Gelfand-Tsetlin theory and Coulomb branches

Let A be your favorite $\mathbb{K}$-algebra, and R a commutative subalgebra.

An A-module V is <u>R-Gelfand-Tsetlin</u> if $\dim_{\mathbb{K}} R\cdot v < \infty$ for all $v \in V$.

Let $\mathcal{W}_\lambda(V) = \{v \in V \mid m_\lambda^k \cdot v = 0 \text{ for } k \gg 0\}$ for $\lambda \in \operatorname{MaxSpec}(R)$.

<u>Lemma</u> V is R-G-T iff $V \cong \bigoplus_{\lambda \in \operatorname{MaxSpec}(R)} \mathcal{W}_\lambda(V)$

$A = U(\mathfrak{g})$ $R = U(\mathfrak{h})$ generalized weight modules

$A = U(\mathfrak{gl}_n)$ $R =$ Gelfand-Tsetlin subalgebra
$= \langle Z(U(\mathfrak{gl}_1)), Z(U(\mathfrak{gl}_2)), Z(U(\mathfrak{gl}_3)), \dots \rangle$

$A = \mathcal{A}(G,N)$ $R = H^*(B(G \times \mathbb{C}^*))$ Coulomb branch

Basic Q: can you classify simple G-T modules?

Once you have a 2-d Ext^1, this becomes wild.
More plausible Q: classify simples.

Drozd-Futorny-Ovsienko give a general answer.

<u>Def</u> We say (A,R) has the <u>Harish-Chandra property</u> if RaR is finitely generated as a L/R module $\forall a \in A$.

All examples I listed on the previous page had this property.

<u>Thm</u> The category of R-G-T modules is equivalent to the category of discrete modules over the category $\mathcal{C}$ with

objects: $\lambda \in \mathrm{MaxSpec}(R)$ $\mathrm{Hom}_{\mathcal{C}}(\lambda, \mu) = \lim A/(A m_\mu^k + m_\lambda^k A)$

If $X \subset \mathrm{MaxSpec}(R)$, then $\mathcal{C}_X\text{-mod} \cong \dfrac{R\text{-mod}_{GT}}{\{M \mid \mathcal{W}_\lambda(M) = 0 \ \forall \lambda \in X\}}$

In particular, $\{$simple R-G-T-modules w/ $\mathcal{W}_\lambda(S) \neq 0\}$

$\Updownarrow$

$\{$simple discrete modules over $\hat{A}_\lambda = \mathrm{Hom}_{\mathcal{C}}(\lambda,\lambda)\}$

This is a beautiful idea, but not very practical iff you can't figure out what the algebras $\hat{A}_\lambda$ and bimodules ${}_\lambda\hat{A}_\mu := \mathrm{Hom}_{\mathcal{C}}(\mu, \lambda)$ are.

Luckily, for Coulomb branches, there is a simple, general answer.

If we consider a Coulomb branch w/ $h=1$, then $\mathrm{MaxSpec}(R)$ = semi-simple elements $(X,1)\in\mathfrak{g}\times\mathbb{C}$ up to conjugacy.

Let $N_\lambda \subset N$ be the subspace where this action integrates.

Let $N_\lambda^- \subset N_\lambda$ be the subspace where this action has non-positive weights.

Let $G_\lambda \subset G$ be the Levi subgroup and $P_\lambda \subset G_\lambda$ the parabolic obtained by exponentiating $\mathfrak{g}_\lambda$ and $\mathfrak{p}_\lambda = \mathfrak{g}_\lambda^-$.

<u>Thm</u> (Nakajima, W.) If $A = \mathcal{A}(G,N)$, then

$$\hat{A}_\lambda = \hat{H}_*^{G_\lambda, BM}(X_\lambda) \quad \text{where}$$

$$X_\lambda = \{(g_1P_\lambda, g_2P_\lambda, n) \mid n \in g_1N_\lambda^- \cap g_2N_\lambda^-\} \subset (G/P_\lambda)^2 \times N_\lambda$$
$$= Y_\lambda \times_{N_\lambda} Y_\lambda \quad \text{for } Y_\lambda = \{(gP_\lambda, n) \mid n \in gN_\lambda^-\}$$

