Research Description
Ben Webster

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1. Introduction

My research centers around investigating questions in representation theory, knot theory and combinatorics from a geometric perspective. My most important tool at the moment is certain categories, constructed using the techniques of geometric representation theory, which generalize the BGG category $\mathcal{O}$ of a semisimple Lie algebra. These categories are relevant to problems in physics, homological knot invariants, canonical basis theory, and the study of finite dimensional algebras.

1.1. **Categorification.**

One important perspective on these categories is as categorifications of objects already the interest to mathematicians.

It is now a well-established principle of mathematics that any set of interest to mathematicians has a good chance of being the set of objects of an even more interesting category, and that often natural structures on the level of sets (maps, group actions, etc.) are shadows of functors between said categories. The pretentious among us might be tempted to label this a principle of "categorification," upgrading from a 0-category (a set) to a 1-category, but at this point, it seems too obvious to the modern mathematician to require a name.

A fact which is a touch more subtle is that a similar principle exists for abelian groups or for vector spaces. Rather than look for a category whose objects are in bijection with elements of the group A, we should search for a category whose Grothendieck group is A. This approach can lead us to see that structures which look strange from the original perspective are in fact quite natural.

For example, Lusztig constructed the canonical basis of $U_q(\mathfrak{n})$ by constructing a categorification of this quantum group and considering the classes of simple objects in that categorification [Lus91], to construct a basis whose existence is not at all clear from an algebraic perspective.

Categorification is also natural from the perspective of extended (T)QFT. An extended n-dimensional TQFT associates an $n-d$-category to each d-manifold. Thus decategorification sends a n-dimensional TQFT to an $n-1$-dimensional. Thus the existence of the categorification of the structure arising in TQFT (such as representations of a quantum group) is evidence that this TQFT is in fact a shadow of a higher dimensional structure.

1.2. **Categories $\mathcal{O}$.** Lusztig's construction is a special case of a more general construction, which allows us to produce a large number of interesting categories, which can be thought of as analogues of category $\mathcal{O}$. While some of these have previously appeared in the literature, there seems to have been no systematic study of them as a class. My intent is to study these categories and the geometry of symplectic singularities from this perspective.

This construction takes as input a symplectic resolution of singularities $Y \to X$, and a $\mathbb{C}^*$-action on $\xi : \mathbb{C}^* \to \mathsf{Aut}(Y \to X)$ which acts with positive weight on the symplectic form ω_Y. The result is a category $\mathcal{O}_Y^\xi$.

This a generalization of a several categories already of interest to representation theorists. For example:

- If $Y = T^*(G/B)$ for a semi-simple Lie group G and Borel B, and $X = \mathcal{N}_\mathfrak{g}$, then this is a block of the original category $\mathcal{O}$ defined by Bernstein-Gelfand-Gelfand, by the localization theorem of Beilinson-Bernstein [BB81].
- If X is a slice to a nilpotent orbit, which carries an appropriate $\mathbb{C}^*$ action, we obtain category $\mathcal{O}$ for a finite W-algebra [Gin].
- If $Y = \mathsf{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma})$ and $X = \mathbb{C}^{2n}/(S_n \wr \Gamma)$, we obtain category $\mathcal{O}$ for the rational Cherednik algebra of $S_n \wr \Gamma$ by work of Ginzburg-Gordon-Stafford [GGS].

In addition to their representation theoretic interest, these categories are connected to a number of interesting problems:

- There seems for a connection between these categories and 3-dimensional, $N = 4$ supersymmetric field theories in physics. In particular, several singularities which are dual to each other under a mirror duality in physics (different from the mirror symmetry for 2-dimensional field theories usually studied by mathematicians) have associated categories $\mathcal{O}$ which are Koszul dual. Interestingly, these sorts of equivalences seem not to have yet been observed by physicists. However, we conjecture that this duality is a general phenomenon, which extends beyond these particular examples.
- The derived categories $D^b(\mathcal{O}_Y^\xi)$ carry an action by autoequivalences of the fundamental group of a certain hyperplane complement (which generalize Arkhipov's twisting functors on the BGG category $\mathcal{O}$). We expect this hyperplane complement is connected to the space of stability conditions for this derived category, and thus understanding these examples may given insight into the structure of the space of stability conditions for a highest weight category.
- In the case where Y is one of the quiver varieties defined by Nakajima [Nak94], category $\mathcal{O}$ gives a categorification of tensor products of integrable representations of quantum groups. The geometry of quiver varieties should allow us to investigate the structure of these categories, give a new perspective on the canonical basis theory of quantum groups, and construct categorical analogues of many interesting objects in representation theory, including the braiding on the category of representations of $U_q(\mathfrak{g})$ and thus the Reshetikhin-Turaev tangle invariants.

