Three-dimensional mirror symmetry: a mathematical perspective

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To quote James Stockdale: “Who am I? Why am I here?”

In my younger days, I started as a representation theorist. In that line of business, the absolute most basic question is “what are the different representations of a given Lie algebra $\mathfrak{g}$?”

For finite dimensional representations, this is mostly settled (though some might disagree), but for infinite dimensional representations is still very interesting.

These have received very intense study for the last 100 years or so, but we still don’t understand them as well as we would like.

The question is why I would bother you with them.

The algebra $U(\mathfrak{g})$ is closely tied to the geometry of the variety $\mathcal{N}_{\mathfrak{g}}$.

It's the universal deformation of $\mathbb{C}[\mathcal{N}_{\mathfrak{g}}]$ compatible with the scaling $\mathbb{C}^*$-action and usual Poisson structure.

Gaiotto and Witten tell us that there is a three dimensional conformal field theory $T(G)$ whose Higgs branch is $\mathcal{N}_{\mathfrak{g}}$. The algebra $U(\mathfrak{g})$ arises if we quantize the Higgs branch, using an Ω-background (deforming a supersymmetry relation to $Q^2 = \epsilon(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$, the rotation vector field around the z-axis in $\mathbb{R}^3$).

Thus, we could hope to study the representations of $U(\mathfrak{g})$ using boundary conditions in this theory or *vice versa*.

But better yet, there are a lot more similar theories out there.

Given a compact gauge group G and complex representation V, there's an attached $\mathcal{N} = 4$ supersymmetric 3d gauge theory.

These have quite interesting Higgs and Coulomb branches, which include a lot of spaces of interest to mathematicians.

- The Higgs branch is simply a hyperkähler quotient: you have a moment map valued in imaginary quaternions for the action of G with 3 moment maps, and take a level modulo G.
- The Coulomb branch is a bit harder to understand, since it has quantum corrections, but recent work of Braverman-Finkelberg-Nakajima and Bullimore-Dimofte-Gaiotto has put it on much firmer footing.

It might be worth saying that these are examples of a class of varieties which is getting very extensive attention from mathematicians: symplectic varieties which are conical (have a contracting $\mathbb{C}^*$-action).

Theorem (Namikawa)

Every conic symplectic variety Y has a universal deformation $\mathcal{Y}$ that smoothes it out as much as possible while staying symplectic (which is thus rigid). The base of this deformation is a vector space H.

For example, $\mathfrak{g}^*$ comes from the nilcone $\mathcal{N}$.

In the cases of interest to us, these come from changing the complex FI (for the Higgs) or mass (for the Coulomb) parameters.

These two branches are also deformed by (different) Ω-backgrounds, and thus give us interesting non-commutative algebras. You can also think of these as the endomorphism algebra of the canonical coisotropic brane on a resolution of this branch.

- The quantized Coulomb branch of the theory for $GL(n)$ on $\mathfrak{gl}_n \oplus (\mathbb{C}^n)^{\oplus \ell}$ gives the spherical rational Cherednik algebra of $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$.
- The quantized Coulomb branch of the theory the quiver gauge theory attached to a Dynkin diagram (built of bifundamentals) and fundamentals gives a (truncated shifted) version of the corresponding Lie algebra.
- The quantized Higgs and Coulomb branches for abelian theories give a "hypertoric enveloping algebra" which had been defined earlier by Musson and van der Bergh.

This quantization has also attracted attention from mathematicians.

Theorem (Bezrukavnikov-Kaledin, Braden-Proudfoot-W., Losev)

Every conical symplectic variety Y has a unique quantization A of its universal Poisson deformation $\mathcal{Y}$; this is an almost commutative algebra such that $\operatorname{gr} A \cong \mathbb{C}[\mathcal{Y}]$.

The center $Z(A)$ is the polynomial ring $\mathbb{C}[H]$; the quotient A_λ for a maximal ideal $\lambda \in H$ has an isomorphism $\operatorname{gr} A_\lambda \cong \mathbb{C}[Y]$. This gives a complete irredundant list of quantizations of $\mathbb{C}[Y]$.

- If Y is a nilcone, then A is the universal enveloping algebra.
- If $Y \cong \mathbb{C}^{2n}/\Gamma$, then A is a spherical symplectic reflection algebra (Cherednik algebras are a special case).

Symplectic singularities are the Lie algebras of the 21st century. - Okounkov

It rapidly leads us to wonder: which theorems of the classical representation theory of Lie algebras carry over (possibly with serious modification), and which fail?

- The finite dimensional representations of a semi-simple Lie algebra are completely decomposable. This fails miserably for a general symplectic singularity.
- However, maybe finite dimensionals are too touchy. They're very sensitive and tend to blink out of existence when you change the quantization parameter. Vermas are much more robust.

