Research Statement
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1. Introduction

My research centers around investigating questions in representation theory, knot theory and combinatorics from a geometric perspective. My most important tool at the moment is certain non-commutative algebras constructed using the techniques of geometric representation theory, and categories of modules over them which generalize the BGG category $\mathcal{O}$ of a semisimple Lie algebra. These categories are relevant to problems in physics, homological knot invariants, canonical basis theory, and the study of finite dimensional algebras.

These algebras are constructed by deformation quantization of certain symplectic varieties, the most interesting case being when this variety is a resolution of an affine singularity. Universal enveloping algebras $U(\mathfrak{g})$ are obtained in the case where the variety in question is T^*G/B for the corresponding group. In view of this analogy, one perspective on my research plans is that I intend to extend to these algebras many theorems and constructions of classical Lie theory, such as the Beilinson-Bernstein localization theorem, the theory of primitive ideals, and the structure of the category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand [?].

While these questions are interesting for their own sake, I will also investigate their applications in other fields. One classical example, which suggests a model for my work, is the construction of Lusztig's canonical basis of $U_q(\mathfrak{n})$ [?]. Lusztig showed the existence of this basis by constructing a category of sheaves on a quiver variety whose Grothendieck group is this quantum group. Considering the classes of simple objects in that categorification gives a basis whose existence is not at all clear from an algebraic perspective.

Lusztig's construction can be generalized in a way which allows the production a large number of interesting categories. These categories can be thought of as analogues of category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand, which is the subcategory of representations of a semi-simple (or Kac-Moody) Lie algebra $\mathfrak{g}$ generated in a certain sense by the Verma modules [?].

1.1. **Symplectic resolutions.** The geometric objects which will appear in this construction are symplectic resolutions of singularities, which have attracted great interest in recent years as their theory has been developed by Kaledin and others (see the survey [?]).

Definition 1. *A* **symplectic resolution of singularities** *is a pair consisting of*

- *a resolution of singularities $\pi : Y \to X$ and*
- *a holomorphic symplectic form $\omega \in \wedge^2 T^*Y$.*

For simplicity, I will assume throughout that X is affine. Examples of such singularities include:

- The nilpotent cone $X = \mathcal{N}_{\mathfrak{g}}$ for a semi-simple Lie group G has a resolution given by $Y = T^*(G/B)$ for any Borel B.
- For any finite subgroup $\Gamma \subset \mathrm{SL}(2)$, the affine quotient surface $\mathbb{C}^2/\Gamma$ has a unique symplectic resolution $\widetilde{\mathbb{C}^2/\Gamma}$.
- More generally, the symmetric power $X = \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ has a resolution given by the Hilbert scheme $Y = \mathsf{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma})$.
- For any action of a compact group K on a complex vector space V, there is a hyperkähler structure on T^*V, and we can consider the hyperkähler quotient $\mathfrak{M}_{\boldsymbol{\alpha}} = T^*V////_{\boldsymbol{\alpha}}K$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ are a triple of moment map values. This quotient may be singular, but it always carries a hyperkähler structure on its generic locus. In fact, if $\mathfrak{M}_{\boldsymbol{\alpha}}$ is smooth, it is a symplectic resolution of $\mathfrak{M}_{\boldsymbol{\alpha}'}$ where $\boldsymbol{\alpha}' = (0, \alpha_2, \alpha_3)$. This allows us to construct a large number of examples of symplectic resolutions, including
  - the quiver varieties of Nakajima [?, ?, ?] and
  - the hypertoric varieties studied by Bielawski-Dancer, Hausel-Sturmfels, Konno and others [?, ?, ?], as well as in joint work of myself and Proudfoot [?] (for more information, see the survey [?]).

1.2. **Categories $\mathcal{O}$ and categorification.** I will describe (in Section ??) a construction which takes as input a symplectic resolution of singularities, and a pair of compatible $\mathbb{C}^*$-actions on X and Y, which we denote $\xi : \mathbb{C}^* \to \mathsf{Aut}(Y \to X)$. The action is assumed to act with positive weight on the symplectic form ω on Y. The result is a category $\mathcal{O}_Y^\xi$. My intent is to study these categories and the geometry of symplectic resolutions from this perspective.

Several of these categories have already appeared in the literature and are of interest to representation theorists. For example:

- If $Y = T^*(G/B)$, then by the localization theorem of Beilinson-Bernstein [?], this is a block of the original BGG category $\mathcal{O}$ for $\mathfrak{g}$.

- If X is a slice to a nilpotent orbit, which carries an appropriate $\mathbb{C}^*$ action, we obtain category $\mathcal{O}$ for a finite W-algebra by a theorem of Ginzburg [**?**].
- If $Y = \mathsf{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma})$, we obtain category $\mathcal{O}$ for the rational Cherednik algebra of the wreath product $S_n \wr \Gamma$ by work of Ginzburg-Gordon-Stafford [**?**].

