$$

is an equivalence to the subcategory of nilpotent finite dimensional quiver representations.

Given such a quiver representation, we can build a representation V over W by declaring that $V_n \cong V_{-1}$ if $n < 0$ and $V_n \cong V_0$ if $n \geq 0$.

If $\mu \neq 0$, we let

- $\partial\colon V_\mu \to V_{\mu-1}$ act by the identity, and
- $x\colon V_{\mu-1} \to V_\mu$ act by multiplication by $\mu + h$ where h is the (nilpotent) action of the loop in the quiver representation.

You can easily check that this satisfies $[\partial, x] = 1$ and that V_μ actually is the generalized μ-weight space.

The next example to understand is the representations of $U(\mathfrak{sl}_2)$ on which $C = EF + FE + H^2/2$ act by a fixed scalar $\frac{1}{2}\alpha(\alpha+2)$ with $\Re(\alpha) \geq -1$. In this case, the relations of $\mathfrak{sl}_2$ become:

$$HE = E(H+2) \qquad EF = -\frac{1}{4}(H-\alpha-2)(H+\alpha)$$
$$HF = F(H-2) \qquad FE = -\frac{1}{4}(H-\alpha)(H+\alpha+2)$$

Taking associated graded, I get functions on nilpotent 2×2 matrices, with coordinates given by $\begin{bmatrix} h/2 & e \\ f & -h/2 \end{bmatrix}$.

For $\mu \in \mathbb{C}$ and a rep V, let

$$V_\mu = \{v \in V \mid (H-\mu)^N v = 0 \text{ for } N \gg 0\}$$

be the generalized weight space for μ.

On a generalized weight module, Lusztig's idempotented version doesn't act. Instead, we let $\widehat{U(\mathfrak{sl}_2)}_\alpha$ be $U(\mathfrak{sl}_2)_\alpha = U(\mathfrak{sl}_2)/\langle C - \frac{1}{2}\alpha(\alpha+2)\rangle$ with idempotents $\mathbf{1}_\mu$ for $\mu \in \mathbb{C}$ added, allowing formal power series in $H-\mu$ acting on $\mathbf{1}_\mu$.

In particular, a polynomial $p(H)$ acts invertibly on V_μ if and only if $p(\mu) \neq 0$. Thus, we can define

$$H'\mathbf{1}_\mu = (H-\mu)\mathbf{1}_\mu$$

$$F'\mathbf{1}_\mu = \begin{cases} -F\frac{4}{(H-\alpha-2)(H+\alpha)}\mathbf{1}_\mu & \mu \notin \{\alpha+2, -\alpha\} \\ -F\frac{4}{(H-\alpha-2)}\mathbf{1}_\mu & \mu = -\alpha \neq \alpha+2 \\ -F\frac{4}{(H+\alpha)}\mathbf{1}_\mu & \mu = \alpha+2 \neq -\alpha \\ -4F\mathbf{1}_\mu & \mu = \alpha+2 = -\alpha \end{cases}$$

These satisfy the relations:

$$[H',E]=0 \quad EF'\mathbf{1}_\mu = \begin{cases} \mathbf{1}_\mu & \mu \notin \{\alpha+2,-\alpha\} \\ H'\mathbf{1}_\mu & \mu \in \{\alpha+2,-\alpha\}, -\alpha \neq \alpha+2 \\ (H')^2\mathbf{1}_\mu & \mu = \alpha+2=-\alpha \end{cases}$$

$$[H,F']=0 \quad F'E\mathbf{1}_\mu = \begin{cases} \mathbf{1}_\mu & \mu \notin \{\alpha,-\alpha-2\} \\ H'\mathbf{1}_\mu & \mu \in \{\alpha,-\alpha-2\}, -\alpha-2 \neq \alpha \\ (H')^2\mathbf{1}_\mu & \mu = \alpha=-\alpha-2 \end{cases}$$

Thus, we can define a equivalence relation on $\mathbb{C}$ by $\mu \sim \mu-2$ if $\mu \notin \{-\alpha,\alpha+2\}$; if two numbers are equivalent, the corresponding weight spaces are isomorphic in all representations.

