An extended category for the BFN construction

Ben Webster
Waterloo/PI

General context:

I like symplectic singularities.
One nice way these come up is in $\mathcal{N}=4$ $d=3$ gauge theory, w/ gauge group G and matter V (a G-rep).

Higgs branch: $T^*V \supseteq \mu^{-1}(0)^{ss} \to \mathcal{M}_H$
Ex: Nakajima quiver varieties

Coulomb branch: close to $T^*T^\vee/W$ ($T \subseteq G$ a maximal torus). Exactly how? ¯_(ツ)_/¯

Luckily, some people didn't just shrug. Let:

$$G_{\mathcal{O}} := G[[t]] \qquad G_K := G((t))$$
$$V_{\mathcal{O}} := V[[t]] \qquad V_K := V((t))$$

Consider moduli of Hecke modifications on the formal disk D at the origin o together w/ compatible sections of the associated bundle w/ fiber V.

That is, $\mathcal{Y} = G_K \times^{G_{\mathcal{O}}} V_{\mathcal{O}}$ is the associated bundle for $V_{\mathcal{O}}$, (i.e. principal bundles on D w/ section of associated V bundle and trivialization on $D \setminus o$), then we are considering:

$$\boxed{\frac{\mathcal{Y} \times_{V_K} \mathcal{Y}}{G_K}}$$

pairs of bundles w/ sections such that the sections agree on $D \setminus o$, modulo changes of trivialization.

Def The BFN algebra is the equivariant homology group $A_0 = H_*^{G_K}(\mathcal{Y} \times_{\mathcal{V}_K} \mathcal{Y})$.

The Coulomb branch is the spectrum of the commutative algebra $\mathcal{M} = \operatorname{Spec}(A_0)$.

This can naturally be deformed by considering equivariant cohomology.

Def Let $F = \operatorname{Aut}_G(V)$ be the flavor group and $\mathbb{C}^*$ act on G_K and V_O by loop rotation.

$$H^{F \cdot G_K \rtimes \mathbb{C}^*}(\mathcal{Y} \times_{\mathcal{V}_K} \mathcal{Y}) \cong A_\hbar^F$$ where $\hbar$ is the equivariant parameter for $\mathbb{C}^*$.

The center of this algebra is $H^*_{F/Z(G)}(pt) = Z$

Modding out by a maximal ideal in Z gives a quantization of $\mathcal{M}$.

But how are we supposed to understand either one?

Note: We can instead consider the Coulomb branch for $T_F \cdot G \curvearrowright V$. This has an action of $H = \mathfrak{t}(T_F T / T)$ and $\mathcal{M}$ is related to $\mathcal{M}_F$ by Hamiltonian reduction:

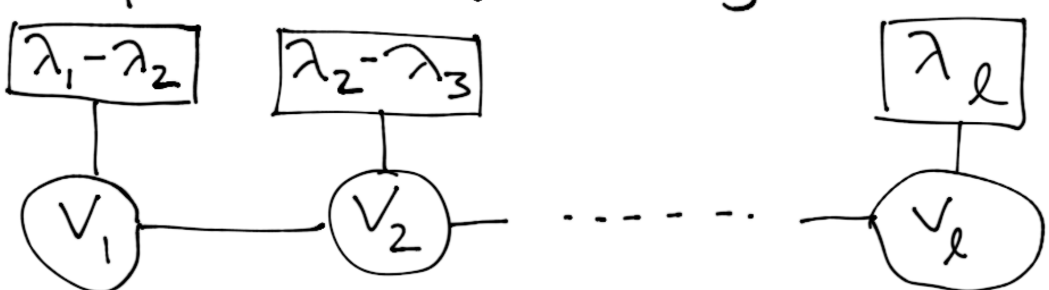
$$\mathbb{C}[\mathcal{M}] \cong \mathbb{C}[\mathcal{M}_F] / (\mu_F = 0)$$

You can instead do GIT quotient, and construct a partial symplectic resolution $\widetilde{\mathcal{M}}$. Don't worry about this: just remember that any construction with Coulomb branches comes along for the ride to $\widetilde{\mathcal{M}}$.