  It seems likely in this case, studying the space of stability conditions will allow us to recover the Knizhnik-Zamolodchikov equations.
- More generally, these categories allow us to construct well-behaved canonical bases in the cohomology of symplectic resolutions of singularities.

2. Categories $\mathcal{O}$ and geometric representation theory

2.1. The definition of $\mathcal{O}_Y^\xi$. The categories $\mathcal{O}$ have a number of different manifestations, which I will attempt to describe briefly.

If $Y \to X$ is a symplectic resolution of singularities with X affine, equipped with an appropriate ("good") $\mathbb{C}^*$-action, then by work of Bezrukavnikov and Kaledin [BK04], we have the following theorem:

Theorem 1. *For each $\eta \in H^2(Y, \mathbb{C})$, we have a canonical deformation A_η of $\mathbb{C}[X]$, the coordinate ring of X.*

- In the case of $X = \mathcal{N}_\mathfrak{g}$, this deformation is the quotient of $U(\mathfrak{g})$ by a central character depending on η.
- In the case of $X = \mathbb{C}^{2n}/(S_n \wr \Gamma)$, this deformation is the specialization of the spherical rational Cherednik algebra.

- If X is a hyperkähler analogue (a hyperkähler reduction $T^*V////G$ for some complex representation V of a compact group G), a case which covers hypertoric and quiver varieties, then the algebra A_η is a quantum hamiltonian reduction of $\mathcal{D}_V$, the algebra of differential operators on V.

Given a $\mathbb{C}^*$-action $\xi : \mathbb{C}^* \to \mathsf{Aut}(Y \to X)$ which acts with the weight 1 on the symplectic form, by naturality we obtain a corresponding C^*-action on A_η. Let A_η^+ be the subalgebra of positive weight under this $\mathbb{C}^*$ action.

Definition 1. *Let $\mathcal{O}_Y^\xi$ be the category of A_η-modules M such that:*

- *M is finitely generated over A_η.*
- *The action of A^+ on M is locally finite.*

2.2. **Localization.** In fact, there is a natural sheaf $\mathcal{A}_\eta$ on Y which is a deformation of the structure sheaf of Y such that $\Gamma(Y, \mathcal{A}_\eta) \cong A_\eta$. Thus we have a localization functor $A_\eta - \mathsf{mod} \to \mathcal{A}_\eta - \mathsf{mod}$, adjoint to the functor of taking sections. This functor sends category $\mathcal{O}$ to the category of sheaves supported (set-theoretically) on the subvariety

$$Y(\xi) = \{y \in Y | \lim_{t\to\infty} \xi(t) \cdot y \text{ exists}\}.$$

We note that for any symplectic resolution of an affine singularity, the Chern class map $c_1 : \mathrm{Pic}(Y) \to H^2(Y, \mathbb{Z})$ is an isomorphism. Let L be a relatively ample line bundle on Y. It is known that for all η, for sufficiently large integers n, this functor is an equivalence on category $\mathcal{O}$ for $A_{\eta+n\cdot c_1(L)}$,

Conjecture 1. *If η lies in the interior of the ample cone $\mathcal{A} \subset H^2(Y, \mathbb{R}) \cong \mathrm{Pic}(Y) \otimes \mathbb{R}$ then the functor $\Gamma : \mathcal{A}_\eta - \mathsf{mod} \to A_\eta - \mathsf{mod}$ is an equivalence.*

This allows us to understand category $\mathcal{O}$ as a category of sheaves on Y, and use techniques similar to those applied to $\mathcal{D}$-modules in more classical geometric representation theory.

As with $\mathcal{D}$-modules, each $\mathcal{A}_\eta$ module has a characteristic cycle in the Chow group with coefficients over $\mathbb{Z}[q, q^{-1}]$ defined by the classes of the components of its support variety, with multiplicity given by the graded dimension of the residue at that point. Since these classes are not compact, we should interpret their fundamental classes as lying in Borel-Moore homology $H_*^{BM}(Y)$. Identifying $H^*_{\mathbb{C}^*}(pt) \cong \mathbb{Z}[q]$, we ultimately obtain a characteristic cycle map $CC : K_q^0(\mathcal{O}_Y^\xi) \to H_*^{BM,\mathbb{C}^*}(Y)[q^{-1}] \cong H^*_{\mathbb{C}^*}(Y)[q^{-1}]$.