For each set μ, there's a unique simple with $V_{\mu'} \cong \mathbb{C}$ for $\mu' \sim \mu$ and $V_{\mu'}=0$ if $\mu' \not\sim \mu$.

These sets are always of the form

- $\{\alpha, \alpha - 2, \dots, -\alpha\}$ (if $\alpha \in \mathbb{Z}_{\geq 0}$),
- $\{-\alpha - 2, -\alpha - 4, \dots\}$ or (if $\alpha \notin \mathbb{Z}_{\geq 0}$) $\{\alpha, \alpha - 2, \dots\}$
- $\{\dots, \alpha + 4, \alpha + 2\}$ or (if $\alpha \notin \mathbb{Z}_{\geq 0}$) $\{\dots, -\alpha + 2, -\alpha\}$
- $\{\dots, \mu + 2, \mu, \mu - 2, \dots\}$ (if $\mu \pm \alpha \notin 2\mathbb{Z}$).

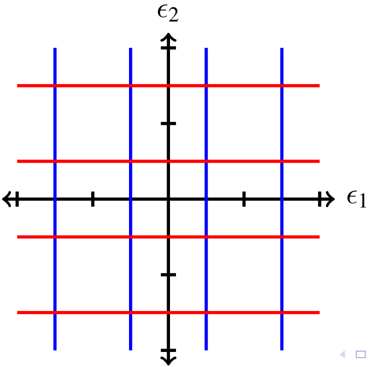
Each coset $\mu + 2\mathbb{Z}$ is a union of 1, 2 or 3 of these, and we can easily draw a quiver with relations that describes the category.

Philosophy

I've gone through this case in a lot of detail, because I want to generalize it, and be able to wave my hands a bit. The secret story here is that $U(\mathfrak{sl}_2)_\alpha$ is a **quantum Coulomb branch**.

Let T be a torus acting on a v. s. V. Let $\varphi_1, \dots, \varphi_d$ be the weights.

Now, draw a picture of $\mathfrak{t}_{\mathbb{R}}$ and draw in all the hyperplanes $\varphi_i(X) \in \mathbb{Z} + 1/2$. For $D \subset GL(2)$ acting on $\mathbb{C}^2 \oplus \mathbb{C}^2$, the weights are $\varphi_1 = \varphi_2 = \epsilon_1, \varphi_3 = \varphi_4 = \epsilon_2$



The coordinate ring of the Coulomb branch has an explicit description in these terms:

Theorem

$\mathbb{C}[\mathfrak{M}] := \mathsf{A}$ *is a* $\mathrm{Sym}(\mathfrak{t}^*)$ *algebra with free basis* r_ν *for* $\nu \in \mathfrak{t}_{\mathbb{Z}}$*, and multiplication rule*

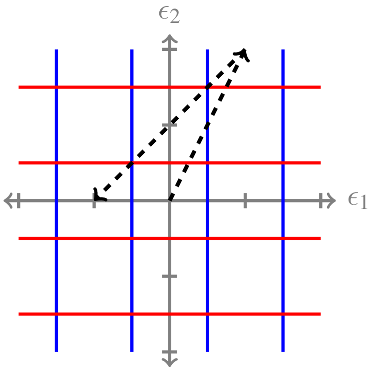
$$r_\nu r_\mu = \prod \varphi_i^{\rho_i(\mu,\nu)} r_{\mu+\nu}$$

where $\rho_i(\mu,\nu)$ *is the number of hyperplanes* $\varphi_i(X) \in \mathbb{Z} + 1/2$ *crossed twice by the path* $0 \to \mu \to \mu + \nu$.

If $G = \mathbb{C}^*$, and $V = \mathbb{C}^2$ with the scalar action, then $e = r_1, f = r_{-1}$ and $h/2 = \varphi_1 = \varphi_2$, the weight 1, then we get the relation $ef = -h^2/4$ again, and thus get the functions on the $\mathfrak{sl}_2$ nilcone.