Notable examples:

- **hypertoric varieties:** when G is a torus, then $\mathcal{M}$ is a hypertoric variety, which is the Higgs branches of another theory. Let D be diagonal matrices on V, then we have $G^\perp \subseteq D$, and $\mathcal{M} \cong \mu_{G^\perp}^{-1}(0)^{ss}/G^\perp$. The fancy word here is "Gale duality."

- **Slodowy slices:** if $\lambda=(\lambda_1 \geq \cdots \geq \lambda_\ell \geq 0)$ and $\mu=(\mu_1 \geq \cdots \geq \mu_\ell \geq 0)$ are two partitions of n, the slice to the μ-orbit in the closure of the λ-orbit is the Slodowy slice attached to the quiver gauge theory for



Where $V_i = \sum_{k=1}^{i} \lambda_k - \mu_k$.

- **affine Grassmannians:** for ADE quivers, the Coulomb branches of these theories are isomorphic to slices between Gr^λ and Gr^μ, where $w_i = \alpha_i^\vee(\lambda)$ and $\lambda-\mu = \sum v_i \alpha_i$. For other quivers, these are standins for such slices. In affine type A, NQV for rank-level dual data.

One idea: Monopole formula.

The orbits of G_K on $Gr \times Gr$ are indexed by dominant coweights.

The preimage in $Y \times_{V_K} Y$ of the orbit of λ is equivariantly homotopy equivalent to $*/G_\lambda$, where G_λ is centralizer of the coweight.

This allows us to compute the Hilbert series of A_0, and to construct a basis of A_K but not how to multiply in this basis.

This is very nice piece of information (physicists had guessed it before the BFN construction gave us the actual algebra), but not terribly useful for analyzing the rep. theory.

Another approach: consider the Iwahori Subgroup $I = \{g(t) \in G_{\mathcal{O}} \mid g(0) \in B\}$. The quotient G_K/I is the affine flag variety; for $G = GL_n$, $G_K/I = \{V_0 \supset V_1 \supset \cdots \supset V_{n-1} \supset tV_0 \mid V_i \subset \mathbb{C}^n((t))$ a lattice$\}$. The associated bundle $\mathcal{X} = G_K \times^I V_0$ can be identified with the moduli of

- a principal bundle P on D w/ a section on the associated V-bundle
- a trivialization on $D \setminus 0$ and a "flag" in P_0/B.

Def: Let $\mathcal{A}^F_\hbar = H_*^{F \cdot G_K \rtimes \mathbb{C}^*}(\mathcal{X} \times_{V_K} \mathcal{X})$, and $\mathcal{A}_\hbar$ the quotient by a maximal ideal in Z.

The map $\mathcal{X} \to \mathcal{Y}$ is a bundle w/ fiber given by G/B.
As a consequence:

Prop: $A^F_\hbar$ and $\mathcal{A}^F_\hbar$ are Morita equivalent via the bimodules $H_*^{F G_K \rtimes \mathbb{C}^* \hbar}(\mathcal{X} \times_{V_K} \mathcal{Y})$ and vice versa.

This Morita equivalence descends to quotients by maximal ideals in Z.

Why is this preferable? Flags are better than Grassmannians, since the Bruhat decomposition is simpler.

We have natural homology classes φ_i for the simple roots α_i given by the locus where the principal bundle and section differ by relative position $s s_i$

For GL_n, this means that the periodic flag $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{n-1} \supseteq tV_0 \supseteq \cdots$ only changes by V_i (and all other V_j stay the same).

These act on $H_*^?(\mathcal{X}) \cong \begin{cases} H^*(BG) & ? = G(K) \\ H^*(B(FG))[h] & ? = FG_K \rtimes \mathbb{C}^* \end{cases}$

by $f \longmapsto \dfrac{s_i f - f}{\alpha_i}$.

But we need something that changes the actual principal bundle, and that's tricky, since that changes the acceptable sections.

