Representation theory (and a little bit of quantum field theory)

Ben Webster

University of Waterloo
Perimeter Institute

April 28, 2022

For a lot of history, it seemed as though commutative rings were maybe the most natural framework in which to view mathematics.

- Obvious context for number theory, algebraic geometry.
- In physics, observable quantities form a commutative ring.

Then quantum mechanics came along, and the picture looked a bit different. The algebra of observables becomes non-commutative, but with a classical limit (it's "**almost commutative**").

On the classical side, physicists had already noticed a hint of the non-commutativity of quantum mechanics: the Poisson bracket (which is often called "**semi-classical**")

Hamilton's equation (for an observable): $$\frac{\partial f}{\partial t} = \{H, f\}$$

Heisenberg's equation (for an operator): $$i\hbar\frac{\partial \hat{f}}{\partial t} = [\hat{H}, \hat{f}]$$

"Almost" commutative algebras

commutative semi-classical almost commutative non-commutative

"Almost" commutative algebras

algebraic geometry: commutative, semi-classical

X

representation theory: almost commutative, non-commutative

↑ where I'd like to be

Definition

An **almost commutative ring** *is a ring A with a filtration $A_0 \subset A_1 \subset \cdots$ and an integer $n > 0$ such that*

$$A_i A_j \subset A_{i+j} \qquad [A_i, A_j] \subset A_{i+j-n}$$

In particular, the ring $\mathrm{gr}(A) \cong \oplus_{i=0}^{\infty} A_i / A_{i-1}$ is commutative and $\mathbb{Z}_{\geq 0}$-graded.

- The Weyl algebra W_n, generated by $x_1, \dots, x_n$ and $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ as differential operators, since $[\frac{\partial}{\partial x_i}, x_j] = \delta_{ij}$.
- Universal enveloping algebra $U(\mathfrak{g})$ for $\mathfrak{g}$ a Lie algebra.

The ring $\mathrm{gr}(A)$ inherits a semi-classical structure:

Definition

A **conical Poisson ring** *is a* $\mathbb{Z}_{\geq 0}$*-graded commutative ring* R *with a second operation* $\{-,-\}\colon R \times R \to R$*, homogeneous of degree* $-n$*, that satisfies the relations of a Lie bracket (bilinear, anti-symmetric, Jacobi) such that the Leibniz rule holds:*

$$\{ab, c\} = a\{b, c\} + b\{a, c\}.$$

There's a **classical limit** functor $A \mapsto (\mathrm{gr}(A), \{-,-\})$ from almost commutative algebras to conical Poisson algebras, with the Poisson bracket given by

$$\{\bar{a}, \bar{b}\} = \overline{[a, b]} \in A_{i+j-n}/A_{i+j-n-1}.$$

“Almost” commutative algebras

- Weyl algebra W_n has classical limit $\mathbb{C}[x_1, \ldots, x_n, p_1, \ldots p_n]$ with the Poisson bracket that Poisson would recognize:

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial x_i} \qquad \{p_i, x_j\} = \delta_{ij}$$

- Similarly, $U(\mathfrak{g})$ has classical limit $\mathbb{C}[\mathfrak{g}^*] = \mathrm{Sym}(\mathfrak{g})$ with the KKS Poisson structure:

$$\{f_X, f_Y\} = f_{[X,Y]}$$

This is corresponds to the (in)famous symplectic structure on coadjoint orbits.

We call an affine variety Y conical Poisson if its coordinate ring has that structure.

Definition

We call Y a **conical symplectic variety** *(i.e. conical variety w/ symplectic singularities) if the Poisson bracket induces a symplectic structure on the smooth locus (+silly technical conditions).*

- If $\mathbb{C}^{2n}$ has the usual symplectic structure, and Γ is a finite group preserving ω, then $Y \cong \mathbb{C}^{2n}/\Gamma$ is an example.
- The variety of nilpotent matrices (and more generally, nilpotent cone of a semi-simple Lie algebra $\mathfrak{g}$) has a natural symplectic structure.

The correspondence between almost commutative and semi-classical is particularly nice in this case.

Definition

If R is an conical Poisson algebra, then a **quantization** *of R is an almost commutative algebra A whose classical limit is R.*

You can easily check that A_1 is the unique quantization of $\mathbb{C}[x, p]$. (Hint: $A_1 \cong \mathbb{C} \cdot \{x, p, 1\}$ as 3-dimensional Lie algebras).

Theorem

Every conical symplectic variety Y has a universal family of quantizations A.

The center $Z(A)$ is a polynomial ring; the quotients A_λ by maximal ideals of the center give a complete irredundant list of quantizations of $\mathbb{C}[Y]$.

