Coulomb branches and cylindrical KLRW algebras II

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Related recent talks:

- Part I of this talk: `youtu.be/ZGO9wC-L0ho`
- More on GT modules for $\mathfrak{gl}_n$: `youtu.be/tWyFM-Fbcc0`
- More on the physics perspective: `youtu.be/CfyNLZeP5iU` and `www.fields.utoronto.ca/video-archive/static/2019/11/2541-21802/mergedvideo.ogv`
- a recent talk of Mina Aganagić on knot homology from the physics perspective: `youtu.be/4TxtJzcRn0U`

The usual KLR (+KLRW variations) algebras appear as convolution algebras for moduli of quiver representations on finite dimensional $\mathbb{C}$-vector space with compatible flags:

$$R_{\mathbf{v}} = \bigoplus_{\mathbf{i},\mathbf{j}} H_*^{BM}(X_{\mathbf{i}} \times_{Y_{\mathbf{v}}} X_{\mathbf{j}}) \cong \mathrm{Ext}^*(\bigoplus_{\mathbf{i}} \pi_* \mathbb{C}_{X_{\mathbf{i}}})$$

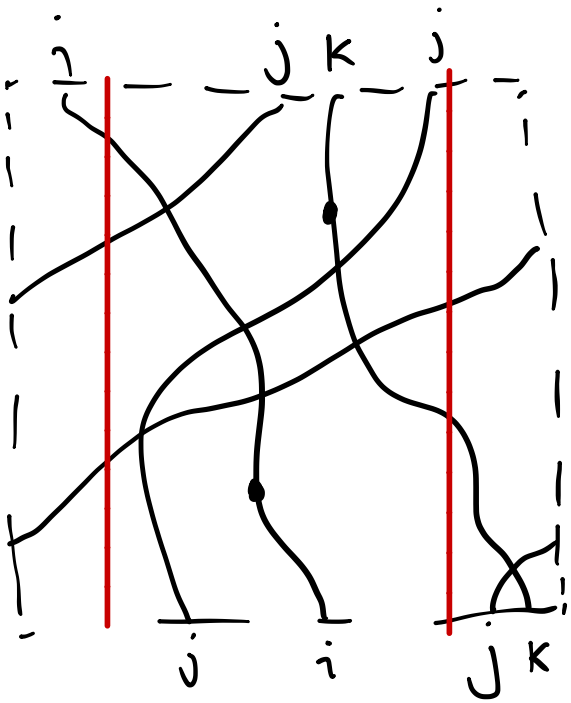
The Coulomb branch is an affine version of this: switch to quiver representations over $\mathcal{C} = \mathbb{C}((t))$ and replace flags with $\mathsf{C} = \mathbb{C}[[t]]$-lattices:

$$A = H_*^{BM}(\mathsf{Y} \times_{\mathcal{Y}} \mathsf{Y})$$

More KLR-like to consider affine flags instead of lattices:

$$\mathsf{R} = \bigoplus_{\mathsf{i},\mathsf{j}} H_*^{BM}(\mathsf{X}_{\mathsf{i}} \times_{\mathcal{Y}} \mathsf{X}_{\mathsf{j}}).$$

These affinized algebras have a similar geometric presentation, just incorporating wrapping around a cylinder/periodic permutations.

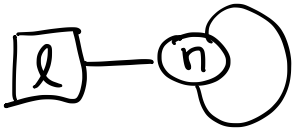


Why am I interested in this construction? Mainly because lots of examples recover interesting varieties, always symplectic:

- my favorite: nilcone of $\mathfrak{gl}_n$

- more generally, Slodowy slices in type A

- symmetric power $\mathrm{Sym}^n(\mathbb{C}^2/\mathbb{Z}_\ell)$



All of these varieties are Nakajima quiver varieties, but for potentially different quivers and dimension vectors. The quiver variety and Coulomb branch for a given quiver should be related by "3d mirror symmetry"/"symplectic duality."

What do we learn from thinking about these familiar varieties as Coulomb branches?

BFN construct resolutions of examples: Springer resolution and Hilbert scheme. This works for (G, N) an affine type A quiver gauge theory, but not in most other cases.

These varieties also have non-commutative resolutions:

Definition

A **noncommutative symplectic resolution** *of a symplectic singularity* $\mathfrak{M} = \operatorname{Spec} A$ *is a ring* R *such that* $A = eRe$ *for some idempotent, and the functor* $M \mapsto eM \colon R\text{-mod} \to A\text{-mod}$ *"looks like" pushforward by a crepant resolution of singularities.*

For a symplectic singularity, symplectic resolution=crepant resolution.