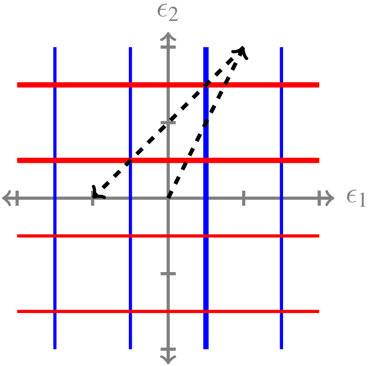
For example:

$$r_{(-2,-2)}r_{(1,2)} = \varphi_1\varphi_2\varphi_3^2\varphi_4^2 \cdot r_{(-1,0)}$$



For example:

$$r_{(-2,-2)}r_{(1,2)} = \epsilon_1^2\epsilon_2^4 \cdot r_{(-1,0)}$$



The quantized Coulomb branch has a very similar presentation; the only difference is that we pay attention to where the hyperplanes are. We fix scalars $m_1, \dots, m_d$. These are the **flavors**.

Theorem

$A_{\mathbf{c}}$ is a $\mathrm{Sym}(\mathfrak{t}^*)$ algebra with free basis r_ν for $\nu \in \mathfrak{t}_{\mathbb{Z}}$, and multiplication rules

$$r_\nu r_\mu = r_{\mu+\nu} \prod_{n \in D_{\nu,\mu}} (\varphi_i + m_i + n)$$

$$(\varphi - \langle \varphi, \nu \rangle) r_\nu = r_\nu \varphi$$

where $D(\mu, \nu)$ is the set of values of φ_i on hyperplanes $\varphi_i(X) \in \mathbb{Z} + 1/2$ crossed twice by the path $0 \to \mu \to \mu + \nu$.

If you shift $m_i \mapsto m_i + \langle \varphi_i, \nu \rangle$ for some $\nu \in \mathfrak{t}$, the result will be isomorphic.

Let $G = \mathbb{C}^*$, and $V = \mathbb{C}$, and we let

$$x = r_1 \qquad \partial = r_{-1} \qquad H = \varphi + m_1.$$

Proposition

The quantum Coulomb branch for this representation is W with isomorphism given by the formulas above.

Similarly, let $G = \mathbb{C}^*$, and $V = \mathbb{C}$, and we let

$$E = r_1 \qquad F = r_{-1} \qquad \frac{1}{2}H = \varphi.$$

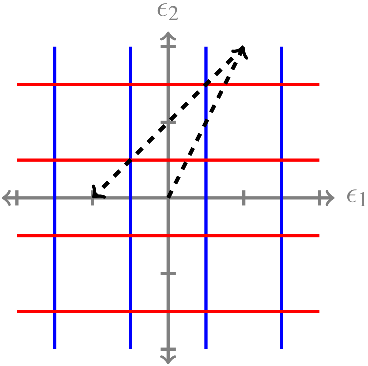
Proposition

The quantum Coulomb branch with $m_1 = \frac{1}{2}(\alpha + 1)$ *and* $m_2 = -\frac{1}{2}(\alpha + 1)$ *is* $U(\mathfrak{sl}_2)_\alpha$ *with isomorphism given by the formulas above.* See $\mathfrak{sl}_2$

As noted, if we shift $m_i \mapsto m_i + p$ for any p, we get an isomorphic algebra, so we can always just take $\alpha = m_1 - m_2 - 1$.

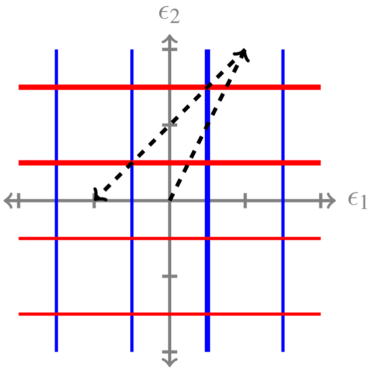
For example if $m_1 = m_3 = 1/2, m_2 = m_4 = -1/2$, then we have

$$r_{(-2,-2)}r_{(1,2)} = r_{(-1,0)}\varphi_1(\varphi_2 - 1)\varphi_3(\varphi_3 - 1)(\varphi_4 - 1)(\varphi_4 - 2)$$



For example if $m_1 = m_3 = 1/2, m_2 = m_4 = -1/2$, then we have

$$r_{(-2,-2)}r_{(1,2)} = r_{(-1,0)}\epsilon_1(\epsilon_1 - 1)\epsilon_2(\epsilon_2 - 1)^2(\epsilon_2 - 2)$$



This presentation is particularly well suited to understanding the action of this algebra on weight modules.