In addition to their representation theoretic interest, these categories are connected to a number of interesting problems:

- There is a connection between these categories and 3-dimensional, $N = 4$ supersymmetric field theories in physics. In particular, several singularities which are dual to each other under a mirror duality in physics (different from the mirror symmetry for 2-dimensional field theories familiar to mathematicians) have associated categories $\mathcal{O}$ which are Koszul dual, and in particular, derived equivalent. Interestingly, these sorts of connections are rather different from the aspects of these varieties physicists have considered. However, my collaborators and I conjecture that this duality is a general phenomenon, which extends beyond these particular examples (see Section **??**).
- One prediction from physics is that this category $\mathcal{O}$ will embed inside the Fukaya category of Y. This would follow from an analogue of Nadler-Zaslow's results embedding constructable sheaves on M in the Fukaya category of T^*M [**?**].
- The derived categories $D^b(\mathcal{O}^\xi_Y)$ carry an action by autoequivalences of the fundamental group of a certain hyperplane complement (which generalize Arkhipov's twisting functors [**?**, **?**] on the BGG category $\mathcal{O}$).

  They also have a second action by the fundamental group of a second hyperplane complement, given by "parallel transport" in the generic points of the base of a universal deformation of Y (this notion is more precise for the Fukaya category $\mathrm{Fuk}(Y)$).

  These functors are quite interesting in examples, and I conjecture that the duality mentioned above interchanges these actions.
- In the case where Y is one of the quiver varieties defined by Nakajima [**?**], category $\mathcal{O}$ gives a categorification of tensor products of integrable representations of quantum groups. The geometry of quiver varieties allows us to investigate the structure of these categories, and I conjecture they can be used to construct categorical analogues of many interesting objects in representation theory, including the braiding on the category of representations of $U_q(\mathfrak{g})$ and thus the Reshetikhin-Turaev tangle invariants.
- In the case where Y is a hypertoric variety, Braden, Licata, Proudfoot and I have constructed a combinatorial version of category $\mathcal{O}$ [**?**], which we will show in forthcoming work is equivalent to the geometric definition. One

can show algebraically that this category has many of the properties we expect from a generalization of category $\mathcal{O}$: it is Koszul, highest-weight (i.e. has generalizations of Verma modules), and one can algebraically construct the twisting functors described above.
- More generally, the Grothendieck group of $\mathcal{O}_Y^\xi$ has a natural inclusion into the equivariant cohomology of Y, giving a new perspective on this cohomology, and the intersection cohomology of X. This categorical perspective gives a "cellular" structure to $H^*(Y)$ analogous to that described by Lusztig on $H^*(G/B) \cong \mathbb{C}[W]$, the group algebra of the Weyl group. These structures correspond to the decomposition of $H^*(Y)$ under the action of a finite dimensional algebra constructed by convolution, which was studied for quiver varieties by Nakajima [?] and for hypertoric varieties by myself and Proudfoot [?].

2. Categories $\mathcal{O}$ and geometric representation theory

2.1. **The definition of $\mathcal{O}_Y^\xi$.** The categories $\mathcal{O}$ have a number of different manifestations, which I will attempt to describe briefly.

If $Y \to X$ is a symplectic resolution of singularities with X affine, then by work of Bezrukavnikov and Kaledin [?], we have the following theorem:

Theorem 1. *For each $\eta \in H^2(Y, \mathbb{C})$, we have a canonical deformation A_η of $\mathbb{C}[X]$, the coordinate ring of X.*

- In the case of $X = \mathcal{N}_\mathfrak{g}$, this deformation is the quotient of $U(\mathfrak{g})$ by the kernel of a central character depending on η.
- In the case of $X = \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$, this deformation is a specialization of the spherical rational Cherednik algebra, also at a parameter depending on η.
- If X is a hyperkähler reduction (as discussed on page 2), for example, a quiver variety, then the algebra A_η is a noncommutative hamiltonian reduction of $\mathcal{D}_V$, the algebra of differential operators on V.

Given a $\mathbb{C}^*$-action $\xi : \mathbb{C}^* \to \mathsf{Aut}(Y \to X)$ which acts with positive weight on the symplectic form, we can construct (by naturality of the deformation) a corresponding $\mathbb{C}^*$-action on A_η. Let A_η^+ be the subalgebra of non-negative weight under this $\mathbb{C}^*$ action.

Definition 2. *Let $\mathcal{O}_Y^\xi$ be the category of A_η-modules M such that:*

- *M is finitely generated over A_η.*
- *The action of A_η^+ on M is locally finite.*

As we discussed in the introduction, this generalizes the usual category $\mathcal{O}$ for $U(\mathfrak{g})$ or a Cherednik algebra, though the equivalence of this definition and the usual one requires a short argument using the Poincaré-Birkoff-Witt theorem.

2.2. **Localization.** One of the most interesting themes of geometric representation theory is the interplay between sheaves of algebras such as the structure sheaf or differential operators, and their sections, most famously captured by the localization theorem of Beilinson-Bernstein [?].

In fact, there is a natural sheaf $\mathcal{A}_\eta$ on Y which "localizes" A_η: $\mathcal{A}_\eta$ is a deformation of the structure sheaf of Y, and A_η can be constructed from the sections of $\mathcal{A}_\eta$ is straightforward way in examples with appropriate $\mathbb{C}^*$ actions.