Theorem

Whenever a BFN resolution exists, the ring R *is a non-commutative symplectic resolution of* A *and* $D^b(\mathsf{R}\text{-mod}) \cong D^b(\mathsf{Coh}(\tilde{\mathfrak{M}}))$ *for* $\mathfrak{M}$ *any symplectic resolution of the Coulomb branch* $\operatorname{Spec} A$.

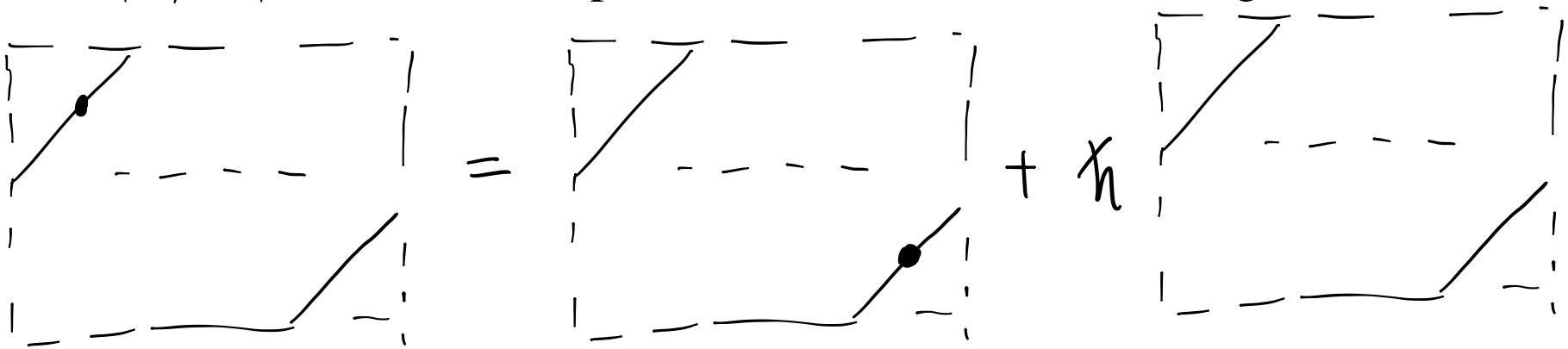
- "Noncommutative Springer resolution" in type A is a special case; this gives such resolutions for all parabolic Slodowy slices in type A.
- In the case of Hilbert scheme (or more generally, affine type A) need to account for extra $\mathbb{C}^*$ acting by scaling on the loop (symplectic $\mathbb{C}^*$ on $\mathbb{C}^2$). Need to use "weighted" version of R. Recovers BFG resolution based on Cherednik algebra.

It's a long road to get these resolutions, but it starts with a noncommutative deformation of A and R.

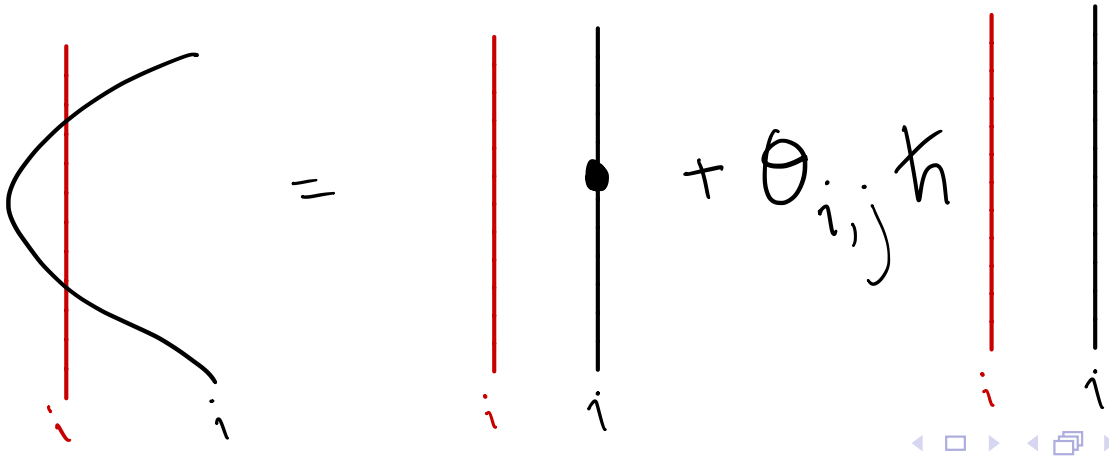
The disk $D = \operatorname{Spec} \mathsf{C}$ has a $\mathbb{C}^*$ action by rotation (so the parameter t has weight 1). Combining this with the action on N/G via φ, we obtain compatible $\mathbb{C}^*$-actions on $\mathsf{Y} = \mathsf{N}/\mathsf{G}, \mathcal{Y} = \mathcal{N}/\mathcal{G}, \mathsf{X}_\mathsf{i}$. We can thus consider deformed versions of these algebras, with $\hbar$ the equivariant parameter for $\mathbb{C}^*$:

