Tensor products and categorification

Ben Webster

University of Waterloo
Perimeter Institute for Mathematical Physics

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UNIVERSITY OF WATERLOO
FACULTY OF MATHEMATICS
Department of Pure Mathematics

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS
PI
INSTITUT PÉRIMÈTRE DE PHYSIQUE THÉORIQUE

Basic question of representation theory:

What do categories of representations of Lie (super)algebras $\mathfrak{g}$ ($\mathfrak{G}$) or quantum groups $U_q(\mathfrak{g})$ look like?

If you're used to finite dimensional modules over Lie algebras, maybe doesn't sound that exciting, though these are very interesting as tensor categories.

Basic principle of this talk:

Interesting (non-semi-simple) categories of representations become easier to understand when you think of them as modules for the action of finite-dimensional representations (as a tensor category).

Category of fd modules is "big" but it's not as big as you might think.

Most interesting way to break it down is to restrict to modules locally finite over the center $Z = Z(U)$ of U, a (quantized) universal enveloping algebra.

Any such module which is indecomposable is killed by some power of a maximal ideal $I \subset Z$. Let $I_0 \in \mathrm{MaxSpec}(Z)$ be the maximal ideal annihilating the trivial module.

For $U = U(\mathfrak{g})$, Harish-Chandra shows $\mathrm{MaxSpec}(Z) \cong \mathfrak{h}^*/W$, with the maximal ideal killing a module with highest weight $\lambda + \rho$ or lowest weight $\lambda - \rho$ corresponds to the orbit $[\lambda]$.

Given $\chi \in \mathrm{MaxSpec}(Z)$, let $\mathcal{C}_\chi$ be category of modules killed by I_χ^N for $N \gg 0$.

Note, every object in this category has an action of $\widehat{Z}_\chi = \varprojlim Z/I_\chi^N$.

For every f.d. $\mathfrak{g}$-module V and $\chi, \chi' \in \mathrm{MaxSpec}(Z)$, we have a functor

$$\mathrm{pr}_{\chi'}(V \otimes -)\colon \mathcal{C}_\chi \to \mathcal{C}_{\chi'}$$

Sums of summands of these functors are called **projective functors**.

Theorem (Bernstein-Gelfand, Soergel)

The category of projective functors $\mathcal{C}_0 \to \mathcal{C}_0$ is equivalent to the category of completed Soergel $\widehat{Z}_0$-$\widehat{Z}_0$ bimodules (under isomorphism $\widehat{Z}_0 \cong \mathbb{C}[[\mathfrak{h}^*]]$). In particular, it has $\#W$ indecomposable objects.

Applications:

- shows projective functors are a categorification of the Hecke algebra.
- key to Soergel's description of category $\mathcal{O}$, Koszul self-duality.
- Elias-Williamson's proof of the Kazhdan-Lusztig conjecture.

Unfortunately:

- doesn't cover singular blocks
- tricky to generalize to quantum/super cases

In order to get at those trickier objects, need to sacrifice working with all types at once. Specialize to

$$\mathfrak{g} = \mathfrak{gl}(n) \qquad \mathfrak{G} = \mathfrak{gl}(m|n)$$

Don't try to think about all representations at once either. Just focus on tensor with vector rep $V = \mathbb{C}^n$ ($\mathbb{C}^{m|n}$) and its dual V^*.

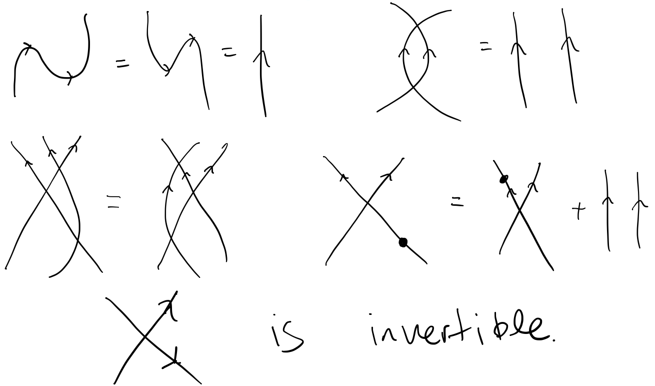
Consider the functors

$\mathcal{E} = V \otimes -$ =  $\mathcal{F} = V^* \otimes -$ = 

Morphisms between compositions of these functors are generated by

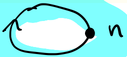
$\Omega = \sum e_{ij} \otimes e_{ji}$    swap    adjunction 

We can describe all the relations between these natural transformations. These define the **level 0 Heisenberg category** or **affine oriented Brauer category**.