We have a natural functor $\Gamma : \mathcal{A}_\eta - \mathsf{mod} \to A_\eta - \mathsf{mod}$, essentially given by taking sections, and an adjoint to this, the localization functor $\mathsf{Loc} : A_\eta - \mathsf{mod} \to \mathcal{A}_\eta - \mathsf{mod}$. The localization functor sends category $\mathcal{O}$ to a category of sheaves supported (set-theoretically) on the subvariety

$$Y(\xi) = \{y \in Y | \lim_{t\to\infty} \xi(t)\cdot y \text{ exists}\}.$$

It follows from Grauert-Riemenschneider that for any symplectic resolution of an affine singularity, the Chern class map $c_1 : \mathrm{Pic}(Y) \to H^2(Y;\mathbb{Z})$ is an isomorphism. Let L be a relatively ample line bundle on Y. It can be shown that for all η, and for sufficiently large integers n, Loc is an equivalence for $A_{\eta+n\cdot c_1(L)}$, that is, it is an equivalence for values of η sufficiently deep within the ample cone $\mathcal{A} \subset H^2(Y;\mathbb{R}) \cong \mathrm{Pic}(Y)\otimes\mathbb{R}$ of Y. I believe this is the asymptotic version of a more precise result:

Conjecture 1. *There is a class $\rho_Y \in H^2(Y;\mathbb{R})$ such that if $\eta+\rho_Y$ lies in the interior of the ample cone $\mathcal{A} \subset H^2(Y;\mathbb{R}) \cong \mathrm{Pic}(Y)\otimes\mathbb{R}$ then the functor Loc : $A_\eta - \mathsf{mod} \to \mathcal{A}_\eta - \mathsf{mod}$ is an equivalence. This condition is also necessary if $\eta \in H^2(Y;\mathbb{Z})$.*

- For the BGG category $\mathcal{O}$, this is precisely the localization theorem [?].
- For the case of a Slodowy slice, a version of this theorem has been proven by Ginzburg [?].
- The proof of this result for hypertoric varieties will appear in future work of Braden, Licata, Proudfoot and myself.

This allows us to understand category $\mathcal{O}$ as a category of sheaves on Y, and use techniques similar to those applied to $\mathcal{D}$-modules in more classical geometric representation theory. For example, the usual techniques of mixed Hodge theory allow us to construct a graded version of $\mathcal{O}_Y^\xi$. While this grading is very important, we will not use it explicitly, and thus will use the same notation for $\mathcal{O}_Y^\xi$ and its graded lift.

It also gives a geometric interpretation of the Grothendieck group $K^0(\mathcal{O}_Y^\xi)$, or more naturally, the graded Grothendieck group $K_q^0(\mathcal{O}_Y^\xi)$ (which has base ring $\mathbb{Z}[q,q^{-1}]$, where q is grading shift). As with $\mathcal{D}$-modules, each $\mathcal{A}_\eta$ module has a characteristic cycle, defined by the sum of the classes of the components of its support variety, with multiplicity given by the graded dimension of the residue at that point. Since these cycles are not compact, we should interpret them as elements of Borel-Moore homology $H_*^{BM}(Y)$. Identifying $H^*_{\mathbb{C}^*}(pt) \cong \mathbb{Z}[q]$, we ultimately obtain a characteristic cycle map

$$CC : K^0_q(\mathcal{O}^\xi_Y) \to H_*^{BM,\mathbb{C}^*}(Y)[q^{-1}] \cong H^*_{\mathbb{C}^*}(Y)[q^{-1}].$$

We could instead consider the quantum cohomology $QH^*(Y)$, but the quantum cohomology of symplectic resolutions is not sufficiently well understood to know whether this is a more natural target for the characteristic cycle map. Moving forward, it would be very interesting to investigate these quantum cohomologies for their own interest, and to understand connections to our categories.

Conjecture 2. *If ξ has isolated fixed points, CC is an isomorphism. In other cases, its image can be geometrically described. This map intertwines the Euler product on the Grothendieck group with the equivariant intersection product (generalizing Ginzburg's index theorem* [**?**]*, which implies the corresponding result for cotangent bundles).*

Thus, at least in the case with isolated fixed points, this would give a canonical basis for $H^*_{\mathbb{C}^*}(Y)[q^{-1}]$ (and, in fact, a dual canonical basis given by the classes of projectives). This basis will satisfy the familiar properties of a canonical basis (i.e. self-duality, semi-orthogonality with standard classes).