$$A_\hbar = H_*^{BM,\mathbb{C}*}(\mathsf{Y} \times_{\mathcal{Y}} \mathsf{Y}). \qquad \mathsf{R}_\hbar = \bigoplus_{\mathsf{i},\mathsf{j}} H_*^{BM,\mathbb{C}*}(\mathsf{X}_\mathsf{i} \times_{\mathcal{Y}} \mathsf{X}_\mathsf{j})$$

Relations only change to account for the fact that $\mathsf{F}_k/\mathsf{F}_{k-1}$ and $\mathsf{F}_{k+m}/\mathsf{F}_{k+m-1}$ are isomorphic, but have different $\mathbb{C}^*$-weight.



We also, to account for the action of φ, have to deform red strand relations by adding θ_i, weight of $\mathbb{C}^*$ on the corresponding basis vector in $\mathbb{C}^{w_i}$.



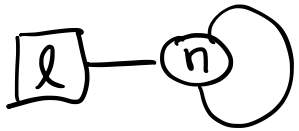
The parameters $H_*^{BM,\mathbb{C}^*}(\mathsf{Y}) \cong \mathbb{C}[\mathfrak{g}, \hbar]^G$ give a maximal commutative subalgebra S of $A_\hbar$.

These result in well-known quantizations of these varieties; we get more familiar algebras if we consider the specialization A_1 setting $\hbar = 1$.

- my favorite: $U(\mathfrak{sl}_n)$ with S the Gelfand-Tsetlin subalgebra

- more generally, W-algebras in type A

- spherical Cherednik algebras for S_n or $G(\ell, 1, n)$, with S the subalgebra generated by the Dunkl-Opdam operators.



Definition

We call an A_1-module M **Gelfand-Tsetlin** *if the subalgebra S acts locally finitely on M, i.e. $\dim(S \cdot m) < \infty$ for all $m \in M$.*

Theorem

Over a base field $\Bbbk$ of characteristic 0, the category of Gelfand-Tsetlin A_1-modules with "integral weights" is equivalent to the category of weakly graded (gradeable after passing to associated graded) R-modules.

So, passing to GT A_1-modules undoes the affinization!

A few words on the proof:

- a GT module satisfies $M = \bigoplus_{\gamma \in \mathrm{MaxSpec}(S)} W_\gamma(M)$ for

$$W_\gamma(M) = \{m \in M \mid \mathfrak{m}_\gamma^N m = 0 \text{ for all } N \gg 0\}.$$

  Note that we can think of $\gamma \in \mathrm{MaxSpec}(S)$ as a conjugacy class of cocharacters $\mathbb{C}^* \to N_{GL(V)}(G)$. (integrality!)
- The category is thus controlled by natural transformations $W_\gamma \to W_{\gamma'}$.
- We have an isomorphism (by localization in equivariant homology)

$$\mathrm{Hom}(W_\gamma, W_{\gamma'}) \cong H_*^{BM}(X_{\gamma'} \times_Y X_\gamma)$$

- This gives the desired R-action.

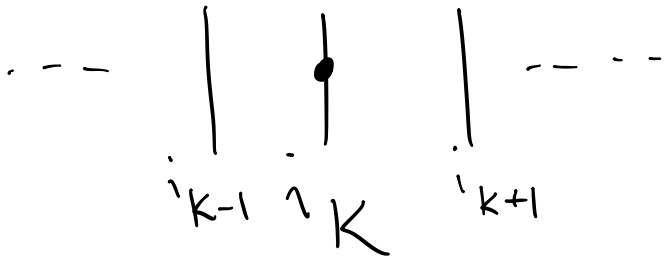
Slight complication: if ϕ_k's encoding φ aren't generic, you might not be able to get every $\mathbf{i}$ with an integral weight, so some simple R-modules might be lost.