What I want to do is separate changing the acceptable sections and changing the principal bundle.

For $\lambda \in t_{\mathbb{R}}$, we let

$$\mathcal{U}_\lambda = \mathrm{span}\left\{ t^a v \,\middle|\, v \in V_\mu,\ a > -\langle \lambda, \mu \rangle - \tfrac{1}{2} \right\}$$

If λ lies in the standard alcove,

$\alpha_i(\lambda) > 0$ $\qquad\qquad$ $\alpha_{top}(\lambda) \leq 1$

then this space is I-invariant, and it's always invariant under a parahoric I_λ.

Def The extended BFN category $\mathcal{B}$ is the category whose:

- Objects are $\lambda \in \mathfrak{t}_{\mathbb{R}}$.
- Morphisms are
$$\mathrm{Hom}(\lambda, \lambda') \cong \mathcal{X}_\lambda \times_{V_K} \mathcal{X}_{\lambda'}$$

where $\mathcal{X}_\lambda = G_K \times^{I_\lambda} U_\lambda$.

Goal: thinking about all of these spaces together actually makes understanding each easier.

- Easier to write generators, since we can break things into morphisms changing λ.
- Similarly, relations can be written in a more basic way.
- The modules $\mathrm{Hom}(0, \lambda)$ for $\mathrm{Hom}(0,0) \cong A_K$ themselves play an important role.

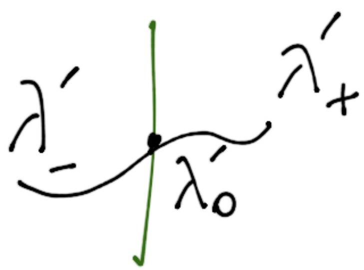
So, what are these morphisms?

$\lambda \xrightarrow{w} \lambda' = w\lambda'$ for $w \in \widehat{W} = W \ltimes \mathfrak{t}_{\mathbb{Z}}$ the extended affine Weyl group.

This corresponds to the graph of the isomorphism $\mathcal{X}_\lambda \cong \mathcal{X}_{w\lambda}$

For each path λ —— λ''

We attach to this path the space where we fix the section $v(t)$ on $D \setminus 0$ and choose compatible elements of $U_{\lambda'}$ for all λ' on the path, which only change when we pass $\alpha(\lambda') = n$ for $n \in \mathbb{Z}$.



$$G/I_{\lambda'_-} \xrightarrow{\mathbb{P}^1} G/I_{\lambda'_0} \xleftarrow{\mathbb{P}^1} G/I_{\lambda'_+}$$

