Coulomb branches and cylindrical KLRW algebras II

Ben Webster

University of Waterloo
Perimeter Institute for Mathematical Physics

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Related recent talks:

- **Part I of this talk:** [youtu.be/ZGO9wC-L0ho](https://youtu.be/ZGO9wC-L0ho)
- **More on GT modules for \( \mathfrak{gl}_n \):** [youtu.be/tWyFM-Fbcc0](https://youtu.be/tWyFM-Fbcc0)
- **a recent talk of Mina Aganagić on knot homology from the physics perspective:** [youtu.be/4TxtJzcRn0U](https://youtu.be/4TxtJzcRn0U)
The usual KLR (+KLRW variations) algebras appear as convolution algebras for moduli of quiver representations on finite dimensional $\mathbb{C}$-vector space with compatible flags:

$$R_v = \bigoplus_{i,j} H_{BM}^*(X_i \times_{Y_v} X_j) \cong \text{Ext}^*(\bigoplus_i \pi_* \mathbb{C}_{X_i})$$

The Coulomb branch is an affine version of this: switch to quiver representations over $\mathcal{C} = \mathbb{C}((t))$ and replace flags with $\mathcal{C} = \mathbb{C}[[t]]$-lattices:

$$A = H_{BM}^*(Y \times_y Y)$$

More KLR-like to consider affine flags instead of lattices:

$$R = \bigoplus_{i,j} H_{BM}^*(X_i \times_y X_j).$$
These affinized algebras have a similar geometric presentation, just incorporating wrapping around a cylinder/periodic permutations.
Why am I interested in this construction? Mainly because lots of examples recover interesting varieties, always symplectic:

- my favorite: nilcone of $\mathfrak{gl}_n$
  
  ![Diagram of nilcone of gl_n]

- more generally, Slodowy slices in type A
  
  ![Diagram of Slodowy slice in type A]

- symmetric power $\text{Sym}^n(\mathbb{C}^2/\mathbb{Z}_\ell)$
  
  ![Diagram of symmetric power]

All of these varieties are Nakajima quiver varieties, but for potentially different quivers and dimension vectors. The quiver variety and Coulomb branch for a given quiver should be related by “3d mirror symmetry”/“symplectic duality.”
What do we learn from thinking about these familiar varieties as Coulomb branches?

BFN construct resolutions of examples: Springer resolution and Hilbert scheme. This works for $(G, N)$ an affine type A quiver gauge theory, but not in most other cases.

These varieties also have non-commutative resolutions:

**Definition**

A noncommutative symplectic resolution of a symplectic singularity $\mathcal{M} = \text{Spec} A$ is a ring $R$ such that $A = eRe$ for some idempotent, and the functor $M \mapsto eM : R\text{-mod} \to A\text{-mod}$ “looks like” pushforward by a crepant resolution of singularities.

For a symplectic singularity, symplectic resolution=crepant resolution.
Theorem

Whenever a BFN resolution exists, the ring $R$ is a non-commutative symplectic resolution of $A$ and $D^b(R\text{-mod}) \cong D^b(\text{Coh}(\tilde{M}))$ for $\tilde{M}$ any symplectic resolution of the Coulomb branch $\text{Spec} A$.

- “Noncommutative Springer resolution” in type A is a special case; this gives such resolutions for all parabolic Slodowy slices in type A.
- In the case of Hilbert scheme (or more generally, affine type A) need to account for extra $\mathbb{C}^*$ acting by scaling on the loop (symplectic $\mathbb{C}^*$ on $\mathbb{C}^2$). Need to use “weighted” version of $R$. Recovers BFG resolution based on Cherednik algebra.
It’s a long road to get these resolutions, but it starts with a noncommutative deformation of $A$ and $R$.

The disk $D = \text{Spec } \mathbb{C}$ has a $\mathbb{C}^*$ action by rotation (so the parameter $t$ has weight 1). Combining this with the action on $N/G$ via $\varphi$, we obtain compatible $\mathbb{C}^*$-actions on $Y = N/G$, $\mathcal{Y} = N/\mathcal{G}$, $X_i$. We can thus consider deformed versions of these algebras, with $\hbar$ the equivariant parameter for $\mathbb{C}^*$:

$$A_{\hbar} = H_{BM, \mathbb{C}^*}^* (Y \times_\mathcal{Y} Y), \quad R_{\hbar} = \bigoplus_{i,j} H_{BM, \mathbb{C}^*}^* (X_i \times_\mathcal{Y} X_j)$$
Relations only change to account for the fact that $F_k/F_{k-1}$ and $F_{k+m}/F_{k+m-1}$ are isomorphic, but have different $\mathbb{C}^*$-weight.