This can be fixed by considering R_1 in place of A_1. Always have $A_1 = e\mathsf{R}_1 e$, but might not be a Morita equivalence.

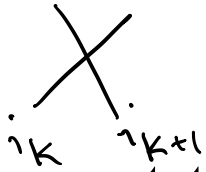
For a fixed periodic word i, we can again consider weight functors

$$W_{\mathsf{i},\mathbf{a}}(M) = \{m \in e(\mathsf{i})M | (y_k - a_k)^N m = 0 \text{ for all } N \gg 0\}.$$

$y_k =$



In this context, there's a reasonable algebraic proof.

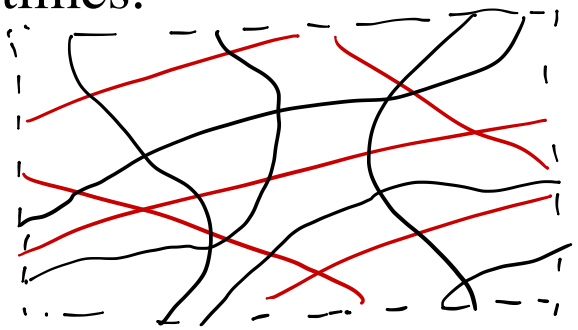
- If $a_k \neq a_{k+1}$, then  acts invertibly, and the functor $W_{\mathsf{i},\mathbf{a}}$ is unchanged by the permutation $(k, k+1)$.
- If we extend by $a_{k+m} = a_k - 1$, then pulling one strand around the cylinder shows that $W_{\mathsf{i},\mathbf{a}}$ is unchanged by shifts of the sequences i and $\mathbf{a}$.
- For a red strand, if $a_i \neq \phi_j$, we can also do a swap. 

Combining all these observations, we can reduce to the cases where all $a_i = \theta_j = 0$, and use inclusion $R \to \mathsf{R}$.

Applications:

- Gives Koszul duality between categories $\mathcal{O}$ for Coulomb branches and quiver varieties/hyperähler quotients attached to a given (G, N).
- First classification of GT modules for $\mathfrak{gl}_n$, and character formulae for them (Kamnitzer-Tingley-W.-Weekes-Yacobi, Silverthorne-W.).
- analogous classification for modules over Cherednik algebras of $G(\ell, p, n)$. (LePage-W.)

For any φ, φ', there is a bimodule relating the two different quantizations where we wrap the red lines around the cylinder the appropriate number of times.



Because of the shift when a dot goes around, we need to change θ_i.

Derived tensor product with this bimodule gives "twisting functors."

This is a special case of a construction for all symplectic resolutions.

Theorem

Twisting functors give a (finite) braid group action on the categories of modules over different quantizations.

This can be upgraded to an action of tangles; the resulting link homology $\mathscr{D}_q(K)$ recovers my old work on categorified Reshetikhin-Turaev (in particular, Khovanov-Rozansky in type A).

In type A, can even upgrade this to an action of the foam category.

This seems to be a version of Witten's prediction of a knot homology constructed with A-branes on a space of Hecke modifications.

For different people, this next part will have different motivations:

- You might want to understand coherent sheaves on a resolution of $\operatorname{Spec} A$.
- You might be the kind of person who says "what if $\Bbbk$ had characteristic p"?
- You might have gone to some recent talks of Aganagić and gotten confused once cigars came up.

Interestingly, either way, you should do the same thing.

Over $\mathbb{F}_p$, you can try to analyze finite dimensional modules over A_1 by diagonalizing S again. Again, let's restrict to integral maximal ideals.

Problem?

If we wrap a strand around the cylinder p times, the shift of the dot is trivial.