We could instead consider the quantum cohomology $QH^*(Y)$, but the quantum cohomology of symplectic resolutions is not sufficiently well understood to know whether this is a more natural target for the characteristic cycle map. Moving forward it would be very interesting to investigate These quantum cohomologies for their own interest, and to understand connections to these categories.

Conjecture 2. *If ξ has isolated fixed points, CC is an isomorphism. In other cases, its image can be geometrically described. This map intertwines the Euler product on the Grothendieck group, and the equivariant intersection product.*

The latter statement is essentially Ginzburg's index theorem [Gin86].

Thus, at least in the case with isolated fixed points, this would give a canonical basis for $H^*_{\mathbb{C}^*}(Y)[q^{-1}]$ (and, in fact, a dual canonical basis given by the classes of projectives). This basis will satisfy the familiar properties of a canonical basis (i.e. self-duality, semi-orthogonality with standard classes), though it is unclear at the moment whether this is the most natural perspective.

2.3. **Twisting functors.** One interesting question to be studied is the relationship between categories $\mathcal{O}$ for different $\mathbb{C}^*$-actions on Y. The categories $\mathcal{O}_Y^\xi$ and $\mathcal{O}_Y^{\xi'}$ are both subcategories of $A_\eta - \mathsf{mod}$ (or when thinking geometrically $\mathcal{A}_\eta - \mathsf{mod}$), and thus there are projection functors $\pi_\xi : D^b(A_\eta - \mathsf{mod}) \to D^b(\mathcal{O}_Y^\xi)$, left adjoint to the inclusion ι_ξ. One can define a canonical functor $\Phi_{\xi,\xi'} = \pi_{\xi'} \circ \iota_\xi : D^b(\mathcal{O}_Y^\xi) \to D^b(\mathcal{O}_Y^{\xi'})$.

Fix a $\mathbb{C}^*$ action ξ with positive weight on the symplectic form, and let $T \subset \mathsf{Aut}_\omega(Y)$ be the maximal subgroup of algebraic symplectomorphisms of Y which commute with $\xi(\mathbb{C}^*)$, such that $Y^T \supset Y^\xi(\mathbb{C}^*)$. Let ξ', ξ'' be actions which differ from ξ by a cocharacter into T, such that $Y^{\xi(\mathbb{C}^*)} = Y^{\xi'(\mathbb{C}^*)} = Y^{\xi''(\mathbb{C}^*)}$. Let $\mathfrak{t}$ be the Lie algebra of T, and let

$$\mathfrak{t}_\xi = \{x \in \mathfrak{t} | Y^{x+d_\xi} = Y^{d_\xi}\}$$

where d_ξ is the action of a generator of the Lie algebra of $\mathbb{C}^*$, and Y^x for x a vector field, denotes the vanishing set of that vector field.

Definition 2. *We call all functors obtained by compositions of $\Phi_{\xi',\xi''}$* **twisting functors** *for (Y, ξ_0).*

Conjecture 3. *All twisting functors are equivalences and they define a map $\pi_1(\mathfrak{t}_\xi) \to \mathsf{Aut}(D^b(\mathcal{O}_Y^\xi))$. (More generally, there is a surjective map of the Deligne groupoid of this space to the groupoid of twisting functors between the categories for different choices of ξ', ξ'').*

In the usual BGG category $\mathcal{O}$, these specialize to Arkhipov's twisting functors.

As we mentioned in the introduction, we expect this action of a fundamental group, which at the moment lacks a good geometric explanation (instead, it passes through the combinatorial description of the fundamental group in terms of the Deligne groupoid), to be connected to the structure of the space of stability conditions on $D^b(\mathcal{O}_Y^\xi)$, but the moment even a conjecture along these lines eludes us.

2.4. **The Fukaya category and A-branes.** This category also has an identity in physics. The sheaf of the algebras $\mathcal{A}_\eta$ can be thought of as the endomorphisms of the "canonical coisotropic brane" corresponding to a hyperkähler structure on Y (as before, $\eta = [\omega_\mathbb{R}]$). Thus, $\mathcal{A}_\eta$-modules should correspond to A-branes on Y, and we have a functor $\mathrm{Fuk}(Y) \to \mathcal{A}_\eta - \mathsf{mod}$ given by $\mathrm{Hom}(A_{c.c.}, -)$. Using techniques similar to Nadler and Zaslow [NZ], one can, at least in some cases, construct enough objects in the Fukaya category to generate a subcategory on which this functor is an equivalence to $\mathcal{O}_Y^\xi$.