Definition

An A module is called a **weight module** *if* $\mathrm{Sym}(\mathfrak{t}^*)$ *acts locally finitely, with finite dimensional (generalized) eigenspaces.*

For $\in \mathfrak{t}$, the (generalized) ν-weight space for A is

$$M_\nu = \{m \in M \mid \text{for all } \mu \in \mathfrak{t}^*, \text{ we have } (\mu - \langle \nu, \mu \rangle)^N m = 0 \text{ for } N \gg 0\}.$$

Proposition

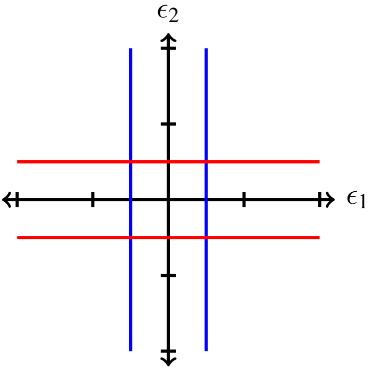
The operator $\mu - \beta$ for $\mu \in \mathfrak{t}^*$ and $\beta \in \mathbb{C}$ is invertible on M_ν if and only if $\langle \nu, \mu \rangle \neq \beta$.

Now consider the (finite) arrangement $\mathcal{A}$ where you only keep hyperplanes of the form $\varphi_i = -m_i$. Define an equivalence relation on $\mathfrak{t}_{\mathbb{Z}}$ by $\nu \sim \nu'$ if they lie in the same chamber of $\mathcal{A}$.

Theorem (Musson-van der Bergh)

The simple weight modules with all weights in $\nu' \in \mathfrak{t}_{\mathbb{Z}}$ are classified up to isomorphism by the chambers in $\mathcal{A}$ which contain elements of $\mathfrak{t}_{\mathbb{Z}}$, i.e. by equivalence classes. The weight spaces all have dimension 1.

In our running example, the hyperplane arrangement $\mathcal{A}$ is:



so there are 9 different projectives.

If instead $m_1 = \pi$, one of these hyperplanes will vanish, and there will only be 6.

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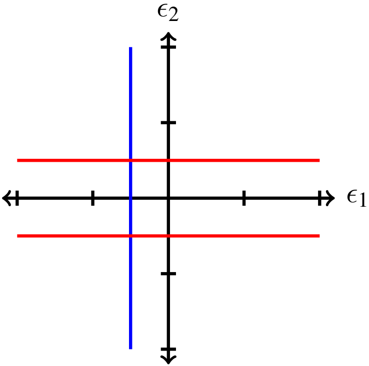

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If instead $m_1 = \pi$, one of these hyperplanes will vanish, and there will only be 6.

This leads to a presentation of the endomorphisms of these projectives:

Theorem (Musson-van der Bergh)

The category of weight modules with weights in $\nu' \in \mathfrak{t}_{\mathbb{Z}}$ is equivalent to the category of modules over the $\mathrm{Sym}(\mathfrak{t}^*)$-algebra Y generated by $r(C, C')$ for C, C' chambers in $\mathcal{A}$, with the multiplication rule

$$r(C, C')r(C', C'') = r(C, C'') \prod_i \varphi_i^{\rho_i(C,C',C'')}.$$

where $\rho_i(C, C', C'')$ is the number of hyperplanes labeled by φ_i crossed twice by the path $C \to C' \to C''$.

You can check this matches the presentation from before using E, H', F'. See $\mathfrak{sl}_2$

Let G be a (nonabelian) group acting on V with T its maximal torus.

Usually, we can think of the non-abelian case as abelian + Weyl group, but with a few extra kinks.

For the Coulomb branch, we have to account for the fact that the affine Grassmannian of T is a discrete set of points in the affine Grassmanian of G; we have to incorporate the tangent spaces to these points.