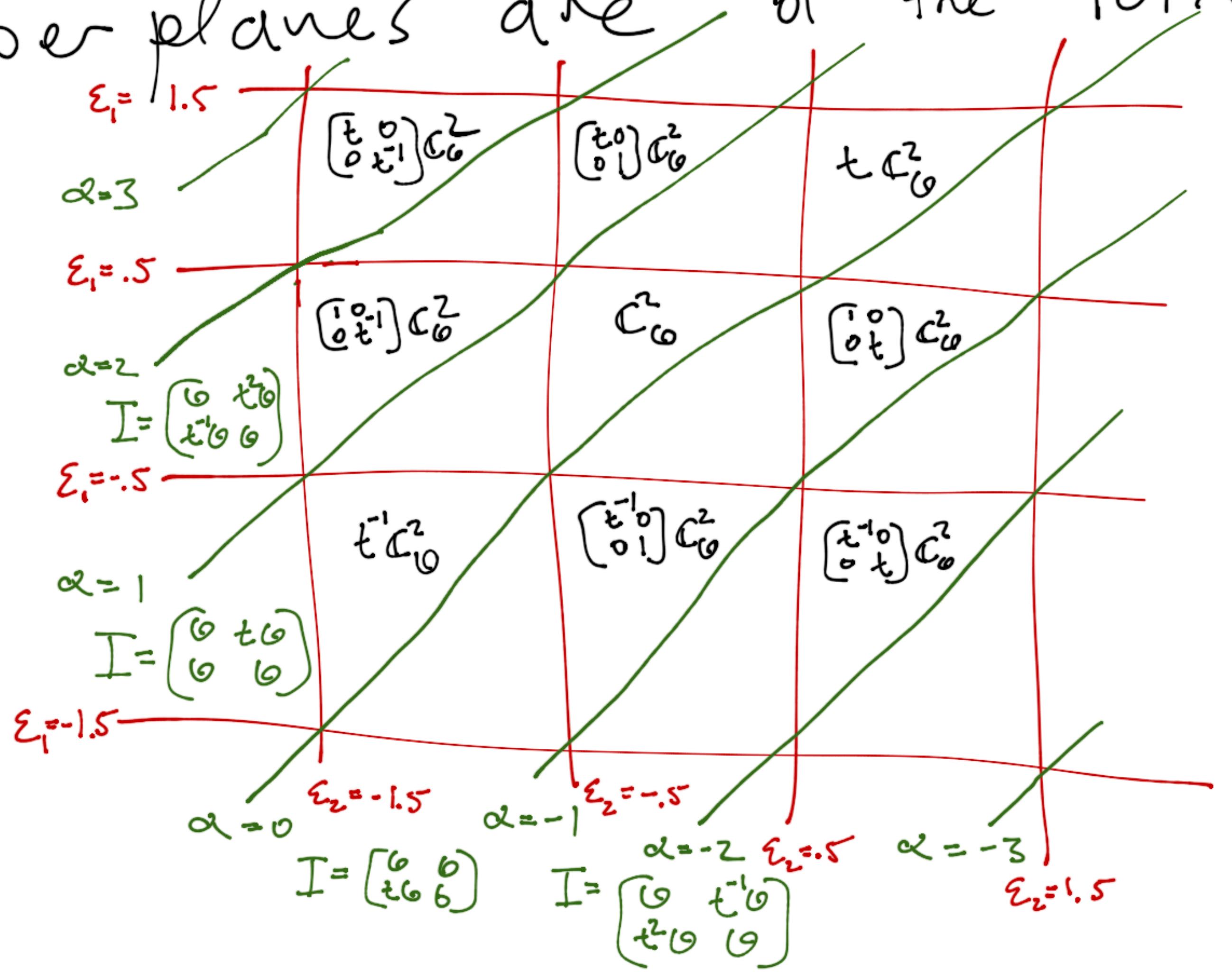
The attached homology class is the pushforward by the map only remembering endpoints.

We have to be careful about what this means.

If the path doesn't double back over hyperplanes $\alpha(x) = n$ or $\mu(\lambda) = n + \frac{1}{2}$ for $n \in \mathbb{Z}$, then you can interpret this naively, but otherwise non-transverse intersections are involved, and you have to push forward a "virtual fundamental class."

Don't worry about this: just decompose the path into pieces without double-backs and multiply those.

So, for $G = GL(2)$, $V = \mathbb{C}^2$. The hyperplanes are of the form



Theorem The generators in

$\underline{W} : \lambda \to w\lambda$ for $w \in W$.

$p : \lambda \to \lambda''$ for a path λ  λ''

$H^*_{F}(\mathfrak{X}_\lambda) \cong \mathrm{Sym}(\mathfrak{t}^*)^{W_\lambda} : \lambda \to \lambda$.

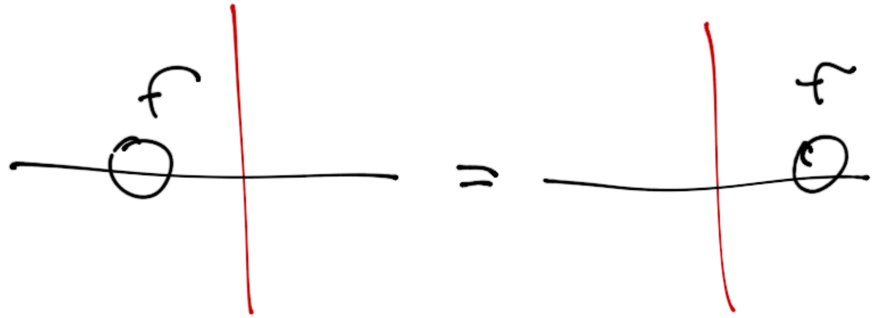
generate the morphisms of $\mathcal{B}$, subject to the relations:

