Coulomb branches and cylindrical KLRW algebras

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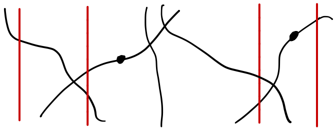
Related recent talks:

- More on GT modules for gl_n: youtu.be/tWyFM-Fbcc0
- More on the physics perspective: youtu.be/CfyNLZeP5iU and www.fields.utoronto.ca/video-archive/static/2019/11/2541-21802/mergedvideo.ogv

For many years, my research has had two related but different looking tracks:

geometric and **diagrammatic**

- homology
- perverse sheaves
- quantum coherent sheaves



How are these things related?

One connection: let Γ be a quiver, and $\mathbf{v}\colon I = \mathcal{V}(\Gamma) \to \mathbb{Z}_{\geq 0}$ a dimension vector. Let

$$N_{\mathbf{v}} = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \qquad G_{\mathbf{v}} = \prod_i GL(\mathbb{C}^{v_i})$$

The quotient $Y_{\mathbf{v}} = N_{\mathbf{v}}/G_{\mathbf{v}}$ is the moduli space of quiver representations of dimension $\mathbf{v}$.

Consider a word $\mathbf{i} = (i_1, \dots, i_n) \in I^n$ where $i \in I$ appears v_i times. We say that a homogeneous complete flag F_k on $\bigoplus_{i\in I} \mathbb{C}^{v_i}$ has type $\mathbf{i}$ if

$$\dim(F_k \cap \mathbb{C}^{v_j}) = \dim(F_{k-1} \cap \mathbb{C}^{v_j}) + \delta_{j,i_k}.$$

Let $X_{\mathbf{i}}$ be the moduli space of quiver representations equipped with a flag of subrepresentations of type $\mathbf{i}$.

Basic geometric object to consider:

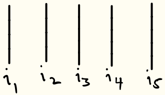
$$R_{\mathbf{v}} = \bigoplus_{\mathbf{i},\mathbf{j}} H_*^{BM}(X_{\mathbf{i}} \times_{Y_{\mathbf{v}}} X_{\mathbf{j}}) \cong \mathrm{Ext}^*\Big(\bigoplus_{\mathbf{i}} \pi_* \mathbb{C}_{X_{\mathbf{i}}}\Big)$$

as an algebra under convolution.

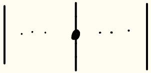
Theorem (Varagnolo-Vasserot, Rouquier)

The algebra $R_{\mathbf{v}}$ is generated by the homology classes:

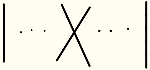
- *the diagonal in $X_{\mathbf{i}} \times_{Y_{\mathbf{v}}} X_{\mathbf{i}}$*



- *the 1st Chern class of the tautological bundle*

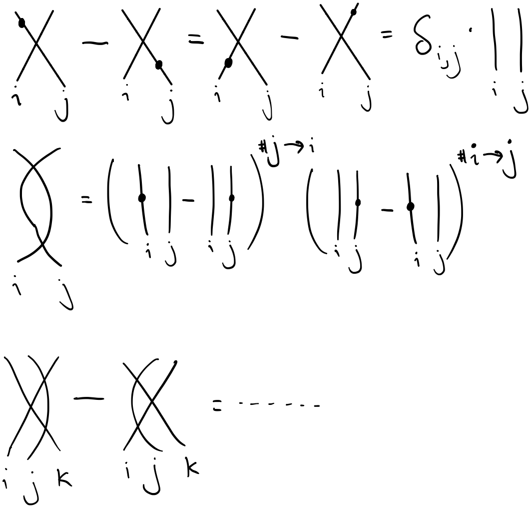


- *push-pull from a partial flag version*



modulo the relations on the next page:

KLR relations:



To generalize KLR algebra, need to give more geometric definition:

Consider a generic cocharacter $\xi\colon \mathbb{C}^* \to G$; we have a resulting complete flag $\{F_w\}$ of some type $\mathbf{i}$ given by the sum of vectors of weight $\leq w$ for each w.