The natural category where Verma modules live is category $\mathcal{O}$.

We call $\xi \in A_\lambda$ a flavor if $[\xi, -]\colon A_\lambda \to A_\lambda$ is semi-simple with integral eigenvalues. This is the lift of a Hamiltonian flavor $\mathbb{C}^*$-action.

Definition

Category $\mathcal{O}_\lambda^\xi$ for ξ over A_λ is the subcategory of modules where ξ acts with finite length Jordan blocks, and f.d. eigenspaces, and eigenvalues bounded above.

Note this category depends on ξ and λ; it's richest if λ is "integral" in some appropriate sense.

These switch roles between Higgs and Coulomb: a grading element on one side corresponds to an integral quantization parameter on the other (via identification with complex FI and mass parameters).

Definition

We call a graded algebra A **Koszul** *if it only has degree 1 generators, degree 2 relations, degree 3 relations between relations, degree 4 relations between relations between relations, etc.*

A category is **Koszul** *if it's equivalent to modules over A.*

A Koszul category has a Koszul dual, which is the category of complexes of projectives with all maps degree 1. In this category, single projectives are simples, and the projective resolution of a simple is injective. You can even talk about this category if A isn't Koszul.

You can also think of this as the representations of an algebra generated by A_1^* with relations given by the annihilator of the degree 2 relations in A.

One of the classic theorems of Lie theory I'm interested in generalizing is:

Theorem (Soergel, 1990)

The category $\mathcal{O}$ for the nilcone $\mathcal{N}_{\mathfrak{g}}$ is Koszul dual to the category $\mathcal{O}$ of the nilcone of the Langlands dual $\mathcal{N}_{^L\mathfrak{g}}$

Of course, this would require some sort of generalization of Langlands duality.

Observation (Braden-Licata-Proudfoot-W., ∼2007)

The category $\mathcal{O}$ for one conic symplectic variety is often Koszul, and its Koszul dual is often category $\mathcal{O}$ for a different variety.

Actually, when the variety is the Higgs/Coulomb branch of a 3-d theory, the one giving the Koszul dual is the Coulomb/Higgs branch of the same theory!

Note that Soergel's result is a special case of this. The name we came up with for this phenomenon was "symplectic duality." The second observation means it's an aspect of S-duality.

Theorem (W., 2016)

The category $\mathcal{O}$ attached to the Higgs branch of the gauge theory for (G, V) is Koszul to that for the Coulomb branch.

The key idea is to pay attention to where Neumann boundary conditions go on each side. These are more general than the boundary conditions considered in most recent work of Bullimore-Dimofte-Gaiotto-Hilburn, since they break symmetry to the torus.

Using some well known approaches from mathematics we can compute the appropriate version of endomorphisms of these boundary conditions on the two sides and see that they match.

I have to confess: at this point the proof becomes pure computation. However BDGH suggest that one can see a more general explanation for this match using reduction to 2-d.

Personally, I think the pure computation is pretty interesting though. The idea is describe both categories using a quiver with relations.

Theorem

The Coulomb category $\mathscr{O}$ is isomorphic to representations of an algebra T described as a quiver algebra with relations as follows:

- *The nodes of the corresponding quiver are chambers in the space of lifted masses cut out by the hyperplane arrangement of weights for the representation V.*
- *The generators of the algebra are generated by weights of the gauge group G, arrows which cross hyperplanes separating chambers, the Weyl group W acting on everything, and the additional Demazure operator $\frac{s_\alpha - 1}{\alpha}$ acting on chambers fixed by s_α, for all roots α.*

Personally, I think the pure computation is pretty interesting though. The idea is describe both categories using a quiver with relations.

Theorem

The Coulomb category $\mathscr{O}$ is isomorphic to representations of an algebra T described as a quiver algebra with relations as follows:

- *The relations include things like:*
  - *crossing a hyperplane twice multiplies by the corresponding weight*
  - *any two paths around a codimension 2 subspace are equal*
  - *Weyl group elements commute past things in the obvious way*

The Neumann boundary conditions give projectives in $\mathcal{O}_{\text{Coulomb}}$ (which is how the previous theorem is proved) and give (semi-)simples in $\mathcal{O}_{\text{Higgs}}$. These modules match under Koszul duality.

Theorem

The algebra T is isomorphic to the Ext algebra of the sum of all the simple modules (with some multiplicities) in the category $\mathcal{O}$ of the Higgs branch for (G, V).

Thus the category $\mathcal{O}_{\text{Higgs}}$ is Koszul dual to the category $\mathcal{O}_{\text{Coulomb}}$.

Quantizations in characteristic p lead one to construct tilting bundles. There are some subtleties, but this combinatorial perspective on Coulomb branches actually resolves many of them.