$$= \prod_{\varphi_i = \mu} (\mu - n - \tfrac{1}{2} + m_i)$$

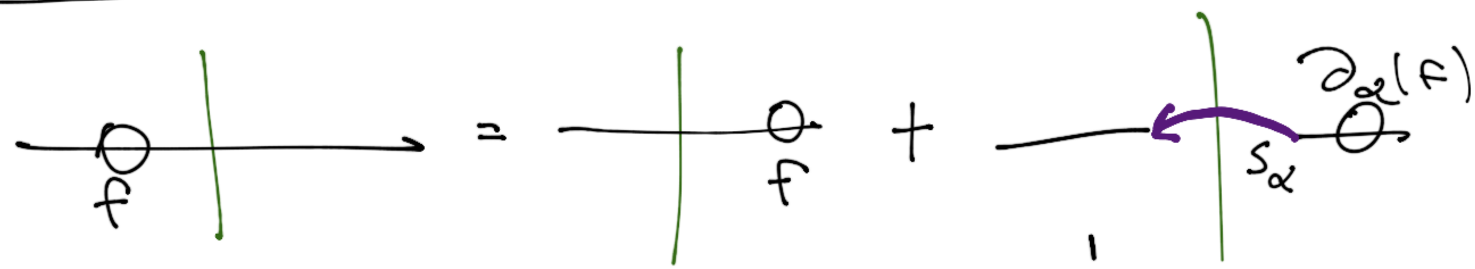
$\mu(\lambda') = n + 1/2$

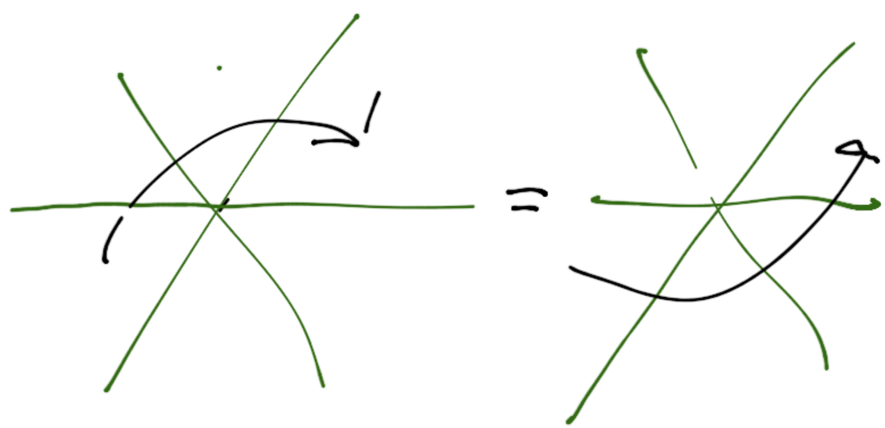
$\alpha(\lambda') = n$

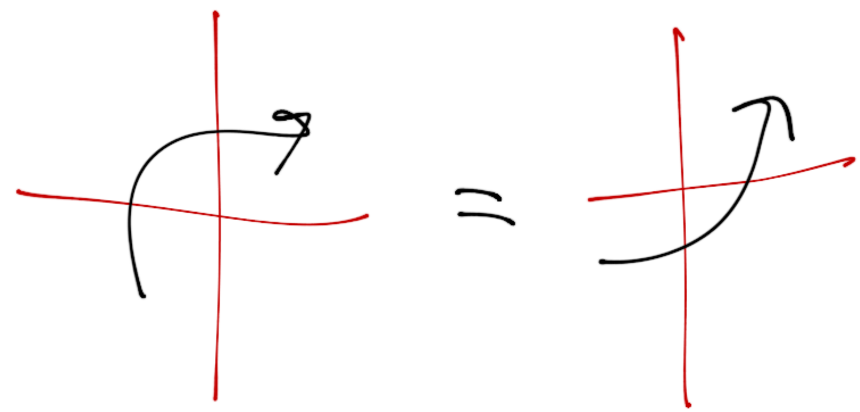
Here φ_i are the weights of V w/ mult. and m_i the corresponding weight of F acting on the φ_i weight space. In practice, we'll often specialize by a homomorphism $H^*_F(pt) \to \mathbb{C}$.