Let $N_{\mathbf{i}}^-$ be the elements of $N_{\mathbf{v}}$ of negative weight under ξ. Let $P_{\mathbf{i}}^- \subset G_{\mathbf{v}}$ be the subgroup preserving the flag $F_\bullet$.

Proposition

We have an isomorphism $X_{\mathbf{i}} \cong N_{\mathbf{i}}^-/P_{\mathbf{i}}^-$.

Can do this for **any** representation N and group G.

Important twist: can consider the different $\xi : \mathbb{C}^* \to \mathrm{Norm}_{GL(V)}(G)$ which lift a fixed $\mathbb{C}^*$-action on $Y = N/G$.

Can similarly define $X_\xi = N_\xi^- / P_\xi^-$, and consider

$$R = \bigoplus_{\xi,\xi'} H_*^{BM}(X_\xi \times_Y X_{\xi'}) \cong \mathrm{Ext}^*(\bigoplus_{\xi} \pi_* \mathbb{C}_{X_\xi}).$$

Of course, there are infinitely many ξ, but only finitely many N_ξ^- up to conjugacy.

Theorem (Sauter, W.)

The algebra R always has a "KLR-type" presentation.

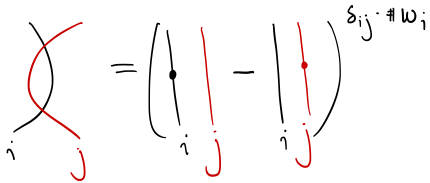
To obtain the algebras of ultimate interest to us, we have to add w_i copies of the representation $\mathbb{C}^{v_i}$ to $N_{\mathbf{v}}$ (i.e. $\mathrm{Hom}(\mathbb{C}^{w_i}, \mathbb{C}^{v_i})$). This is sometimes called "framing."

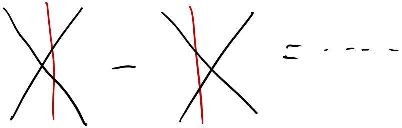
Moduli spaces of framed quiver representations are closely related to Nakajima quiver varieties.

We'll want to choose $\varphi\colon \mathbb{C}^* \to \prod GL(\mathbb{C}^{w_i}) \subset \mathrm{Aut}_G(N^{\mathbf{w}}_{\mathbf{v}})$. This puts an order on the basis vectors of the $\mathbb{C}^{w_i}$'s, which we can record as a word in I with w_i copies of i.

A choice of ξ corresponds to interleaving this with a word in I containing v_i copies of i.

The corresponding KLRW algebra has a very similar presentation. The red strands can never cross (since φ is fixed), and it's optional whether to allow dots on them. Relations:





I was interested in these algebras to construct categorifications of tensor products and knot invariants, and their connections to quiver varieties.

I'll say more about this later, but let me just mention that the bimodules that correspond to braiding can be gotten by taking Ext between pushforwards for different φ's.

But then I learned that there was a quite different lens to view them through: Coulomb branches.

Now affinize everything:

Taylor series $\mathsf{C} = \mathbb{C}[[t]]$	$\mathsf{G} = G[[t]]$	$\mathsf{N} = N[[t]]$
Laurent series $\mathcal{C} = \mathbb{C}((t))$	$\mathcal{G} = G((t))$	$\mathcal{N} = N((t))$

Relevant spaces:

$$\mathsf{Y} = \mathsf{N}/\mathsf{G} = \mathrm{Map}(D = \operatorname{Spec}\mathsf{C} \to N/G)$$

$$\mathcal{Y} = \mathcal{N}/\mathcal{G} = \mathrm{Map}(D^* = \operatorname{Spec}\mathcal{C} \to N/G)$$

These can be interpreted as spaces of principal G bundles with a section of the associated N-bundle on D and D^*.

Thus, the fiber product $\mathsf{Y} \times_{\mathcal{Y}} \mathsf{Y}$ is the space of such bundles on the "raviolo" gluing two copies of D along D^*.

Previous experience tells us it would be fun to consider

$$A = H_*^{BM}(\mathsf{Y} \times_{\mathcal{Y}} \mathsf{Y}).$$

Using factorization arguments, we can see that A is a commutative $\mathbb{C}$-algebra of finite type.

Definition

The **Coulomb branch** *is the spectrum* $\mathfrak{M} = \operatorname{Spec} A$.