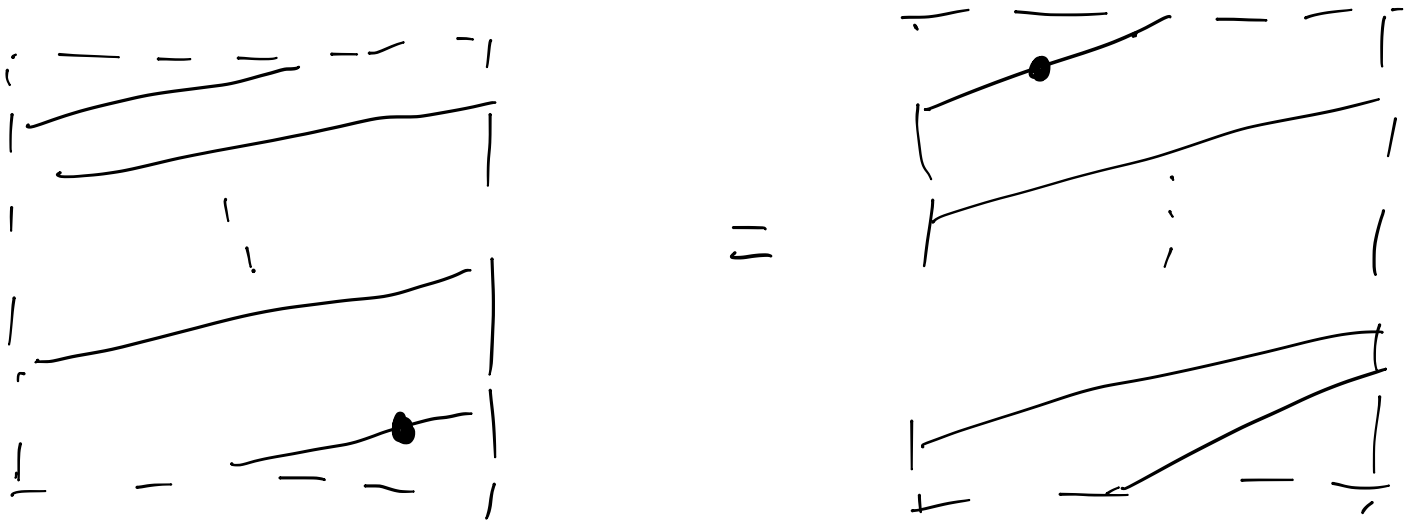
Theorem

Let $\Bbbk = \mathbb{F}_p$. For generic φ (and p big enough), the category of finite dimensional A_1-modules with "integral weights" is equivalent to the category of finite dimensional weakly graded R_0-modules.

So, still affinized, but resolved now.

Similar arguments to last time:

- geometric proof: localization to μ_p-fixed points on Y.
- algebraic proof: same calculations as last time, but now we have natural transformations $W_{\mathsf{i},0} \to W_{\mathsf{j},0}$ as endomorphisms given by any diagram on the cylinder with i where all strands have winding number divisible by p.



Why does this have anything to do with coherent sheaves?

Fancy char p stuff: there's a quantum Frobenius map $A_0 \to Z(A_1)$. This is actually the sections of a map of sheaves $\mathscr{O}_{\tilde{\mathfrak{M}}} \to \mathscr{Q}_{\tilde{\mathfrak{M}}}$ of structure sheaf to a localization of A_1 on any resolution $\tilde{\mathfrak{M}}$.

Applying results of Bezrukavnikov and Kaledin, we can construct a very special vector bundle $\mathcal{T}$ on $\tilde{\mathfrak{M}}$ by "diagonalizing the action of $S \subset A_1$."

A lift of this vector bundle also exists in char 0, so can forget about characteristic p story.

A **tilting generator** is a vector bundle T such that $\mathrm{Ext}^{>0}(T,T)=0$, and $\langle T\rangle = D^b\mathsf{Coh}(\tilde{\mathfrak{M}})$.

Theorem

Assume that G is a torus, or (G,N) corresponds to an affine type A quiver gauge theory. The vector bundle $\mathcal{T}$ is a tilting generator for $\tilde{\mathfrak{M}}$ and $\mathrm{End}(\mathcal{T})=\mathsf{R}$.

The fact that R is a non-commutative resolution is a corollary.

While all commutative and non-commutative resolutions are derived equivalent, these equivalences are not unique. Instead, they generate an action of the affine braid group on this category; these descend from twisting functors in char p:

Theorem

The twisting affine braid group action on $D^b(\mathsf{R}\text{-mod})$ is generated by cylindrical versions of R-matrix bimodules.

In fact, this extends to an action of affine tangles, by affine versions of the cup and cap bimodules, and in type A to affine foams. This gives a link homology $\mathscr{D}_{coh}(K)$.

Theorem

The following link homologies are all the same:

- $\mathscr{D}_{coh}(K)$, *constructed from the affine tangle action above.*
- $\mathscr{D}_q(K)$, *constructed from the tangle action on quantum coherent sheaves.*
- *the invariant constructed in my older knot homology work (which matches Khovanov-Rozansky in type A).*
- *Aganagić's physical construction.*

Thanks for listening.