2.3. **Twisting functors.** One interesting question to be studied is the relationship between categories $\mathcal{O}$ for different $\mathbb{C}^*$-actions on Y. The categories $\mathcal{O}^\xi_Y$ and $\mathcal{O}^{\xi'}_Y$ are both subcategories of $A_\eta-\mathsf{mod}$ (or when thinking geometrically $\mathcal{A}_\eta-\mathsf{mod}$). One can show the existence of projection functors $\pi_\xi : D^b(A_\eta - \mathsf{mod}) \to D^b(\mathcal{O}^\xi_Y)$, left adjoint to the inclusion ι_ξ. This allows one to define a canonical functor

$$\Phi_{\xi,\xi'} = \pi_{\xi'} \circ \iota_\xi : D^b(\mathcal{O}^\xi_Y) \to D^b(\mathcal{O}^{\xi'}_Y).$$

Given a $\mathbb{C}^*$-action ξ with positive weight k on ω, we can take the corresponding vector field $\log\xi$, which satisfies $L_{\log\xi}\omega = k\omega$. Thus $\nu = k^{-1}\log\xi$ satisfies $L_\nu\omega = \omega$. We can think of the space of such vector fields as a sort of linearization of set of appropriate $\mathbb{C}^*$-actions.

Definition 3. *We say that a vector field ν such that $L_\nu\omega = \omega$ is* **matched** *to ξ if $[\nu, \log\xi] = 0$ and $Y^{\xi(\mathbb{C}^*)}$ is precisely the zero set of ν.*

Let M_ξ be the space of vector fields matched to ξ. This is naturally a complex affine space (modeled on the space of symplectic vector fields commuting with ξ and vanishing on $Y^{\xi(\mathbb{C}^*)}$) minus some affine subspaces where the zero sets are too large.

Conjecture 3. *If $\log\xi'$ is matched to ξ, then the functor $\Phi_{\xi,\xi'}$ is an equivalence.*

Interestingly, this equivalence is not canonical. For example, $\Phi_{\xi,\xi'} \circ \Phi_{\xi',\xi} \not\cong \mathrm{Id}$. Instead, these functors generate an interesting groupoid.

Definition 4. *We call a autofunctor of $D^b(\mathcal{O}_Y^{\xi_0})$ a* **twisting functor** *if it is of the form $\Phi_{\xi_n,\xi_0} \circ \Phi_{\xi_{n-1},\xi_n} \circ \cdots \circ \Phi_{\xi_0,\xi_1}$ for some sequence of $\mathbb{C}^*$-actions such that $\log \xi_i$ is matched to ξ_0 for all i.*

Conjecture 4. *There is a natural map $\pi_1(M_\xi) \to \mathsf{Aut}(D^b(\mathcal{O}_Y^\xi))$ whose image is the twisting functors.*

This map is typically nontrivial; for example, the Serre functor should be the image of a distinguished element of $\pi_1(M_\xi)$. In the usual BGG category $\mathcal{O}$, these specialize to Arkhipov's twisting functors. The twisting functors for a hypertoric variety are explicitly constructed (in terms of bimodules) by Braden, Licata, Proudfoot and myself in [**?**].

As we mentioned in the introduction, this action of a fundamental group at the moment lacks a good geometric explanation (instead, my understanding of Conjecture **??** passes through a combinatorial description of $\pi_1(M_\xi)$). In Section **??**, we will describe a conjectural duality between symplectic singularities, which gives a derived equivalence between $\mathcal{O}_Y^\xi$ to category $\mathcal{O}$ for another variety $Y^\vee$, and I conjecture this equivalence intertwines the twisting functors with geometric functors given by monodromies in a deformation of $Y^\vee$ as a symplectic variety with base M_ξ.

2.4. **The Fukaya category and A-branes.** This category also has an identity in physics. The sheaf of the algebras $\mathcal{A}_\eta$ can be thought of as the endomorphisms of the "canonical coisotropic brane" B_{can} corresponding to a hyperkähler structure on Y such that $\eta = [\omega_\mathbb{R}]$. Thus, $\mathcal{A}_\eta$-modules should correspond to A-branes on Y, and we have a functor $\mathrm{Fuk}(Y) \to \mathcal{A}_\eta - \mathsf{mod}$ given by $\mathrm{Hom}(B_{can}, -)$. Analogously with the work of Nadler-Zaslow [**?**], I expect this functor will be full and faithful, with image given by $\mathcal{A}_\eta$-modules which are "regular" and have Lagrangian support (that is, are "holonomic").

Of course, this functor currently only exists at "the physical level of rigor," but given the success of the work of Nadler-Zaslow, I believe this connection to symplectic geometry can be made precise, and in many cases will illuminate the structure of the category $\mathcal{O}_Y^\xi$.

3. Geometric knot homology

3.1. **Quiver varieties.** One interesting direction one is led in the study of these categories is the construction of homological knot invariants. Stroppel [**?**] and Sussan [**?**] have given constructions of knot invariants for fundamental representations of $\mathrm{SL}(n)$ using the BGG category $\mathcal{O}$ for the Lie algebra $\mathfrak{sl}_m$, building on and generalizing earlier work of Bernstein-Frenkel-Khovanov [**?**], Frenkel-Khovanov-Stroppel [**?**], Khovanov [**?**, **?**] and Khovanov-Rozansky [**?**].

I plan to generalize of this construction using category $\mathcal{O}$ for the quiver varieties described by Nakajima [**?**]. Nakajima shows how each tensor product of integrable representations of Kac-Moody algebras has an associated quiver variety equipped with a $\mathbb{C}^*$ action [?], and Zheng has shown (in different language) how the category $\mathcal{O}$ associated to this variety is a categorification of the aforementioned tensor product, with the action of the quantum group defined by correspondences [?].