Define the Demazure operator $\partial_\alpha \colon \operatorname{Sym}(\mathfrak{t}^*) \to \operatorname{Sym}(\mathfrak{t}^*)$ by

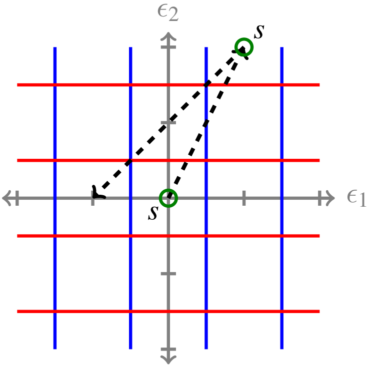
$$\partial_\alpha(f) = \frac{s_\alpha f - f}{\alpha}.$$

To construct the non-abelian Coulomb branch, take the picture from the abelian case, and add in the hyperplanes $\alpha \in \mathbb{Z}$. We'll now change our running example to $GL(2)$ acting on $\mathbb{C}^2 \oplus \mathbb{C}^2$:

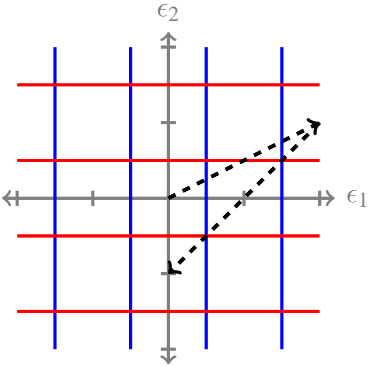
The basis vectors of our algebra over now correspond not just to paths but paths precomposed with an element of the Weyl group. You can think of the relations as concatenating and then a straightening process:

- Paths and $w \in W$ commute by the usual action.
- Undoing two crossings of a weight hyperplane multiplies by φ_i and two crossings of a root hyperplane gives 0.
- When a polynomial commutes past a root hyperplane, same rule for commuting a polynomial past a Demazure operator.
- Sliding the path through an intersection of a root and a weight hyperplane has a correction term.

$$r_{(-2,-2)} s r_{(1,2)} s = r_{(-2,-2)} r_{(2,1)} = \varphi_1^2 \varphi_2^2 \varphi_3 \varphi_4 \cdot r_{(0,-1)}$$



$$r_{(-2,-2)}sr_{(1,2)}s = r_{(-2,-2)}r_{(2,1)} = \epsilon_1^4\epsilon_2^1 \cdot r_{(0,-1)}$$



The resulting algebra $\mathsf{E} \supset \mathbb{C}[\mathfrak{M}_{\mathrm{ab}}]$ is now noncommutative.

Definition

The coordinate ring $\mathbb{C}[\mathfrak{M}] := Z(\mathsf{E}) \cong \mathsf{eEe}$ *of the Coulomb branch is the centralizer of the idempotent* e *projecting to* $\mathrm{Sym}(\mathfrak{t}^*)^W$.

As in the abelian case, we can construct a quantization $E_{\mathbf{c}} \supset A_{\mathbf{c},\mathrm{ab}}$ where $\mathbf{c}$ is compatible with G.

This comes from paying attention to where the hyperplane is, and shifting the weights. $E_{\mathbf{c}}$ contains a "projection" idempotent e.

Theorem

The algebra $A_{\mathbf{c}} = eE_{\mathbf{c}}e$ *is a quantization of the Coulomb branch.*

Examples:

- If $G = GL(n)$ and $V = (\mathbb{C}^n)^{\oplus \ell} \oplus M_{n\times n}(\mathbb{C})$, then $A_{\mathbf{c}}$ is the spherical rational Cherednik algebra for $G(\ell, 1, n)$. (Braverman-Etingof)
- If Γ is a Dynkin quiver w/ dimension vectors d_i, w_i, and $G = \prod_i GL(i)$ and

$$V = \bigoplus_{i\to j} \operatorname{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}) \oplus \bigoplus_i (\mathbb{C}^{d_i})^{\oplus w_i},$$

then $A_{\mathbf{c}}$ is a truncated shifted Yangian for G for the highest weight $\lambda = \sum w_i \omega_i$ and weight space for $\mu = \lambda - \sum d_i \alpha_i$. (Braverman-Finkelberg-Kamnitzer-Kodera-Nakajima-W.-Weekes)

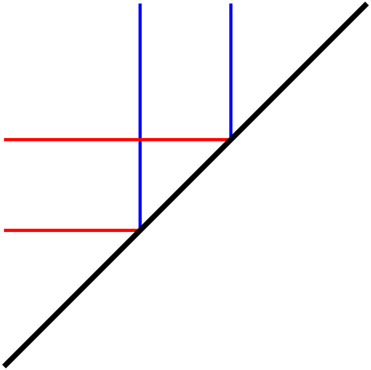
Rather than working with $eE_{\mathbf{c}}e$, I'd rather stick with $E_{\mathbf{c}}$. This will be Morita equivalent via the functor $M \mapsto eM$.