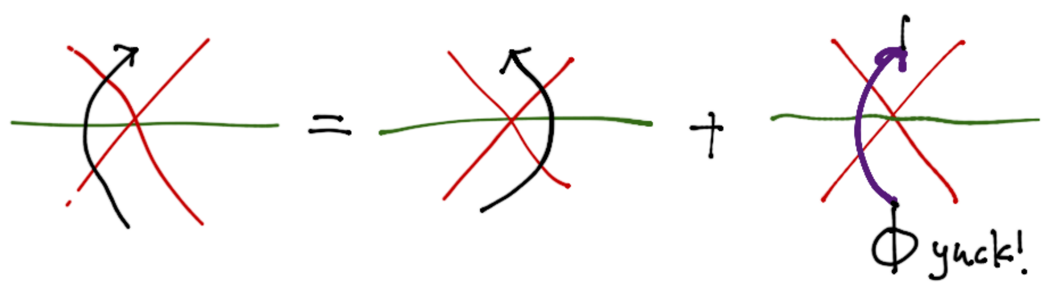


commutation w/ $\hat{W}$ is as you would expect.









So, what are we hoping to achieve?
Well, I wanted to understand the representation theory of this algebra.

How does this way of thinking about things help?

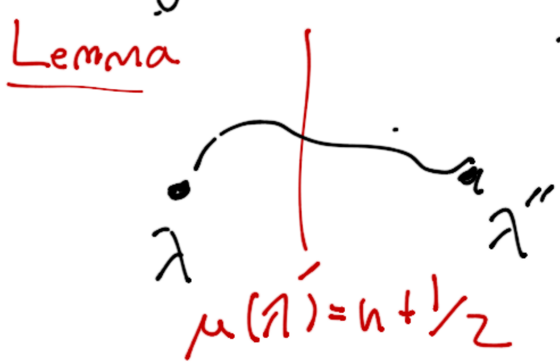
We can instead consider representations of $\mathcal{B}$ rather than just A_n. We specialize to $n=1$, for simplicity.

Def We call a f.d. representation $M(\lambda)$ of $\mathcal{B}$ weight if for each generic λ, $H^*(\mathfrak{X}_\lambda) = \mathbb{C}[t]$ acts locally finitely.
Note: We have a weight space of a point $(\lambda, \nu) \in \mathfrak{t} \times \mathfrak{t}$, looking at $M(\lambda)/m_\nu^N M(\lambda)$ for $N \gg 0$.

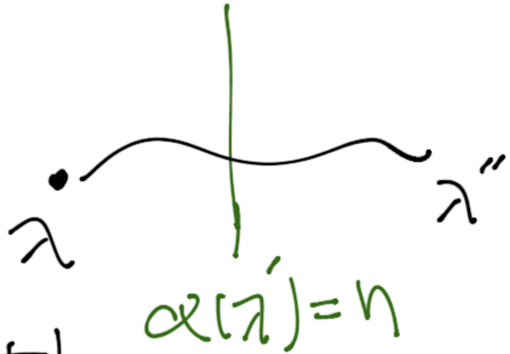
This is weird since $M(\lambda)$ is constant on small enough balls, whereas the dependence on ν is "spiky." For each λ, finitely many cosets of $\mathfrak{t}_{\mathbb{Z}}$ have non-zero weight spaces.

Note: $M(\lambda)_\nu = M(w\lambda)_{w\nu}$. Thus $M(0)_\nu \cong M(0)_{w\nu}$ for $w \in W$, and $M(\lambda)_\nu \cong M(\lambda-\nu)_0$ if ν is integral.