There's also a q-deformed version of these relations for $U_q(\mathfrak{gl}(m|n))$.

But how do we see the projective functors and block decomposition?

The "bubbles" 

generate Harish-Chandra center. Thus, we can separate the blocks by the eigenvalues of these bubbles.

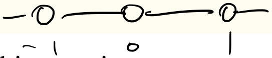
We can also diagonalize (the ss part of) the natural transformation Ω:

$$\mathcal{E} = \bigoplus_{i \in \mathbb{C}} \mathcal{E}_i \qquad \mathcal{F} = \bigoplus_{i \in \mathbb{C}} \mathcal{F}_i$$

Each $\mathcal{E}_i$ is $\mathcal{E}$ composed with projection to a block depending on i, so this has roughly the same effect as the discussion before of projections, but with better controlled combinatorics.

Theorem (Rouquier, Brundan-Savage-W.)

On integral blocks, the functors $\mathcal{E}_i, \mathcal{F}_i$ define a categorical action of $\mathfrak{sl}_\infty$ (with Dynkin diagram identified with $\mathbb{Z}$).



Definition of categorical action: too long for this margin.

Natural categories can thus be understood in terms of which module over $\mathfrak{sl}_\infty$ they categorify:

type of module	$\mathfrak{gl}(n)$	$\mathfrak{gl}(m\|n)$
finite dimensionals	$\bigwedge^n \mathbb{C}^\infty$	$\bigwedge^m \mathbb{C}^\infty \otimes \bigwedge^n (\mathbb{C}^\infty)^*$
category $\mathcal{O}$	$\bigotimes^n \mathbb{C}^\infty$	$\bigotimes^m \mathbb{C}^\infty \otimes \bigotimes^n (\mathbb{C}^\infty)^*$
Whittaker	$\mathrm{Sym}^n(\mathbb{C}^\infty)$	$\mathrm{Sym}^m \mathbb{C}^\infty \otimes \mathrm{Sym}^n(\mathbb{C}^\infty)^*$

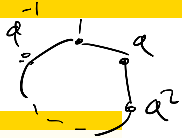
We can also consider intermediate versions of these categories: parabolic $\mathcal{O}$, non-principal Whittaker which give tensors of $\bigwedge$ / Sym. Other interesting categories include Gelfand-Tsetlin modules.

Theorem (Brundan-Losev-W.)

In all the cases above, these categories are the unique categorifications of these modules with a categorical $\mathfrak{sl}_\infty$ action and some additional technical properties. In all of the cases on the previous page, characters of simples can be computed from canonical basis.

- proves existence of graded lift and Koszulity for category $\mathcal{O}$ over $\mathfrak{gl}(m|n)$.
- this gives a super-duality-free proof of Brundan's analogue of the Kazhdan-Lusztig conjecture. Later work of Leonard unifies this with Chen and Wang's proof.
- shows that this $\mathfrak{sl}_\infty$-action replaces Soergel bimodules.
- gives an explicit equivalence to the same categories over $U_q(\mathfrak{g}/\mathfrak{G})$ with q generic.

If q is an pth root of unity, then changes to a $\widehat{\mathfrak{sl}}_p$-action.



The uniqueness argument we used before breaks down, so more thought needed here.

On the other hand, the finite dimensional case is now suddenly interesting.

Theorem (Riche-Williamson)

The category of finite-dimensional modules over $U_q(\mathfrak{gl}(n))$ or $GL_n(\mathbb{F}_p)$ are the unique categorification of $\bigwedge^n \mathbb{C}^\infty$ in this characteristic. The principal block of finite-dimensional modules is the unique categorification of the anti-spherical module over the affine Hecke category.

How does this story generalize to other types?

For types BCD, there is an obvious analogue of the Heisenberg category: **the affine Brauer category**.

This is specifically designed to match the functor

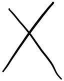
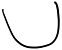
$$\mathcal{E} = V \otimes - =$$

of tensor product with the vector representations V in these types.

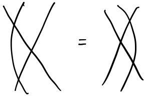
Since this representation is self-dual, only need one functor and no orientations.