One reason this is better is that $A_{\mathbf{c},\mathrm{ab}} \subset E_{\mathbf{c}}$. In particular, I still have a copy of $\mathrm{Sym}(\mathfrak{t}^*)$ to serve as my "torus."

Definition

We call a $E_{\mathbf{c}}$-module M a **weight module**, *if it is a weight module restricted to $A_{\mathbf{c},\mathrm{ab}}$. Similarly, the (generalized) weight space M_ν is as before.*

Since we have the Weyl group action, the weight spaces in $\mathfrak{t}_{\mathbb{Z}}$ are isomorphic if they lie in the same chamber of $\mathcal{A}$ up to the action of the Weyl group. Thus we only need one Weyl chamber:



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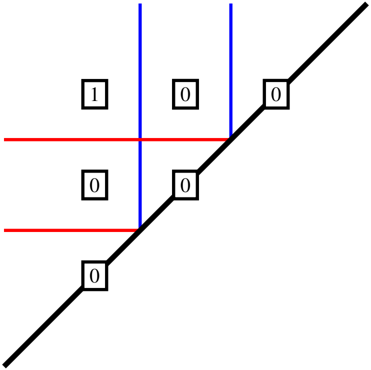

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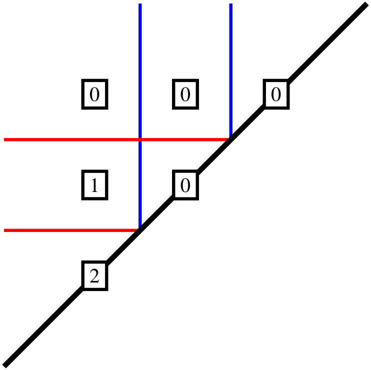

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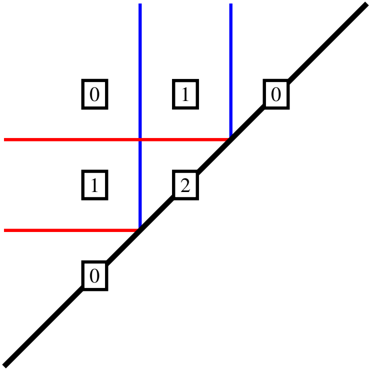

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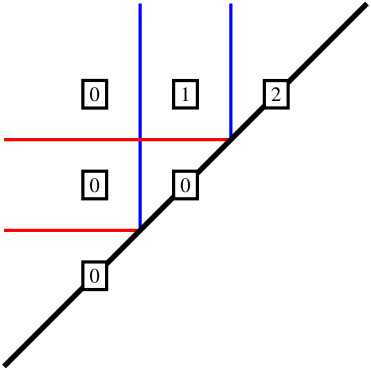


The weight multiplicities in simples are much more complicated now though.

We can run the Musson-van der Bergh argument again in the non-abelian case. The result Y is:

- We have operators $e(C), r(C, C')$ and $\mathrm{Sym}(\mathfrak{t}^*)$ for chambers of the hyperplane arrangement $\mathcal{A}$ discussed before in the positive Weyl chamber (possibly for a Levi).
- When we hit the wall of the positive Weyl chamber defined by α, we can act by $\psi_\alpha(C)$, which commutes past polynomials like a Demazure operator.

Theorem (W.)

The generalized representations of $E_{\mathbf{c}}$ with weights in a fixed coset $\mathfrak{t}_{\mathbb{Z}} + \nu$ are equivalent to the representations of the algebra Y, sending each weight space V_μ to the image of the idempotent $e(C)$ for the corresponding chamber.

In the quiver case, this algebra is already well-known (to me, at least).