· We also have a lot of isomorphisms of weight spaces given by paths.



If the only hyperplane separating λ and λ'' is $\mu(\lambda)=n+1/2$ then $M(\lambda)_\nu \cong M(\lambda'')_\nu$ unless $\langle \mu, \nu\rangle = n+1/2 - m_i$.



If it is $\alpha(\lambda')=n$, then $M(\lambda)_\nu \cong M(\lambda'')_\nu$ (always).

Thus we can define an equivalence rel. on $\mathfrak{t}^\vee$ by $\nu \sim \nu'$ if $M(\cdot)_\nu \cong M(\cdot)_{\nu'}$ for all reps M (i.e. there is an isomorphism of functors).

Combining these observations, we find that:

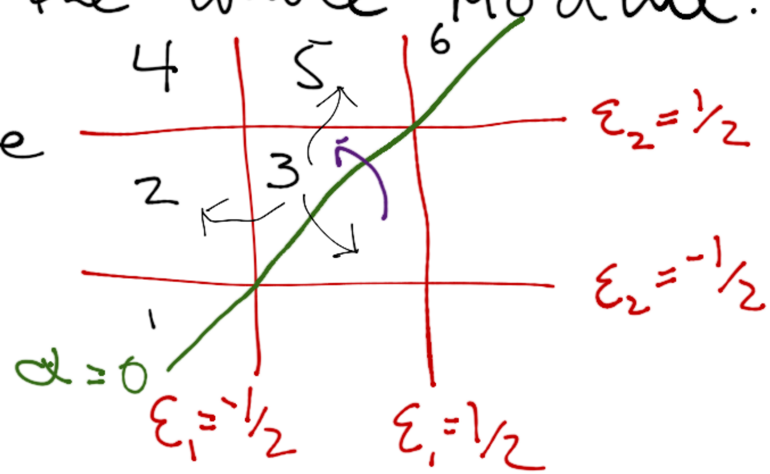
Prop Two dominant weights ν and ν' are related by $\nu \sim \nu'$ if they are not separated by a hyperplane $e_i = -m_i$ for $m_i \in \mathbb{Z} + 1/2$.

What good does this do? I can try to capture the essence of an $\mathcal{A}$-module by just looking at one weight space in each chamber (equivalence class).

Q: What structure do we need on these to reconstruct the whole module?

My favorite example

$GL(2) \circlearrowright \mathbb{C}^2 \oplus \mathbb{C}^2$
$m_* = 1/2 \quad m_* = -1/2$



Info needed: action of paths joining one chamber to another. Need to also consider images of chambers under W.

These paths need to be normalized so that they don't depend on the representative of each equivalence class.

This has additional benefit of simplifying the relations of the algebra, giving it a more combinatorial flavor, and relation to the Higgs.

Thm Cat $\mathcal{O}$'s for Higgs and Coulomb are Koszul dual. More details in Toronto Oct. 6.

Interesting extension: over a field of characteristic p the quantization has large center $\cong \mathbb{F}_p[\widetilde{\mathcal{M}}]$. This gives an Azumaya algebra on $\widetilde{\mathcal{M}}$. Kaledin: with a little fancy footwork, we can turn this in a tilting generator.

Def A tilting generator on a scheme X is a coherent sheaf $\mathcal{F}$ such that $\mathbb{R}\mathrm{Hom}(\mathcal{F},-): D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{End}(\mathcal{F})\text{-mod})$ is an equivalence of categories.

Thm: If $\widetilde{\mathcal{M}}$ is smooth, then the sum of $\mathrm{Hom}(\lambda, 0)$ for $\lambda \in \mathfrak{t}/_{\mathbb{Z}}$ generic, representing all $\widehat{\omega}$-orbits of chambers, gives a tilting generator on $\mathcal{M}$ (over $\mathbb{C}$).

One can even extend this to get a TG for each integral maximal ideal of Z! This is a concrete realization of Bezrukavnikov-Kaledin program