Of course, this functor only exists at "the physical level of rigor," and thus at the moment is only philosophy, but it suggests that our constructions could be phrased in terms of the Fukaya category, and maybe interesting from the perspective of physics. In particular, it would be very interesting to understand how our results on knot homology relate to those of Seidel-Smith [SS06] and Manolescu [Man07].

3. Geometric knot homology

3.1. **Quiver varieties.** One interesting direction one is led in the study of these categories is the construction of homological knot invariants. Stroppel [Str] and Sussan [Sus07] have given constructions of knot invariants for fundamental representations of SL_n using the BGG category $\mathcal{O}$ for the Lie algebra $\mathfrak{sl}_m$, generalizing earlier work of Khovanov [Kho00, Kho02] and Khovanov-Rozansky [KR04].

The correct generalization of this construction should use category $\mathcal{O}$ for the quiver varieties described by Nakajima [Nak98]. Nakajima shows how each tensor product of integrable representations of Kac-Moody algebras has an associated quiver variety equipped with a $\mathbb{C}^*$ action [Nak01], and Zheng has shown (in different language) how the category $\mathcal{O}$ associated to this variety is a categorification of the aforementioned tensor product, with the action of the quantum group defined by pull and push on correspondences [Zhe08].

As mentioned in the introduction, there are twisting functors for these categories, which relate tensor products of the same factors in possibly different orders.

Conjecture 4. *As in the categorification used by Stroppel, the twisting functors descend, on the level of the Grothendieck group, to the usual braiding on the category of $U_q(\mathfrak{g})$-modules, defined by the universal R-matrix.*

There are also good candidates for the evaluation and coevaluation of quantum group representations, given by a certain vanishing cycle functor on a subvariety. These ingredients were enough for Reshetikhin and Turaev to construct quantum invariants of knots and links [RT90], so given a system of categories $\mathcal{O}$ with twisting functors, evaluation, and coevaluation satisfying appropriate relations, one can define a categorical tangle invariant, and in particular, a bigraded vector space to each knot.

Conjecture 5. *For each semi-simple Lie algebra $\mathfrak{g}$, there is an invariant of links labelled with representations of $\mathfrak{g}$, valued in bigraded vector spaces, whose Euler characteristic is the Reshetikhin-Turaev invariant of this knot.*

3.2. **Functoriality.** Another interesting question is that of functoriality. It's expected that interesting homological knot invariants will be functorial, with respect to embedded cobordisms between links. This functoriality was described for original Khovanov homology by Jacobsson [Jac04] and interpreted in the representation-theoretic context by Stroppel [Str05, Str].

Conjecture 6. *The homological knot invariant described earlier is functorial (up to sign) for oriented cobordisms labelled with representations of $\mathfrak{g}$.*

It seems likely that the original construction due to Stroppel using adjunctions between twisting functors will carry through directly. It would be even more interesting to see if the theory of disoriented cobordisms due to Morrison, Walker, and Clark [CMW] could be carried through to this perspective to produce an honestly functorial invariant.

Ultimately, I hope that this construction points the way to categorification of Chern-Simons theory, which is presumably a 4-dimensional TQFT whose dimensional reduction is Chern-Simons theory. While physicists have given some thought to this problem, a mathematically rigorous development of such a structure is sorely lacking.

One particularly interesting possibility is that this may allow us to construct some sort of homological version of Witten-Reshetikhin-Turaev invariants. In this case, the difficulties of defining WRT invariants for q not a root of unity could be explained by the fact that infinite dimensional vector spaces often do not have well defined graded dimensions.

3.3. **HOMFLY homology and generalizations.** Another interesting feature of knot homology theories is the existence of stable limits as the rank of the group in question approaches infinity. The best known of these is HOMFLY homology, first defined by Khovanov and Rozansky [KR08], and further developed by Khovanov [Kho07]. Such theories are expected to exist based on physical reasoning, as noted by Gukov and Walcher [GWb].