This definition has some motivation in 3d QFT (it's the local operators in a topological twist of a gauge theory), but it's also recognizable as an affine version of our construction of KLR algebras, so it should have a KLR type presentation.

In the case of $N = N_{\mathbf{v}}$ and $G = G_{\mathbf{v}}$ from before,

- $\mathcal{N}/\mathcal{G}$ is the moduli of quiver representations over the field $\mathcal{C}$.
- N/G is the moduli of such quiver representations with a choice of lattice $\Lambda_i \cong \mathsf{C}^{v_i} \subset \mathcal{C}^{v_i}$ that gives a subrepresentation.

But our KLR presentation comes from being able to switch consecutive spaces in a flag, so we want flags, not lattices.

Definition

An **affine flag** *in $\mathcal{C}^m$ is a sequence of a lattices $\mathsf{F}_k \subset \mathcal{C}^m$ for $k \in \mathbb{Z}$ such that*

$$\cdots \subset \mathsf{F}_k \subset \mathsf{F}_k \subset \mathsf{F}_{k+1} \subset \cdots \qquad t\mathsf{F}_k = \mathsf{F}_{k-m}$$

Objects describing affine flags are periodic (periodic permutations for Schubert cells, etc.)

So, we can now let i be a **periodic word**: a map $\mathsf{i}\colon \mathbb{Z} \to I$ such that $\mathsf{i}_k = \mathsf{i}_{k+m}$ for all k for $m = \sum v_i$ such that any m consecutive entries contain v_i copies of i.

Any homogeneous affine flag $\mathsf{F}_\bullet \subset \bigoplus_{i\in I} \mathbb{C}^{v_i}$ has a periodic word as its type, defined by

$$\dim(\mathsf{F}_k \cap \mathbb{C}^{v_j} / \mathsf{F}_{k-1} \cap \mathbb{C}^{v_j}) = \delta_{j,\mathsf{i}_k}.$$

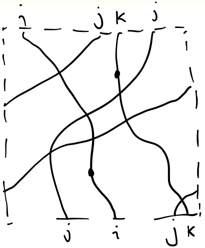
Let X_{i} be the moduli space of quiver reps over $\mathbb{C}$, together with a choice of affine flag of subreps of type i.

Theorem

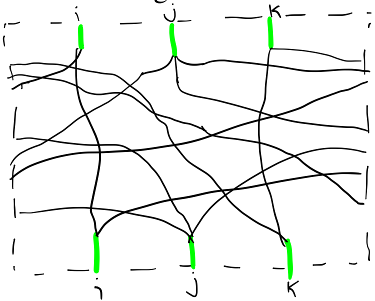
The convolution algebra

$$\mathsf{R} = \bigoplus_{\mathsf{i},\mathsf{j}} H_*^{BM}(\mathsf{X}_\mathsf{i} \times_{\mathcal{Y}} \mathsf{X}_\mathsf{j}) \cong \operatorname{Ext}^*(\bigoplus_{\mathsf{i}} \pi_* \mathbb{C}_{X_\mathsf{i}})$$

has a presentation by cylindrical KLR diagrams with local relations unchanged.



To get Coulomb branch A, need to integrate out finite flag variety back down to a single lattice. This corresponds to having a thick strand bringing together all with label i for each i at top and bottom of diagram (but general cKLR diagram in the middle).



This result generalizes to the KLRW case with addition of red strands.

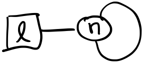
The talk thus far

Why am I interested in this construction? Mainly because lots of examples recover interesting varieties, always symplectic:

- my favorite: nilcone of $\mathfrak{gl}_n$

- more generally, Slodowy slices in type A

- symmetric power $\mathrm{Sym}^n(\mathbb{C}^2/\mathbb{Z}_\ell)$

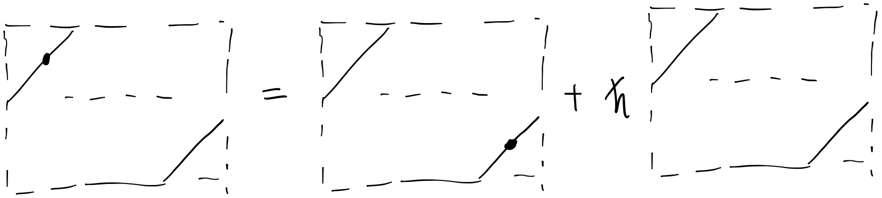


All of these varieties are Nakajima quiver varieties, but for potentially different quivers and dimension vectors. The quiver variety and Coulomb branch for a given quiver should be related by "3d mirror symmetry"/"symplectic duality."