As mentioned in the introduction, there are twisting functors for these categories, relating tensor products of the same factors in possibly different orders, which correspond to different $\mathbb{C}^*$-actions.

Conjecture 5. *The twisting functors descend, on the level of the Grothendieck group, to the usual braiding on the category of $U_q(\mathfrak{g})$-modules, defined by the universal R-matrix (up to a ribbon twist).*

The analogous result holds in the case where $\mathfrak{g} = \mathfrak{sl}_n$, and all representations in the tensor product are fundamental, and this underpins the earlier work on knot homology using category $\mathcal{O}$.

There are also good candidates for the evaluation and coevaluation of quantum group representations, given by a certain vanishing cycle functor on a subvariety. The existence of a braiding and (co)evaluation were enough for Reshetikhin and Turaev to construct quantum invariants of knots and links [?], so given a system of categories $\mathcal{O}$ with twisting functors, evaluation, and coevaluation satisfying appropriate relations, one can define a categorical tangle invariant, and in particular, a bigraded vector space to each knot. Better yet, the special case of original Khovanov homology was shown to be functorial (up to sign) by Jacobsson [?] and this functoriality was interpreted in the representation-theoretic context by Stroppel using the adjunction of twisting functors [?, ?]. This should extend to the quiver variety case, and I expect to be able to use the theory of disoriented cobordisms due to Morrison, Walker, and Clark [?] to produce an honestly functorial invariant (with no sign problems) from this perspective.

Conjecture 6. *For each finite dimensional semi-simple Lie algebra $\mathfrak{g}$, there is a functorial invariant of links labeled with representations of $\mathfrak{g}$, valued in bigraded vector spaces, whose Euler characteristic is the Reshetikhin-Turaev invariant of this knot.*

One particularly interesting possibility is that this may allow us to construct some sort of homological version of Witten-Reshetikhin-Turaev invariants. In this case, the difficulties of defining WRT invariants for q not a root of unity could be explained by the fact that infinite dimensional vector spaces often do not have well defined graded dimensions.

3.2. **HOMFLY homology and generalizations.** Another interesting feature of knot homology theories is the existence of stable limits as the rank of the group in question approaches infinity. The best known of these is HOMFLY homology, first defined by Khovanov and Rozansky [?], and further developed by Khovanov [?]. Such theories are expected to exist based on physical reasoning, as noted by Gukov and Walcher [?].

Studying the stabilization of quiver varieties under diagram expansion, using constructions similar to those which appear in my earlier work on tensor product multiplicity stabilization [?], could be used to construct these limits, in terms of a stable category associated to $\mathfrak{gl}_\infty$.

Interestingly, HOMFLY homology has appeared in a slightly different geometric context, as described by myself and Williamson [?].

It is encoded in the geometry of the space $P_{\boldsymbol{\alpha}} = P_{\alpha_1} \times_B \cdots \times_B P_{\alpha_n}$ where α_i is a simple root of $\mathrm{SL}(n)$, B is the upper triangular subgroup and $P_\alpha = \overline{Bs_\alpha B}$ is the block upper triangular subgroup, where the only block of size bigger than 1 is $\{i, i+1\}$, and $\sigma = \sigma_{\alpha_1}^{\pm} \cdots \sigma_{\alpha}^{\pm 1}$ is a braid representative of our knot. This variety is an analogue of the Bott-Samelson varieties which resolve Schubert varieties, but equipped with the canonical map $m : P_{\boldsymbol{\alpha}} \to \mathrm{SL}(n)$ given by multiplication to the group $\mathrm{SL}(n)$ rather than the flag variety $\mathrm{SL}(n)/B$. The space $P_{\boldsymbol{\alpha}}$ has a natural $B \times B$-action given by left multiplication and right multiplication, twisted by inverse. If we consider the diagonal subgroup $B_\Delta \subset B \times B$ this acts by an analogue of the conjugation action, and, in fact, m intertwines these B_Δ-actions.

Most concretely, HOMFLY homology can be understood in terms of the equivariant cohomology of certain subvarieties of $P_{\boldsymbol{\alpha}}$, corresponding to subwords $\boldsymbol{\alpha}' \prec \boldsymbol{\alpha}$. It is the homology of the complex whose total space is $\bigoplus_{\boldsymbol{\alpha}' \prec \boldsymbol{\alpha}} H^*_{B_\Delta}(P_{\boldsymbol{\alpha}'})$. The differentials are given by combinations of pushforward and pullback of equivariant cohomology, which are determined by the exponents ϵ_i.

A more geometrically natural, but technically more sophisticated approach to describing this complex can be given by considering a filtration on the sheaf of equivariant cochains (subject to compactness conditions determined by ϵ_i) on the open subset

$$Bs_{\alpha_1}B \times_B \cdots \times_B Bs_{\alpha_n}B \subset P_{\boldsymbol{\alpha}}.$$

This is the *weight filtration* which (crudely speaking) measures the failure of the Hodge theorem on small neighborhoods of points (this filtration is discussed from the perspective of Hodge theory in [?], or from an algebraic perspective in [?]).