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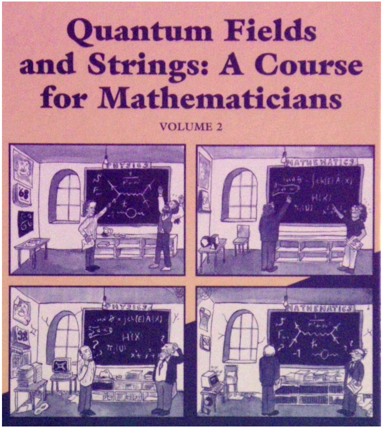
- If Y is a nilcone, then A is the universal enveloping algebra.
- If $Y \cong \mathbb{C}^{2n}/\Gamma$, then A is a spherical symplectic reflection algebra.

Why would we expect something like this exist?

They arise naturally from 3-dimensional quantum field theories, with a topological twist.



You should think of a QFT as a big generalization of the notion of an algebra:

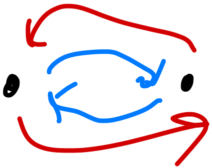
- you have operators that live at different points in a 3-dimensional manifold (the worldsheet), and a rule for multiplying them, which depends on their positions (possibly a metric, etc.).
- in many cases, you can add in an extra differential Q (a "twist") which has degree 1 in a grading on operators, such the change in product as you move operators around is exact.

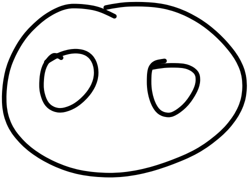
Thus, the product on the cohomology of Q acting on operators is topological.

However, just passing to cohomology loses information; we have secondary products, just like in the cohomology of a topological space.

In a $d = 2$-dimensional topologically-twisted theory, we thus get

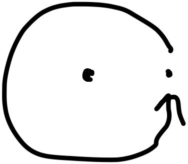
- a commutative algebra A assigned to a point, with a cohomological grading. If you like TQFT, think of this as the vector space for an S^1. This is commutative because the points can swap places.
- a (super)Poisson bracket of degree -1: this is the difference between the homotopies induced by the two ways of switching the points.





In a $d = 3$-dimensional topologically-twisted theory, we thus get

- a commutative algebra A assigned to a point, with a cohomological grading. If you like TQFT, think of this as the vector space for an S^2. This is commutative because the points can swap places.
- a Poisson bracket of degree -2: the $d = 2$ construction on the equator gives a degree -1 chain map; the two hemispheres give homotopies of this chain map to 0, and so their difference is a chain map of degree -2.



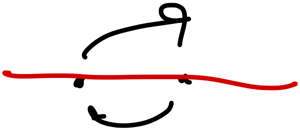
You can construct a deformed algebra $(A_\hbar, \star_\hbar)$ by doing an "equivariant" twist for the S^1-action on $\mathbb{R}^3$. Ω-background

This is no longer commutative, since you can't swap the points S^1-equivariantly. Here $H^*_{S^1}(pt; \mathbb{C}) \cong \mathbb{C}[\hbar]$.

The formula

$$\frac{1}{\hbar}(a \star_\hbar b - b \star_\hbar a) = \{a, b\}$$

is calculating the secondary product by Atiyah-Bott for the S^1-action on S^2.



“Theorem”

Every 3d quantum field theory with a topological twist gives rise to a pair of an almost commutative algebra and its classical limit.

Much remains mysterious about this story, but this is both a valuable way to construct examples, and to understand the phenomena we see in the examples that result.

What questions do we want to ask?

Symplectic singularities are the Lie algebras of the 21st century. - Okounkov

The last third of the 20th century saw a huge development of the theory of infinite dimensional representations of Lie algebras. The crown jewel of this is the proof of the Kazhdan-Lusztig conjecture:

Theorem (Brylinski-Kashiwara)

The characters of simple highest weight $U(\mathfrak{g})$-modules can be written in terms of Verma modules (which have easy characters) with coefficients given by Kazhdan-Lusztig polynomials.

These polynomials have a geometric interpretation in the intersection cohomology of Schubert varieties.

How do we generalize this picture? Unfortunately, the case of a UEA has a lot of special structure we can't expect in other cases.

One other tractable situation:

Beginning with a connected reductive complex group G, and a representation V, there's a 3-d $\mathcal{N} = 4$ supersymmetric field theory you can build from this.

The $\mathcal{N} = 4$ supersymmetry gives us two distinguished twists Q_A, Q_B, related to the isomorphism $\mathrm{Spin}(4) \cong SL_2 \times SL_2$. The symplectic singularities that result from these twists are called the **Higgs** and **Coulomb** branches.