More generally, ${}_\lambda\hat{A}_\mu = 0$ if they don't lie in the same affine Weyl group orbit. If they do, then

$$G_\lambda = G_\mu \qquad N_\lambda = N_\mu$$

$${}_\lambda\hat{A}_\mu = \hat{H}_*^{G_\lambda, BM}(Y_\lambda \times_{N_\lambda} Y_\mu)$$

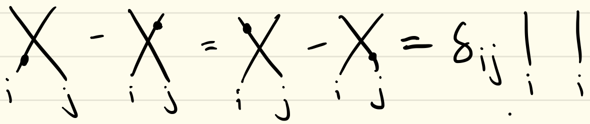
Important special case: $A = U(\mathfrak{gl}_n(\mathbb{K}))$ $R =$ G-T

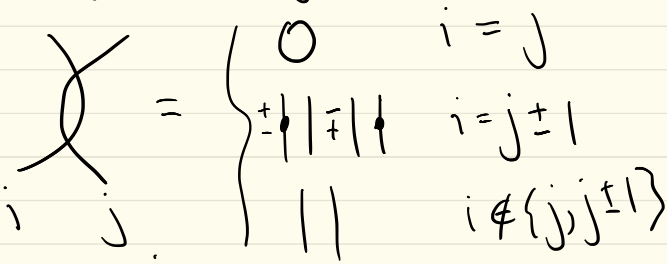
This is a special case of the Coulomb branch w/

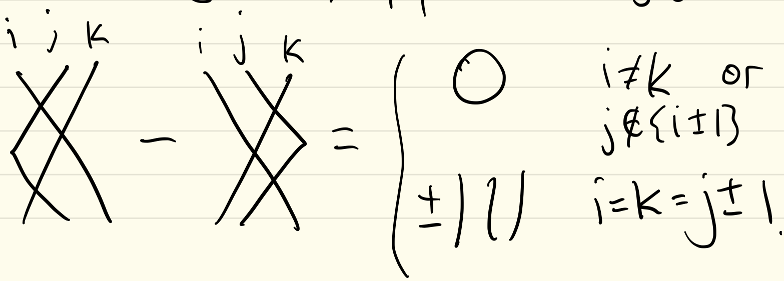
$$G = GL_1 \times GL_2 \times GL_3 \times \cdots \times GL_{n-1}$$
$$N = M_{1\times 2}(\mathbb{C}) \times M_{2\times 3}(\mathbb{C}) \times \cdots \times M_{n-1\times n}(\mathbb{C})$$

The resulting algebra that controls G-T modules w/ integral weights is <u>almost</u> the KLR algebra of type A_n, but I'm missing a factor of GL_n above.

<u>Reminder</u> type A_n KLR algebra is the formal span of string diagrams w/ dots and labels in $[1,n]$ modulo local relations of the form







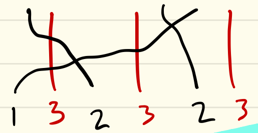
Assume $\mathrm{char}(\mathbb{k})=0$.

If we choose an integral weight λ of G (honest cocharacter), we can diagonalize, and get multisets $C_1, C_2, C_3, \dots C_{\mathbb{Z}}$ of size 1, 2, 3, etc. We also have a C_n corresponding to central character of $U(\mathfrak{gl}_n)$.

Putting these in order, we get a word in $[1,n]$ w/ k copies of k. Let $e(\lambda)$ be the corresponding idempotent in type A_n KLR algebra.

Let $\tilde{T} \subset KLR$ be subalgebra where strands w/ label n never change order. We draw these red.

for $\mathfrak{sl}_3 \to$



<u>Thm</u> (KTWWY) The category of G-T modules w/ integral weights and central character corresponding to regular C_n is equivalent to the category of $\tilde{T}$-mod with dots nilpotent.

With a bit of extra work this allows us to classify simple G-T modules for $\mathfrak{gl}_n$: they correspond to dual canonical basis vectors in $W \otimes_U W \otimes_U \cdots \otimes_U W$ where

$$U = U^-(\mathfrak{sl}_n) \qquad W = U F_n U \subset U^-(\mathfrak{sl}_{n+1})$$
$$\cong U \otimes \mathbb{C}^n$$

Another important example:

Rational Cherednik algebra of $G(\ell,1,n)$.

$$G = GL_n \qquad N = \mathfrak{gl}_n \oplus (\mathbb{C}^n)^{\oplus \ell}$$

$H^*(BGL_n)$ = symmetrized Dunkl-Opdam operators.