We also, to account for the action of $\varphi$, have to deform red strand relations by adding $\theta_i$, weight of $\mathbb{C}^*$ on the corresponding basis vector in $\mathbb{C}^{w_i}$. 
The parameters $H^{BM}_{\ast, \mathbb{C}^*} (Y) \cong \mathbb{C}[g, \hbar]^G$ give a maximal commutative subalgebra $S$ of $A_{\hbar}$.

These result in well-known quantizations of these varieties; we get more familiar algebras if we consider the specialization $A_1$ setting $\hbar = 1$.

- **my favorite:** $U(\mathfrak{sl}_n)$ with $S$ the Gelfand-Tsetlin subalgebra

- more generally, $W$-algebras in type A

- spherical Cherednik algebras for $S_n$ or $G(\ell, 1, n)$, with $S$ the subalgebra generated by the Dunkl-Opdam operators.
Definition

*We call an $A_1$-module $M$ **Gelfand-Tsetlin** if the subalgebra $S$ acts locally finitely on $M$, i.e. $\dim(S \cdot m) < \infty$ for all $m \in M$.*

Theorem

*Over a base field $\mathbb{k}$ of characteristic 0, the category of Gelfand-Tsetlin $A_1$-modules with “integral weights” is equivalent to the category of weakly graded (gradeable after passing to associated graded) $R$-modules.*

So, passing to GT $A_1$-modules undoes the affinization!
A few words on the proof:

- a GT module satisfies $M = \bigoplus_{\gamma \in \text{MaxSpec}(S)} W_{\gamma}(M)$ for

\[ W_{\gamma}(M) = \{ m \in M \mid m^{N}_{\gamma} m = 0 \text{ for all } N \gg 0 \}. \]

Note that we can think of $\gamma \in \text{MaxSpec}(S)$ as a conjugacy class of cocharacters $\mathbb{C}^{*} \to N_{GL(V)}(G)$. (integrality!)

- The category is thus controlled by natural transformations $W_{\gamma} \to W_{\gamma'}$.

- We have an isomorphism (by localization in equivariant homology)

\[ \text{Hom}(W_{\gamma}, W_{\gamma'}) \cong H_{BM}^{\bullet}(X_{\gamma'} \times_{Y} X_{\gamma}) \]

- This gives the desired $R$-action.
Slight complication: if $\phi_k$’s encoding $\varphi$ aren’t generic, you might not be able to get every $i$ with an integral weight, so some simple $R$-modules might be lost.

This can be fixed by considering $R_1$ in place of $A_1$. Always have $A_1 = eR_1e$, but might not be a Morita equivalence.

For a fixed periodic word $i$, we can again consider weight functors

$$W_{i,a}(M) = \{m \in e(i)M | (y_k - a_k)^N m = 0 \text{ for all } N \gg 0\}.$$
In this context, there’s a reasonable algebraic proof.

- If $a_k \neq a_{k+1}$, then acts invertibly, and the functor $W_{i,a}$ is unchanged by the permutation $(k, k + 1)$.

- If we extend by $a_{k+m} = a_k - 1$, then pulling one strand around the cylinder shows that $W_{i,a}$ is unchanged by shifts of the sequences $i$ and $a$.

- For a red strand, if $a_i \neq \phi_j$, we can also do a swap.

Combining all these observations, we can reduce to the cases where all $a_i = \theta_j = 0$, and use inclusion $R \to \mathbb{R}$.
Applications:

- Gives Koszul duality between categories $\mathcal{O}$ for Coulomb branches and quiver varieties/hyperähler quotients attached to a given $(G, N)$.
- First classification of GT modules for $\mathfrak{gl}_n$, and character formulae for them (Kamnitzer-Tingley-W.-Weekes-Yacobi, Silverthorne-W.).
- Analogous classification for modules over Cherednik algebras of $G(\ell, p, n)$. (LePage-W.)
For any $\varphi, \varphi'$, there is a bimodule relating the two different quantizations where we wrap the red lines around the cylinder the appropriate number of times.