While the B_Δ-equivariant cohomology of the whole complex does not depend in an interesting way on the knot we consider, the successive quotients of this filtration on the complex of cochains do. In fact, they are the equivariant cohomologies $H^*_{B_\Delta}(P_{\boldsymbol{\alpha}'})$, and one can show that that 2nd differential in the spectral sequence is precisely the one we did not describe above. Thus, the E^2 term of the spectral sequence converging to the equivariant cohomology of this open subset is the HOMFLY homology of the link.

Williamson and I are currently working on extending this second description of HOMFLY homology to colored HOMFLY homology, the stable limit for arbitrary wedge-powers of the standard representation. The existence of this theory was predicted on physical grounds, and Khovanov, Lauda and Mackaay are currently preparing an algebraic description.

Fix a braid σ and choice of coloring $\mathbf{c} = \{c_1, \cdots, c_n\}$. Let $C = \sum_i c_i$, and let $P \subset \mathrm{SL}(C)$ be the block upper triangular subgroup of block sizes $\mathbf{c}$.

Conjecture 7. *There is a natural filtered complex of sheaves on* $\mathrm{SL}(C)$*, such that the* E_2 *page of the spectral sequence calculating the* P_Δ*-equivarariant cohomology of this complex is a categorification of the colored HOMFLY polynomial. This spectral sequence is a knot invariant.*

This theory is rather difficult to construct algebraically (which is why only the version using V and $\wedge^2 V$ has appeared [**?**]), so in this case, using geometry considerably simplifies the calculations involved in checking invariance, as well as providing motivation.

4. Mirror duality and category $\mathcal{O}$

4.1. **A duality for singularities.** I am also working on another direction of research, jointly with Braden, Licata, and Proudfoot, on understanding a notion of duality between symplectic singularities. While we're still investigating the best set of hypotheses for the conjecture below, I will state a maximally optimistic version here and a more concrete conjecture using quantum field theory in Conjecture **??**.

Conjecture 8 (Braden, Licata, Proudfoot, Webster)**.** *For each affine symplectic singularity* X*, there is a dual singularity* $X^\vee$ *such that*

(1) *there is an order reversing bijection between strata of* X *and* $X^\vee$ *(by* [**?**]*, there is a canonical Whitney stratification given by symplectic leaves).*
(2) *there is a bijection between symplectic* $\mathbb{C}^*$ *actions on* X*, up to conjugacy, and pairs consisting of a partial symplectic resolution of* $X^\vee$*, and a choice of relatively ample line bundle. Thus, to each resolution* $Y \to X$ *with* $\mathbb{C}^*$*-action* ξ*, we have a dual* $Y^\vee \to X^\vee$ *with action* $\xi^\vee$*.*
(3) *Furthermore, the categories* $\mathcal{O}_Y^\xi$ *and* $\mathcal{O}_{Y^\vee}^{\xi^\vee}$ *are Koszul dual, and, in particular, derived equivalent. This duality switches the action of monodromy and twisting functors.*

For example:

- If $Y = T^*G/B$ then $Y^\vee = T^{*L}G/^L B$, and this duality claim is equivalent to the Koszul duality theorem for categories $\mathcal{O}$ due to Beilinson, Ginzburg, and Soergel [**?**].
- If X is a hypertoric singularity associated to a hyperplane arrangement H, then $X^\vee$ is another hypertoric singularity, associated to the Gale dual $H^\vee$. Braden, Licata, Proudfoot, and I have combinatorially constructed

Koszul dual categories associated to hypertoric varieties with $\mathbb{C}^*$ actions which we will show in forthcoming work are, in fact, the geometrically defined category $\mathcal{O}$ [?].

- Conjecturally, if Y is the space of G-instantons on the algebraic surface $\widetilde{\mathbb{C}^2/\Gamma}$, then $Y^\vee$ is the G' instantons on $\widetilde{\mathbb{C}^2/\Gamma'}$ where G and Γ' (resp. G' and Γ) are matched by the MacKay correspondence.
- Conjecturally, if Y is the Nakajima quiver variety for weights λ and μ in type ADE, then $Y^\vee$ is the slice to Gr_μ in $\overline{\mathrm{Gr}_\lambda}$ where Gr_μ denotes an orbit of polynomial loops in the affine Grassmannian of the Langlands dual group.

  Hopefully, this perspective will allow me to relate the homological knot invariants defined in terms of quiver varieties (including Khovanov-Rozansky homology) to those defined using the affine Grassmannian by work of Seidel-Smith [?], Manolescu [?], and Cautis-Kamnitzer [?].