This is one of a hierarchy of dualities here which are all compatible under boundary conditions, and which all have interesting mathematical manifestations:

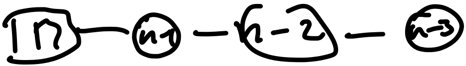
4d mirror symmetry (S-duality, E-M duality)	geometric Langlands
3d mirror symmetry (S-duality)	S(ymplectic) duality
2d mirror symmetry (T-duality)	homological mirror symmetry (Kontsevich)

There are many interesting pairs of theories which are equivalent switching the A and B twists. We call these **mirror dual**.

In both cases, the corresponding symplectic singularity and quantization have mathematical constructions.

- Higgs branches can be realized as a symplectic reduction of T^*V, and a quantization can be constructed by replacing T^*V with differential operators, and performing non-commutative Hamiltonian reduction.
- Coulomb branches are gotten by starting with $T^*\check{T}/W$ and doing some "quantum corrections." One description of the coordinate ring $\mathbb{C}[Y]$ is as the homology of a space constructed from the affine Grassmannian with a convolution giving multiplication. There is a quantization given by $\mathbb{C}^*$-equivariant homology of this space.

Braverman-Finkelberg-Nakajima.



Examples:

(V, G)	Higgs	Coulomb
$(\mathbb{C}^{n+1}, U(1))$	$T^*\mathbb{P}^n$	$\mathbb{C}^2/\mathbb{Z}_{n+1}$
$(\mathbb{C}^{n+1}, U(1)^n)$	$\mathbb{C}^2/\mathbb{Z}_{n+1}$	$T^*\mathbb{P}^n$
$(\mathrm{Mat}_{n\times n} \times \mathbb{C}^n, GL_n)$	$\mathrm{Sym}^n(\mathbb{C}^2)$	$\mathrm{Sym}^n(\mathbb{C}^2)$
$(\mathrm{Mat}_{n\times n-1} \times \mathrm{Mat}_{n-1\times n-2} \times \cdots, GL_{n-1} \times GL_{n-2} \times \cdots)$	$\mathrm{Nil}_{n\times n}$	$\mathrm{Nil}_{n\times n}$
$(0, G)$	pt/G	Toda phase space
quiver gauge theory	Nakajima quiver variety	affine Grassmannian slice

We've already found some dual pairs: (1) and (2) are dual to each other, and (3) and (4) are self-dual.

How are we supposed to study these algebras? With Lie theory, we're starting from having a century of experience, and a rather special set up.

- On the Higgs side, the geometric methods works pretty well. The replacement for D-modules is "quantum coherent sheaves" on a resolution of the Higgs branch. These are hard to work with (no six functor formalism) but close enough to G-equivariant D-modules on V to make things work.
- With Coulomb branches, the algebraic methods are more successful. There's a natural "torus" $\mathbb{C}[\mathfrak{t}/W]$ in the quantization A and you can analyze its weight spaces, with the structure of the representation V influencing how they are related.

Remarkably, the same combinatorics show up in both situations.

We call $\xi \in A_\lambda$ a grading element if $[\xi, -]\colon A_\lambda \to A_\lambda$ is semi-simple with integral eigenvalues.

Definition

Category $\mathcal{O}^\xi_\lambda$ for ξ over A_λ is the subcategory of modules where ξ acts with finite length Jordan blocks, and f.d. eigenspaces, and eigenvalues bounded above.

Note this category depends on ξ and λ; it's richest if λ is "integral" in some appropriate sense.

These switch roles between Higgs and Coulomb: a grading element on one side corresponds to an integral quantization parameter on the other.

We can define a graded algebra T (purely based on the combinatorics of G, V, ξ, λ) such that:

Theorem (W.)

For ξ integral, the categories $\mathcal{O}_\lambda^\xi$ for A_{Coulomb} and $\mathcal{O}_\xi^\lambda$ for A_{Higgs} (assuming certain hypotheses on V and G) can both be reconstructed from T (but in different ways).

The category $\mathcal{O}_\xi^\lambda$ is **Koszul dual** *to $\mathcal{O}_\lambda^\xi$. These categories are not equivalent, their derived categories are, switching simples and projectives.*

In particular, the question of how to write characters of interesting modules in these two categories are in some sense dual.

The algebra T is graded, whereas the Coulomb category $\mathcal{O}$ isn't. This provides a graded lift, which has good positivity properties.

Theorem (W.)

There are "Verma" modules in $\mathcal{O}^{\xi}_{\lambda}$, and the multiplicities of simples in them are given by a version of Kazhdan-Lusztig polynomials/canonical bases.

In general, lots of "canonical bases" (for reps of Lie algebras, for the Hecke algebra, etc.) show up this way.