The disk D has a $\mathbb{C}^*$ action by rotation (so the parameter t has weight 1). Combining this with the action on N/G via φ, we obtain compatible $\mathbb{C}^*$-actions on $\mathsf{Y}, \mathcal{Y}, \mathsf{X}_{\mathsf{i}}$.

$$A_\hbar = H_*^{BM,\mathbb{C}^*}(\mathsf{Y} \times_{\mathcal{Y}} \mathsf{Y}). \qquad \mathsf{R}_\hbar = \bigoplus_{\mathsf{i},\mathsf{j}} H_*^{BM,\mathbb{C}^*}(\mathsf{X}_{\mathsf{i}} \times_{\mathcal{Y}} \mathsf{X}_{\mathsf{j}})$$

Relations only change to account for the fact that $\mathsf{F}_k/\mathsf{F}_{k-1}$ and $\mathsf{F}_{k+m}/\mathsf{F}_{k+m-1}$ are isomorphic, but have different $\mathbb{C}^*$-weight.



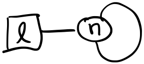
The parameters $H_*^{BM,\mathbb{C}^*}(\mathsf{Y}) \cong \mathbb{C}[\mathfrak{g},\hbar]^G$ give a maximal commutative subalgebra S of $A_\hbar$.

These result in well-known quantizations of these varieties; we get more familiar algebras if we consider the specialization A_1 setting $\hbar = 1$.

- my favorite: $U(\mathfrak{sl}_n)$ with S the Gelfand-Tsetlin subalgebra

- more generally, W-algebras in type A

- spherical Cherednik algebras for S_n or $G(\ell, 1, n)$, with S the subalgebra generated by the Dunkl-Opdam operators.



Definition

We call an A_1-module M **Gelfand-Tsetlin** *if the subalgebra S acts locally finitely on M, i.e. $\dim(S \cdot m) < \infty$ for all $m \in M$.*

Theorem

The category of Gelfand-Tsetlin A_1-modules with "integral weights" is equivalent to the category of weakly graded (gradeable after passing to associated graded) R-modules.

So, passing to GT A_1-modules undoes the affinization!

A few words on the proof:

- a GT module satisfies $M = \bigoplus_{\gamma \in \mathrm{MaxSpec}(S)} W_\gamma(M)$ for

$$W_\gamma(M) = \{m \in M \mid \mathfrak{m}_\gamma^N m = 0 \text{ for all } N \gg 0\}.$$

  Note that we can think of $\gamma \in \mathrm{MaxSpec}(S)$ as a conjugacy class of cocharacters $\mathbb{C}^* \to N_{GL(V)}(G)$. (integrality!)
- The category is thus controlled by natural transformations $W_\gamma \to W_{\gamma'}$.
- We have an isomorphism (by localization in equivariant homology)

$$\mathrm{Hom}(W_\gamma, W_{\gamma'}) \cong H_*^{BM}(X_{\gamma'} \times_Y X_\gamma)$$

- This gives the desired R-action.

Applications:

- Gives Koszul duality between categories $\mathcal{O}$ for Coulomb branches and quiver varieties/hyperkähler quotients attached to a given (G,N).
- First classification of GT modules for $\mathfrak{gl}_n$, and character formulae for them (Kamnitzer-Tingley-W.-Weekes-Yacobi, Silverthorne-W.).
- analogous classification for modules over Cherednik algebras of $G(\ell,p,n)$. (LePage-W.)
- Categorified knot invariants have two Koszul dual constructions; Coulomb side construction seems to be “A-branes on Hecke modifications” proposed by Witten.
- Connection to coherent sheaves next time!

Thanks for listening.