In the first two cases, the Koszul duality results are well understood and should be regarded as evidence for Conjecture **??**. The case of instanton spaces is less well understood. One special case is Hilbert schemes of $\widetilde{\mathbb{C}^2/\Gamma}$ (which are moduli spaces of $U(1)$-instantons) and thus $\mathcal{O}_Y^\xi$ is category $\mathcal{O}$ for a Cherednik algebra. In this case, our conjecture reduces to:

Conjecture 9. *Category $\mathcal{O}$ for the rational spherical Cherednik algebra of $S_n \wr C_\ell$ is Koszul, and its Koszul dual is category $\mathcal{O}$ for the space of $U(\ell)$-instantons on $\mathbb{C}^2$.*

4.2. **Connections to physics.** Remarkably, the same list of examples of "dual" varieties has been known to physicists for some time. They are the Higgs branches of mirror dual $N = 4$ supersymmetric quantum field theories. This was established for T^*G/B by Witten and Gaiotto [?], and for hypertoric singularities by Kapustin and Strassler [?]. Alternatively, this duality switches the Higgs branch and quantum Coulomb branches of these QFT's, the Higgs branches of dual QFT's can be described as the Higgs and quantum Coulomb branches of the same theory.

Conjecture 10. *If X is the Higgs branch of the moduli space of vacua for an $N = 4$ supersymmetric $d = 3$ field theory, then $X^\vee$ is the Higgs branch of the mirror dual field theory, that is, the quantum Coulomb branch of the original field theory.*

Some, though certainly not all, of our expectations about connections between X and $X^\vee$ can be explained in this picture. For example, the connection between resolutions of X and $\mathbb{C}^*$ actions on $X^\vee$ corresponds to the switching of mass and Fayet-Illiopolous parameters under mirror duality.

All quiver varieties are the Higgs branches of appropriate QFT's, so this conjecture does suggest what the dual singularity to a quiver variety should be.

Unfortunately, the quantum Coulomb branch is a very mysterious object, and present techniques in physics do not allow for a description of them in this generality which would satisfy a mathematician. I've had discussions with several physicists on this subject, but it is still a point which requires more investigation.

4.3. **Goresky-MacPherson duality.** There are several other conjectural connections between X and $X^\vee$, which we expect will all be consequences of Conjecture **??**.

For instance, we consider two smooth varieties M and N with torus actions of S on M and T on N, both with isolated fixed points, and both equivariantly formal in the sense of Goresky-Kottwitz-MacPherson [**?**].

Definition 5. *We call M and N* **Goresky-MacPherson dual** *if there is a perfect pairing $H^2_S(M) \times H^2_T(N) \to \mathbb{C}$ such that*

- *$H^2_S(pt)$ and $H^2_T(pt)$ are mutual annihilators.*
- *There is a bijection $\Phi : M^S \cong N^T$ such that $\ker i_m^* \subset H^2_S(M)$ and $\ker i^*_{\Phi(m)} \subset H^2_T(N)$ are mutual annihilators, where i_m is the inclusion map of a fixed point.*

For example, T^*G/B and $T^{*L}G/^LB$ are GM-dual, with the pairing given by the identifications $H^2_T(Y) \cong \mathfrak{t} \oplus \mathfrak{t}$ and $H^2_{^LT}(T^{*L}G/^LB) \cong \mathfrak{t}^* \oplus \mathfrak{t}^*$ (the kernels of maps to points are the graphs of the elements of the Weyl group, and thus mutual annihilators since they are the graphs of dual linear maps).

Let (Y, ξ) and $(Y^\vee, \xi^\vee)$ be duals, in the sense of Conjecture **??**. Let T and S be the maximal Hamiltonian tori acting on Y and $Y^\vee$ which contain the image of ξ and $\xi^\vee$.

Conjecture 11. *The spaces Y and $Y^\vee$ are GM dual.*

Work in progress by Braden, Licata, Proudfoot, Phan, and myself [**?**] shows that Conjecture **??** would be a consequence of conjecture **??**, by showing that an algebraic analogue holds for a large class of Koszul algebras A, relating A to its dual $A^!$.

4.4. **The BBD filtration and cells.** I assume throughout this section that the fixed points of ξ are isolated, to simplify notation and statements.

One of the more interesting structures of the Hecke algebra which is clearer after realizing it as the Grothendieck group of the BGG category $\mathcal{O}$ is that of the cells.

For each nilpotent orbit, there is an ideal of the Hecke algebra which is spanned by the classes of simples whose characteristic varieties in T^*G/B are contained in the preimage of that orbit closure. This is a two-sided cellular ideal.

This notion generalizes to arbitrary symplectic resolutions of affine singularities. The singular variety X has a natural stratification $X = \sqcup_\alpha X_\alpha$ by symplectic leaves [?], which generalizes the orbit stratification on $\mathcal{N}_{\mathfrak{g}}$. This allows us to define a generalization of two sided cells.

Definition 6. *For each stratum X_α of X, let $\mathcal{O}_\alpha \subset \mathcal{O}_Y^\xi$ be the subcategory of objects whose supports are contained in $\bar{X}_\alpha$. Let*

$$\mathcal{O}_\alpha^\perp = \{M|\mathrm{Ext}^\bullet(M,N) = 0 \text{ for all } N \in \mathcal{O}_\alpha\}$$

be the annihilator subcategory.

In particular, we have two dual filtrations of $K_q^0(\mathcal{O}_Y^\xi)$ by subspaces $K_\alpha = K_q^0(\mathcal{O}_\alpha)$ and $B_\alpha = K_q^0(\mathcal{O}_\alpha^\perp)$, which are exchanged by Koszul duality.