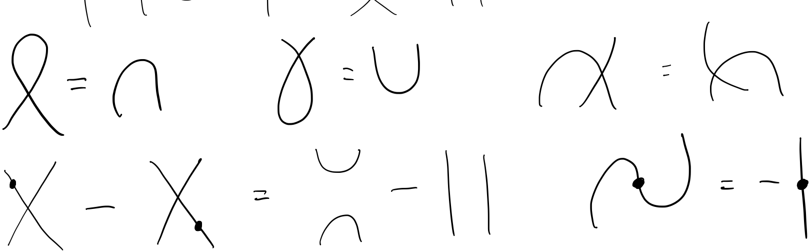
The affine Brauer category

Generators are effectively the same:

Ω'  swap  adjunction  

Relations are those of Nazarov-Wenzl algebras:





Assume $\mathfrak{G} = \mathfrak{osp}(m|2n)$.

Theorem (Rui-Song)

The functor $\mathcal{E}$ of tensor product with the vector representation defines an action of the affine Brauer category on $U(\mathfrak{G})$-modules.

I'm taking some liberties here: they don't state this theorem quite so generally.

Just as in the type A case, the natural thing to do here is decompose with respect to (the ss part of) Ω'.

$$\mathcal{E} = \bigoplus_{i \in \mathbb{C}} \mathcal{E}_i$$

It's convenient to write $\mathcal{F}_i = \mathcal{E}_{-i}$.

This recovers the relation that $\mathcal{F}_i$ is the left and right adjoint of $\mathcal{E}_i$.

In some cases, this lets us go back to the type A case:

Theorem (W.)

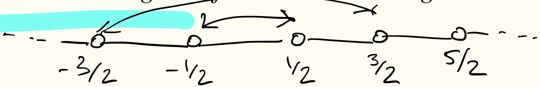
If $\tau \in \mathbb{C}$, and $2\tau \notin \mathbb{Z}$, then $\mathcal{E}_i, \mathcal{F}_i$ for $i \in \mathbb{Z} + \tau$ define a categorical $\mathfrak{sl}_\infty$-action on $U(\mathfrak{G})$-modules locally finite over Z.

Of course, these functors act trivially on integral blocks, so this sounds a little boring, but we can often deform (for example, as in category $\mathcal{O}$), to a more generic category which will carry one of these actions.

The functors which are non-trivial on integral blocks are $\mathcal{E}_i$ for $i \in \frac{m-1}{2} + \mathbb{Z}$.

Theorem (Bao-Shan-W.-Wang, W.)

The functors $\mathcal{E}_i$ for $i \in \frac{1}{2} + \mathbb{Z}$ induce a categorical action of the quantum coideal subalgebra for the Satake diagram below:



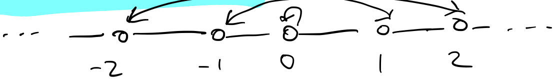
They don't use the strong categorification, but the fact that category $\mathcal{O}$ for $\mathfrak{osp}(m|2n)$ categorifes the tensor product

$$\bigotimes^{\lfloor m/2 \rfloor} \mathbb{C}^{\infty} \otimes \bigotimes^{n} (\mathbb{C}^{\infty})^*$$

is key to Bao and Wang's version of Brundan-Kazhdan-Lusztig in this case.

Categorified coideal subalgebras

Obviously, we expect that $\mathcal{E}_i$ for $i \in \mathbb{Z}$, we will get a categorical action of the quantum coideal subalgebra for the Satake diagram below:



Unfortunately, we don't know a definition of one! There should be hints in Bao-Wang's $\imath$divided powers.

The functor $\mathcal{E}_0$ is a strange beast, and requires more investigation.

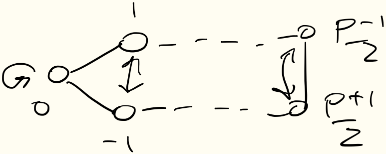
Conjecture

The same categories we considered before (finite-dimensionals, parabolic $\mathcal{O}$, Whittakers) are again characterized and given a graded lift by their structure as categorical modules over categorified coideal subalgebras.

Important motivation for me: fd modules over $U_q(\mathfrak{g})$ for q a pth root of unity or $G(\mathfrak{F}_p)$.

Conjecture

These categories are the unique categorifications of $\otimes^n(\mathbb{C}^\infty)$ *as a module for the appropriate quantum coideal subalgebra in* $\widehat{\mathfrak{sl}}_p$*, with Satake diagram:*



This gives an explicit identification of the principal block with the anti-spherical module over the Langlands dual affine Hecke category.

This last identification is already proven by Achar-Makisumi-Riche-Williamson (but using some much more serious technology).

Thanks

Thanks for listening.