Studying the stabilization of quiver varieties under diagram expansion, using constructions similar to those which appear in my earlier work on tensor product multiplicity stabilization [Web06], could be used to construct these limits, in terms of a stable category associated to $\mathfrak{gl}_\infty$.

Interestingly, HOMFLY homology has appeared in a slightly different geometric context, as described by myself and Williamson [WW08]. Most concretely, it can be understood in terms of the equivariant cohomology $H^*_{B_\Delta}(P_{\alpha_1} \times_B \cdots \times_B P_{\alpha_n})$ of group-like Bott-Samelsons for the conjugation B-action, where α_i is a simple root of $\mathrm{SL}(n)$, B is a Borel and $P_\alpha = \overline{Bs_\alpha B}$ is an almost minimal parabolic.

The HOMFLY homology of a link is given by the cohomology of a complex built from these groups for $\{\alpha_i\}$ a subword of a braid presentation for our link, with differentials given by pushforward and pullback of equivariant cohomology.

A more geometrically natural, but technically more sophisticated approach to describing this complex can be given by considering a filtration on the sheaf of equivariant cochains on the open subset

$$Bs_{\alpha_1}B \times_B \cdots \times_B Bs_{\alpha_n}B \subset P_{\alpha_1} \times_B \cdots \times_B P_{\alpha_n},$$

where the link in consideration is the closure of the braid $\sigma_{\alpha_1}^{\pm} \cdots \sigma_{\alpha}^{\pm 1}$ (subject to compactness conditions determined by the sign of the exponents). This is the weight filtration which (crudely speaking) measures the failure of the Hodge theorem on small neighborhoods of points.

While the cohomology of the whole complex does not depend in an interesting way on the knot we consider, the successive quotients of this filtration on the complex of cochains do. In fact, they are the equivariant cohomologies described above, so the E^2 term of the spectral sequence converging to the equivariant cohomology of this open subset is the HOMFLY homology of the link.

Williamson and myself are currently working on extending this second description of HOMFLY homology to colored HOMFLY homology, the stable limit for arbitrary wedge-powers of the standard representation. The existence of this theory was predicted on physical grounds, and Khovanov, Lauda and Mackaay are currently preparing an algebraic description.

The geometric description should be essentially the same as that for original HOMFLY homology, but using larger parabolics. This theory is rather difficult to construct algebraically (which is why only the version using V and $\wedge^2 V$ has appeared, even in preprint form [MSV]), so in this case, using geometry considerably simplifies the calculations involved in checking invariance.

Conjecture 7. *For each braid σ of index n, there is a natural filtered complex of sheaves on $\mathrm{SL}(n)/P$ such that the E^2 page of the spectral sequence calculating the parabolic equivarariant cohomology of this complex is a categorification of the colored HOMFLY polynomial. This spectral sequence is a knot invariant.*

4. Mirror duality and category $\mathcal{O}$

4.1. **A duality for singularities.** I am also working on another direction of research, jointly with Braden, Licata, and Proudfoot, on understanding a notion of duality between symplectic singularities. While we're still investigating the best set of hypotheses for the conjecture below, I will state a maximally optimistic version here and a more concrete conjecture using quantum field theory in Conjecture 10.

Conjecture 8 (Braden, Licata, Proudfoot, Webster)**.** *For each affine symplectic singularity X, there is a dual singularity $X^\vee$ such that there is a bijection between partial symplectic resolutions of X and symplectic $\mathbb{C}^*$ actions on $X^\vee$.*

Thus, to each resolution $Y \to X$ with $\mathbb{C}^*$-action ξ, we have a dual $Y^\vee \to X^\vee$ with action $\xi^\vee$.

Furthermore, the categories $\mathcal{O}_Y^\xi$ and $\mathcal{O}_{Y^\vee}^{\xi^\vee}$ are Koszul dual, and, in particular, derived equivalent. This duality switches the action of shuffling and twisting functors.

For example:

- If $Y = T^*G/B$ then $Y^\vee = T^{*L}G/^LB$, and this duality claim is equivalent to the Koszul duality theorem for categories $\mathcal{O}$ due to Beilinson, Ginzburg, and Soergel [BGS96].
- If X is a hypertoric singularity associated to a hyperplane arrangement H, then $X^\vee$ is another hypertoric singularity, associated to the Gale dual $H^\vee$, by work of Braden, Licata, Proudfoot, and myself [BLPW].
- Conjecturally, if Y is the space of G-instantons on the algebraic surface $\widetilde{\mathbb{C}^2/\Gamma}$, then $Y^\vee$ is the G' instantons on $\widetilde{\mathbb{C}^2/\Gamma'}$ where G and Γ' (resp. G' and Γ) are matched by the MacKay correspondence.