Because of the shift when a dot goes around, we need to change $\theta_i$.

Derived tensor product with this bimodule gives “twisting functors.”

This is a special case of a construction for all symplectic resolutions.
Theorem

Twisting functors give a (finite) braid group action on the categories of modules over different quantizations.

This can be upgraded to an action of tangles; the resulting link homology $\mathcal{D}_q(K)$ recovers my old work on categorified Reshetikhin-Turaev (in particular, Khovanov-Rozansky in type A).

In type A, can even upgrade this to an action of the foam category.

This seems to be a version of Witten’s prediction of a knot homology constructed with A-branes on a space of Hecke modifications.
For different people, this next part will have different motivations:

- You might want to understand coherent sheaves on a resolution of $\text{Spec } A$.
- You might be the kind of person who says “what if $\Bbbk$ had characteristic $p$”?
- You might have gone to some recent talks of Aganagić and gotten confused once cigars came up.

Interestingly, either way, you should do the same thing.
Over $\mathbb{F}_p$, you can try to analyze finite dimensional modules over $A_1$ by diagonalizing $S$ again. Again, let’s restrict to integral maximal ideals.

**Problem?**

If we wrap a strand around the cylinder $p$ times, the shift of the dot is trivial.

**Theorem**

Let $k = \mathbb{F}_p$. For generic $\varphi$ (and $p$ big enough), the category of finite dimensional $A_1$-modules with “integral weights” is equivalent to the category of finite dimensional weakly graded $R_0$-modules.

So, still affinized, but resolved now.

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Similar arguments to last time:

- **geometric proof**: localization to \( \mu_p \)-fixed points on \( Y \).
- **algebraic proof**: same calculations as last time, but now we have natural transformations \( W_{i,0} \rightarrow W_{j,0} \) as endomorphisms given by any diagram on the cylinder with \( i \) where all strands have winding number divisible by \( p \).
Why does this have anything to do with coherent sheaves?

Fancy char \( p \) stuff: there’s a quantum Frobenius map \( A_0 \to Z(A_1) \). This is actually the sections of a map of sheaves \( \mathcal{O}_{\tilde{M}} \to \mathcal{O}_{\tilde{M}} \) of structure sheaf to a localization of \( A_1 \) on any resolution \( \tilde{M} \).

Applying results of Bezrukavnikov and Kaledin, we can construct a very special vector bundle \( \mathcal{T} \) on \( \tilde{M} \) by “diagonalizing the action of \( S \subset A_1 \).”

A lift of this vector bundle also exists in char 0, so can forget about characteristic \( p \) story.
A tilting generator is a vector bundle $T$ such that $\text{Ext}^0(T, T) = 0$, and $\langle T \rangle = D^b\text{Coh}(\tilde{M})$.

**Theorem**

Assume that $G$ is a torus, or $(G, N)$ corresponds to an affine type A quiver gauge theory. The vector bundle $T$ is a tilting generator for $\tilde{M}$ and $\text{End}(T) = R$.

The fact that $R$ is a non-commutative resolution is a corollary.
While all commutative and non-commutative resolutions are derived equivalent, these equivalences are not unique. Instead, they generate an action of the affine braid group on this category; these descend from twisting functors in char $p$:

**Theorem**

*The twisting affine braid group action on $D^b(\mathcal{R} \text{-mod})$ is generated by cylindrical versions of $R$-matrix bimodules.*
In fact, this extends to an action of affine tangles, by affine versions of the cup and cap bimodules, and in type A to affine foams. This gives a link homology $\mathcal{D}_{coh}(K)$.

**Theorem**

The following link homologies are all the same:

- $\mathcal{D}_{coh}(K)$, constructed from the affine tangle action above.
- $\mathcal{D}_{q}(K)$, constructed from the tangle action on quantum coherent sheaves.
- the invariant constructed in *my older knot homology work (which matches Khovanov-Rozansky in type A).*
- Aganagić’s physical construction.
Thanks for listening.