When massaged correctly, this construction allows us to describe the category of coherent sheaves on a Coulomb branch (over $\mathbb{Q}$) in terms of an quiver with relations. In very rough terms, you define an algebra like T, but with the masses considered in $\mathbb{R}/\mathbb{Z}$ instead of just $\mathbb{R}$.

For hypertoric varieties, this algebra was worked out by myself and Michael McBreen in unpublished work. For affine Grassmannian slices (including Slodowy slices in type A), these are given by versions of KLR algebras on cylinders.

The algebra T is graded, whereas the Coulomb category $\mathscr{O}$ isn't. This provides a graded lift, which has good positivity properties.

Theorem (W.)

There are "Verma" modules in $\mathscr{O}_\lambda^\xi$, and the multiplicities of simples in them are given by a version of Kazhdan-Lusztig polynomials/canonical bases.

In particular, we get a new proof of Rouquier's conjecture that decomposition numbers for rational Cherednik algebras are given by affine parabolic KL polynomials.

In general, lots of "canonical bases" (for reps of Lie algebras, for the Hecke algebra, etc.) show up this way.

If we start with a quiver Γ, and dimension vectors $\mathbf{v}, \mathbf{w}$, then quiver varieties are the Higgs branches in the case of a quiver gauge theory

$$V = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \oplus \bigoplus_{i} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i}) \qquad G = \prod GL(v_i, \mathbb{C}).$$

Theorem (W.)

The category T-gmod categorifies a tensor product of simple representations (depending on $\mathbf{w}$ and ξ) for the quantum group attached to Γ ($q =$grading shift).

There are natural functors categorifying all natural maps in quantum group theory. This allows us to construct a categorified Reshetkhin-Turaev knot invariant for any representation.

Even the Lie algebras not attached to graphs (types BDFG) have an associated algebra T, and this construction can be carried through purely combinatorially.

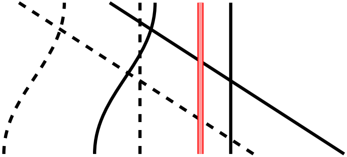
In the case of a quiver gauge theory, the family of algebras which show up already has a rich theory: they are (weighted) Khovanov-Lauda-Rouquier algebras. We can define this algebra in terms of diagrams modulo local relations.

Let's consider the special case of

$$G = GL(n, \mathbb{C}) \qquad V \cong \mathfrak{gl}(n) \oplus (\mathbb{C}^n)^{\oplus \ell}.$$

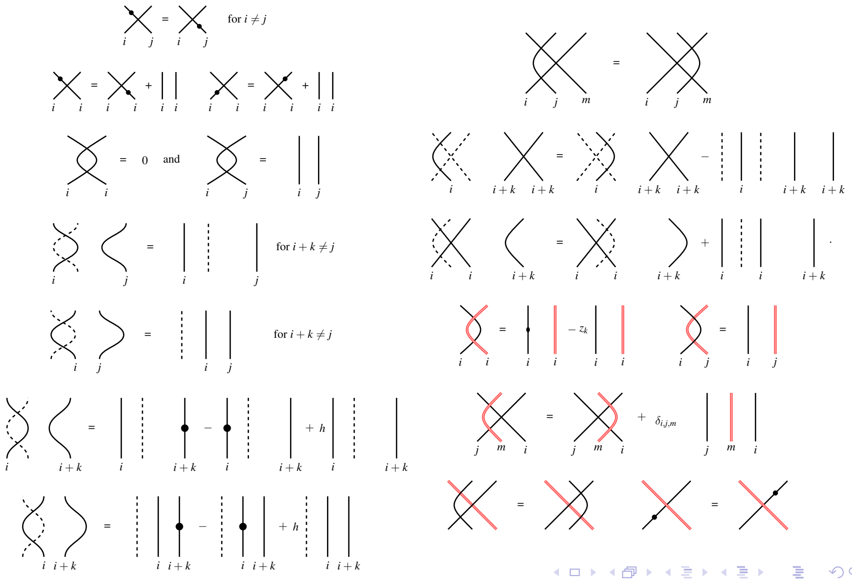
The corresponding Coulomb quantization is the rational Cherednik algebra of $S_n \operatorname{wr} \mathbb{Z}/\ell\mathbb{Z}$.

In this case, the diagrams of T look like:



- The x-values of the strands give a lifted mass.
- The weights of $\mathbb{C}^n$ correspond to the red lines: you cross a corresponding hyperplane when a red and black line cross.
- the weights of $\mathfrak{gl}(n)$ are the distance between points, so you cross one of their hyperplanes when two strands cross, or reach a fixed distance from each other.

KLR algebras



Thanks for listening.