We call the set of simples in $\mathcal{O}_\alpha$ which do not lie in $\mathcal{O}_\beta$ for $\beta < \alpha$ the **two-sided cell** *of α.*

Recall that we have an isomorphism $K_q^0(\mathcal{O}_Y^\xi) \cong H^*_{\mathbb{C}^*}(Y)[q^{-1}]$. Under this isomorphism, we expect this filtration will be geometrically meaningful.

We have a natural filtration on this cohomology induced by the filtration of Y by $\overline{Y_\alpha} = \pi^{-1}(\overline{X_\alpha})$. Let $F_\alpha \subset H^*_{\mathbb{C}^*}(Y)$ be the subspace generated by the Poincaré duals to cycles contained in $\overline{Y_\alpha}$. This filtration matches that given by decomposing $\pi_*\underline{\mathbb{C}}_Y$ as a perverse sheaf (a decomposition which featured prominently in the book of Chriss-Ginzburg [?]). Proudfoot and I studied this filtration for hypertoric varieties, and showed that it is a categorification of the Kook-Reiner-Stanton convolution formula for the Tutte polynomial [?, ?]. For quiver varieties, it contains information about the decomposition of the total cohomology of the quiver variety as a $\mathfrak{g}$-representation [?].

In particular,

$$F_\alpha/F_{<\alpha} \cong IH^*_{\mathbb{C}}(X_\alpha, \mathcal{L}_\alpha)$$

where $\mathcal{L}_\alpha = R\pi_*^{top}(\underline{\mathbb{C}}_{Y_\alpha})$ is the local system given by the top cohomology of the fibers over X_α with the Gauss-Manin connection.

We assume from now on that the local system $\mathcal{L}_\alpha$ is trivial. Note that this holds for hypertoric varieties and quiver varieties. As a consequence of this, we get that

$$IH^*_{\mathbb{C}}(X_\alpha, \mathcal{L}_\alpha) \cong IH^*_{\mathbb{C}^*}(\overline{X_\alpha}) \otimes H^*_{top}(\pi^{-1}(x)),$$

for $x \in X_\alpha$. We anticipate that similar statements will hold in the case when $\mathcal{L}_\alpha$ is not necessarily trivial, but they will be more difficult to state precisely.

Conjecture 12. *Under the isomorphism $K_q^0(\mathcal{O}_Y^\xi) \cong H^*_{\mathbb{C}^*}(Y)$, the filtrations $K_\bullet$ and $F_\bullet$ match.*

Furthermore, we expect that there will be a finer decomposition corresponding to the left and right cells of the Hecke algebra.

As in the BGG/Hecke algebra case, left cells are defined by primitive ideals of the algebra A_η (an ideal is called **primitive** if it annihilates a simple module).

Definition 7. *Two simple modules lie in the same* **left cell** *if their annihilators coincide.*

Since all primitive ideals are two-sided, their associated varieties in X must be Poisson, and thus of the form $\overline{X_\alpha}$ for some α. The two-sided cell for α is a union of left cells corresponding to the primitive ideals whose associated varieties are $\overline{X_\alpha}$.

For each primitive ideal, one can consider, instead of the associated variety, the support of the localization $\mathsf{Loc}(A_\eta/J)$, and the intersection $Q_J = \mathrm{supp}(\mathsf{Loc}(A_\eta/J)) \cap \pi^{-1}(x)$ of this with over a point $x \in X_\alpha$.

Conjecture 13. *The assignment $J \mapsto Q_J$ gives a bijection between primitive ideals J associated to α (that is, left cells in the two-sided cell for α) and components of $\pi^{-1}(x)$. The span in the Grothendieck group of the simples in a single left cell is canonically isomorphic to the intersection cohomology $IH^*(\overline{X_\alpha})$, and the class of these simples to give a unique canonical basis of this space.*

Definition 8. *Two simple modules lie in the same* **right cell** *if they map to the same basis vector in $IH^*(\overline{X_\alpha})$.*

We expect that this cell decomposition will behave well under duality. By Koszul duality, there is a bijection between simple modules of $\mathcal{O}_Y^\xi$ and $\mathcal{O}_{Y^\vee}^{\xi^\vee}$.

Conjecture 14. *This bijection preserves two-sided cells (compatibly with the bijection between strata of X_α and $X_\alpha^\vee$) and sends left cells to right cells. In particular, it induces a basis-preserving isomorphism between $IH^*(\overline{X_\alpha})$ and the top cohomology of a fiber over the dual stratum to X_α in $X^\vee$.*

This result is classical for $T^*(\mathrm{SL}(n)/B)$ (the flag variety of Lie groups of other types do not have trivial local systems on strata). In the case of hypertoric varieties, the match between dimensions of the intersection cohomology of a stratum and the top cohomology of a fiber in the dual resolution was noted by Proudfoot and I [**?**], and an isomorphism is described by Braden in forthcoming work. The proof of the full conjecture for hypertoric varieties is work in progress of Braden, Proudfoot, Licata and myself.