The usual parameter κ corresponds to scaling flavor on $\mathfrak{gl}_n$.

---

due to the scaling, we now need <u>weighted</u> KLR algebras.

If $\kappa \notin \mathbb{Q}$, N_λ / G_λ is moduli of reps for a linear quiver.

If $\kappa = \frac{a}{e}$, N_λ / G_λ is moduli of reps of e-cycles.

Thm (RSVV-L-W) {Simple modules in cat $\mathcal{O}$}

$\Updownarrow$

{dual canonical basis vectors in twisted Fock}

This generalizes to Dunkl-Opdam modules, for an enlarged Fock space

$G(\ell,p,n)$ - in progress (LePage - W.)

My real interest is the fact these same algebras $H_*^{BM;G}(Y_\lambda \times_N Y_\lambda)$ show up somewhere totally different in representation theory.

Consider the category of G-equivariant D-modules on N. Then

$$\boxed{H_*^{BM;G}(Y_\lambda \times_N Y_\mu) \cong \mathrm{Ext}_{N/G}(\pi_* \mathcal{O}_{Y_\lambda}, \pi_* \mathcal{O}_{Y_\mu})}$$

Thus, we have a version of Koszul duality between the category of G-T-modules and this part of $D(N/G)$.

We can make this more symmetric-looking if we pass to category $\mathcal{O}$.

Fix $\xi \in (\mathfrak{g}^*)^G$. This gives a grading element of $\mathfrak{t}(G,N)_{\mathbb{R}}$, i.e. an element whose adjoint action is semi-simple with real eigenvalues.

Def V is in category $\mathcal{O}$ if the generalized eigenspaces of ξ are finite dimensional, span V and the spectrum is bounded above (in real part).

Note that any module in category $\mathcal{O}$ is a G-T module, since ξ-generalized weight spaces are R-invariant. Thus, we have $\mathcal{O} \subset GT$, and we can try to understand $\mathcal{O}$ by picking out this subcategory.

$\boxed{\begin{array}{l}\text{\underline{Thm} (W) The category } \mathcal{O} \text{ for a Coulomb branch is the category of modules over } \mathcal{C}^{\xi}, \\ \text{the quotient of } \mathcal{C} \text{ by all objects such that } \max_{\lambda' \cong \lambda} \xi(\lambda') = \infty.\end{array}}$

For Gelfand-Tsetlin modules, this returns a known description of category $\mathcal{O}$ as a tensor product categorification. In the hypertoric case, this gives previous work of BLPW.

On the Higgs side, we can use ξ to define a GIT quotient of $G \circlearrowright T^*V \to \mathfrak{g}^*_{\xi}$

If you think about stacks, we have $\mathfrak{M}_\xi$ is an open subset of $T^*(V/G) = \mu^{-1}(0)/G$. Pullback of microlocal D-modules $\pi_* \mathcal{O}_{Y_\lambda}$ defines semi-simple DQ-modules S_λ.

<u>Thm</u> $S_\lambda = 0$ iff $\max_{\lambda' \cong \lambda} \xi(\lambda') = \infty$, and in "good cases"

$$\mathrm{Ext}^\bullet(S_\lambda, S_\mu) = \mathrm{Hom}_{\mathcal{C}^\xi}(\lambda, \mu).$$

and the objects S_λ contain all simples in geometric category $\mathcal{O}$ on $\mathfrak{M}_\xi$ as summands.

For these purposes, the hypertoric and quiver cases are good.

This is all a char 0 story, but it's hard to resist saying something about characteristic p.

The algebra $\hat{A}_\lambda$ is "close to" the skew group ring $\mathbb{K}[\mathfrak{t}] \# \hat{W}_\lambda$ where $\hat{W}_\lambda$ is the stabilizer of λ in the affine Weyl group.

Char 0: $\hat{W}_\lambda$ is a finite Coxeter group.
Char p: $\hat{W}_\lambda$ is an affine Coxeter group.

<u>Thm</u> The ring $\hat{A}_\lambda$ is close to a Coulomb branch; its the homology of a slightly different set of triples. (Extended category!)

In the case of $U(\mathfrak{gl}_n)$ this is a cylindrical version of the KLR^C algebra. This leads you down the road to tilting bundle fun, but a story for another time.