Pure Math 945
Combinatorial Representation Theory

Below are some problems it might be interesting to work on. They are purely optional, and I do not expect write-ups of them. I am happy to discuss them if you have problems or questions.

1) Write down all partitions with 6 parts and their Young diagrams.
   - (a) Draw the Bratelli diagram for the inclusion of $S_5 \subset S_6$.
   - (b) Compute the dimensions of each corresponding representation of S_6. (It might help to recall/calculate the dimensions for S_5, and then use the Bratteli diagram.)
   - (c) Compute the character table of S_6 via the Murnaghan-Nakayama rule.

2) Confirm that $V_{\lambda^T/\nu^T} \cong V_{\lambda/\nu} \otimes \mathrm{sgn}$ using the Young semi-normal or normal form.

3) Confirm that the Young semi-normal form for a partition λ also defines a representation of the group algebra $\mathbb{F}_p S_n$ over the finite group with p elements for p a prime if λ does not contain a hook of length $p+1$ (i.e. there are at most p boxes in the union of the longest row and longest column). Show that in this case, the associated representation over $\mathbb{F}_p$ is still irreducible.

4) Show that the representation of S_n associated to a hook $\lambda = (n-m, 1, \ldots, 1)$ is isomorphic to the exterior power $\bigwedge^m V_{(n-1,1)}$ where $V_{(n-1,1)}$ is the unique non-trivial subrepresentation of the usual representation of ^{n}. Show directly in this case that tensoring with the sign representation gives the transpose of partitions.

5) Consider the degenerate affine Hecke algebra H_2 generated by s, x_1, x_2 with the usual relations
$$s^2 = 1 \quad x_1x_2 = x_2x_1 \quad sx_2 - x_1s = 1 = x_2s - sx_1$$
   Consider the quotient of H_2 by the 2-sided ideal generated by $f(x_1)$ for some polynomial f. Show that the possible eigenvalues of a simultaneous eigenvector of x_1 and x_2 in a representation of this quotient are exactly:
   - (a) $(\alpha, \alpha \pm 1)$ with α a root of f,
   - (b) (α, β) with both $\alpha \neq \beta$ roots of f and
   - (c) (α, α) with α a double root of f.