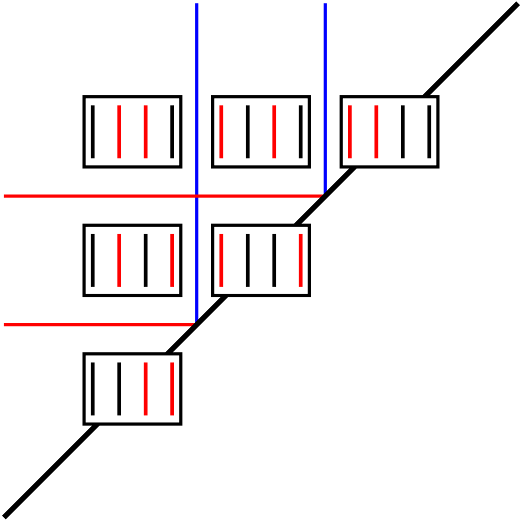
You represent a chamber by using the coordinates in $\mathfrak{t} \subset \oplus_i \mathfrak{gl}(d_i)$ as positions of black dots on the line, with labels corresponding to the factor. The values of m_i are represented by red dots.

You change your chamber when you pass a black dot past a red, or two black dots with adjacent labels. You cross a root hyperplane when two points with the same label pass.

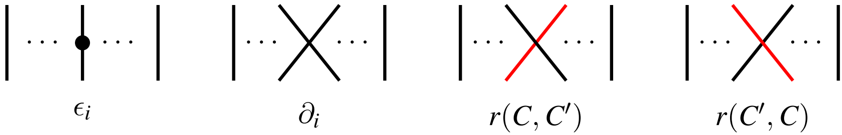
Theorem

The algebra which appears is a reduced weighted KLR algebra for Γ depending on the choice of weights m_i.

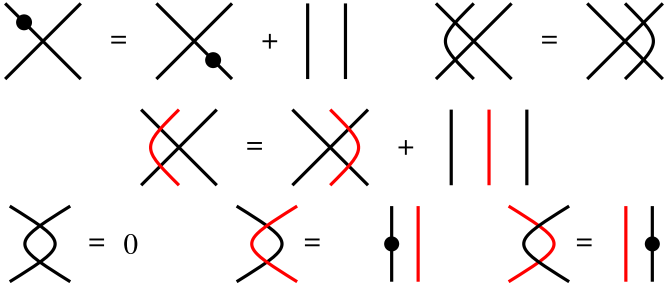
This is a "symplectic dual" appearance of KLR to the usual way of getting it from representations of quivers.



generators:



relations:



The Coulomb branches in this case have a lot of identities:

Proposition (BFKKNWW)

The Coulomb branch for a Dynkin quiver is a slice to Gr^{μ} *in* $\overline{\mathrm{Gr}}^{\lambda}$ *in the affine Grassmannian of that Dynkin type. The Coulomb quantization is a truncated shifted Yangian.*

Corollary

The tensor product categorifications for all tensor products of simples over $\mathfrak{g}$ *of type ADE can be realized as category* $\mathscr{O}$ *over a truncated shifted Yangian.*

Witten proposed a construction of knot homology for arbitrary representations using gauge theory, by considering Morse homology of solutions to certain PDEs with boundary conditions. This is manifestly topologically invariant but not very practical.

However, dimensional reduction suggests we can calculate it by looking at A-branes in these Coulomb branches for quiver gauge theories.

I don't know much about A-branes, but I do know that they are roughly the same thing as modules over quantizations, and now we all know something about those.

So, Witten's work suggests one should able to construct a homological knot invariant using functors between modules over these quantizations.

Given a horizontal slice of a labeled knot, we let Λ be the sum of weights on the strands, and let

$$\Lambda = \sum w_i \omega_i = \sum v_i \alpha_i.$$

Theorem

For each tangle T, there is a complex of bimodules between Coulomb quantizations for the dimension vectors defined above (for integral parameters depending on the order when reading the slice), which are functorial in this tangle.

The action of these on the category of finite dimensional modules over the Coulomb branch categorifies the actions of tangles on the invariants of tensor product representations, and provide categorifications of the Reshetikhin-Turaev invariants.

Thanks.