In the first two cases, the Koszul duality results are well understood and should be regarded as evidence for Conjecture 8. The case of instanton spaces is less well understood. One special case is Hilbert schemes of $\widetilde{\mathbb{C}^2/\Gamma}$ (which are moduli spaces of $U(1)$-instantons) and thus $\mathcal{O}_Y^\xi$ is category $\mathcal{O}$ for a Cherednik algebra. In this case, our conjecture reduces to:

Conjecture 9. *Category $\mathcal{O}$ for the rational spherical Cherednik algebra of $S_n \wr C_\ell$ is Koszul, and its Koszul dual is category $\mathcal{O}$ for the space of $U(\ell)$-instantons on $\mathbb{C}^2$.*

I intend to investigate this conjecture geometrically, using techniques analogous to those of [BGS96] for geometric proofs of Koszulity, as well as the more general question of what hypotheses on X and Y will guarantee the Koszulity of $\mathcal{O}_Y$.

4.2. **Connections to physics.** Remarkably, the same list of examples has been known to physicists for some time. They are the Higgs branches of mirror dual $N = 4$ supersymmetric quantum field theories. This was established for T^*G/B by Witten and Gaiotto [GWa], and for hypertoric singularities by Kapustin and Strassler [KS99].

Conjecture 10. *If X is the Higgs branch of the moduli space of vacua for an $N = 4$ supersymmetric $d = 3$ field theory, then $X^\vee$ is the Higgs branch of the mirror dual field theory, that is, the (quantum) Coulomb branch of the original field theory.*

Some of our expectations about connections between X and $X^\vee$ can be explained in this picture. For example, the connection between resolutions of X and $\mathbb{C}^*$ actions on $X^\vee$ corresponds to the switching of mass and F-I parameters under mirror duality. Interestingly, many of the connections that we expect X and $X^\vee$ have *not* yet been considered by physicists, though hopefully more work will be done from that perspective.

All quiver varieties (in fact, all hyperkähler analogues) are Higgs branches, so this conjecture does suggest what the dual singularity to a quiver variety should be. Unfortunately, the quantum Coulomb branch is a very mysterious object, and present techniques in physics do not allow for a description of them in this generality which would satisfy a mathematician.

Obviously, this is a very interesting area for future research, and fruitful one for interactions between math and physics.

4.3. **Goresky-MacPherson duality.** There are several other conjectural connections between X and $X^\vee$, which we expect will all be consequences of Conjecture 8.

For instance, we consider two smooth varieties M and N with torus actions of S on M and T on N, both with isolated fixed points, and both equivariantly formal.

Definition 3. *We call M and N* **Goresky-MacPherson dual** *if there is a perfect pairing $H^2_S(M) \times H^2_T(N) \to \mathbb{C}$ such that*

- *$H^2_S(pt)$ and $H^2_T(pt)$ are mutual annihilators.*
- *There is a bijection $\Phi : M^S \cong N^T$ such that $\ker i_m^* \subset H^2_S(M)$ and $\ker i^*_{\Phi(m)} \subset H^2_T(N)$ are mutual annihilators, where i_m is the inclusion map of a fixed point.*

For example, T^*G/B and $T^{*L}G/{}^LB$ are GM-dual, with the pairing given by the identifications $H^2_T(Y) \cong \mathfrak{t} \oplus \mathfrak{t}$ and $H^2_{^LT}(T^{*L}G/{}^LB) \cong \mathfrak{t}^* \oplus \mathfrak{t}^*$ (the kernels of maps to points are the graphs of the elements of the Weyl group, and thus mutual annihilators since they are the graphs of dual linear maps).

Let (Y, ξ) and $(Y^\vee, \xi^\vee)$ be duals, as before. Let T and S be the maximal Hamiltonian tori acting on Y and $Y^\vee$ which contain the image of ξ and $\xi^\vee$.

Conjecture 11. *The spaces Y and $Y^\vee$ have torus actions which make them GM dual.*

Work in progress by Braden, Licata, Proudfoot, Phan, and myself [BLP$^+$] shows that Conjecture 11 would be a consequence of conjecture 8, by showing an algebraic analogue holds for all standard Koszul algebras A, relating A to its dual $A^!$.

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