Research Statement

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In this research description, I will discuss 3 problems in the interface between geometry, representation theory and topology which I am working on at the moment and will continue working on over the next few years. While these problems may outwardly look disconnected, they all rest upon wider applications of categorification and quantization.

While the definition of a deformation quantization or quantum group is well-known, what we mean by a categorification requires a bit of discussion: while there is no precise definition, in essence, it means replacing an n-category with an $n+1$ category, which returns our original category when the $n+1$ morphisms are forgotten. Concretely, assuming our categories are all linear over a field k, categorification assigns

a number n (-1-category)	$\longrightarrow$	a vector space V (0-category)
a polynomial $p(q)$ (graded -1-category)	$\longrightarrow$	a graded vector space $\bar{V}$ (graded 0-category)
a vector space V (0-category)	$\longrightarrow$	a category $\mathcal{C}$ (1-category)
a ring R (1-category with 1 object)	$\longrightarrow$	a monoidal category $\mathcal{M}$ (2-category with 1 object)

In these cases, forgetting the top layer of morphisms corresponds to the familiar operations of taking dimension of a vector space or Grothendieck group of a category.

Of course, categorifications are by no means unique, and should not just be any category with the right Grothendieck group, but rather one which also has the important properties of the object we were categorifying. Thus, a categorification of a polynomial-valued knot invariant should be a vector space-valued knot invariant and a categorification of a representation of an algebra should have endofunctors which realize the action of that algebra.

Thus far, categorifications (often in conjunction with quantizations) have made important appearances in knot theory and geometric representation theory, and led not just to new invariants, but also new results about classical invariants. In particular, categories often have notions of simple or indecomposable objects, which can used to give bases of the Grothendieck group with manifest positivity or duality properties. The best example of this is Lusztig's construction of the canonical basis of any representation of a semi-simple Lie group, using the geometry of quiver varieties.

Project 1: *Categorification of higher Teichmüller varieties and TQFT*

For any complex algebraic group G and any Riemann surface with boundary S, Fock and Goncharov [FGb] have defined a moduli space $\mathcal{X}_{G,S}$ of G-local systems on S enriched with certain data at boundary components and punctures. This space forms part of a **cluster ensemble**, as defined in [FGa]. In particular, it is equipped with a system of open embeddings of tori $i_{\mathbf{s}} : (\mathbb{C}^*)^n \hookrightarrow \mathcal{X}_{G,S}$, which are called **clusters**, and the coordinates $\mathbf{s} = \{s_1, \dots, s_n\}$ on the torus, considered as rational functions on $\mathcal{X}_{G,S}$, are called **cluster variables**.

A remarkable property of these clusters is that the transition maps $i_{\mathbf{s}'}^{-1} \circ i_{\mathbf{s}}$ are always given by rational functions with positive real coefficients (these are a "positive atlas"). Thus, if a point in $\mathcal{X}_{G,S}$ has positive real coordinates in one coordinate system, it must have positive coordinates in all of them. We call the set of such points in each variety the **positive part** of $\mathcal{X}_{G,S}$, which we denote by $\mathcal{X}^+_{G,S}$.

Secondly, $\mathcal{X}_{G,S}$ has a natural Poisson structure, which is compatible with these coordinate patches in a natural way. For a fixed cluster $\mathbf{s} = \{s_1, \dots s_n\}$, the coordinate functions have Poisson brackets given by

$$\{s_i, s_j\} = a_{ij} s_i s_j.$$

for some skew-symmetric matrix $A = \{a_{ij}\}$.

If $G = \mathrm{PSL}_2(\mathbb{C})$, then $\mathcal{X}_{\mathrm{PSL}_2,S}$ is naturally a finite covering space of Teichmüller space. Thus, we refer to the $\mathcal{X}^+_{G,S}$ as **higher Teichmüller space.**

In previous work with Reshetikhin and Yakimov [RYW], I constructed the Poisson structure on $\mathcal{X}_{G,S}$ using graph connections on the surface S, and the standard Poisson-Lie structure on G. I plan to investigate whether this construction lifts can be applied to a construct a deformation quantization of $\mathcal{X}_{G,S}$ using the quantized function algebra of G, which would probably match that described by Fock and Goncharov.

Such a quantization would be particularly interesting, since it expected that $\mathcal{X}_{G,S}$ is the semi-classical limit of a topological quantum field theory, whose primary avatar is an action of the mapping class group Γ_S of the surface S on $\mathcal{X}_{G,S}$.

One useful tool for understanding this mapping class action would be a categorification of the cluster structure on $\mathcal{X}_{G,S}$ analogous to the categorification for acyclic cluster algebras [BMR$^+$, CKa, CKb]. Since Geiss, Leclerc, and Schröer [GLS] recently described a categorification of the cluster algebra structure on the unipotent radical of G, which is closely related to that on $\mathcal{X}_{G,S}$, it is reasonable to conjecture that their construction, which uses the representations of the preprojective algebra on the Dynkin quiver of G, can be extended to $\mathcal{X}_{G,S}$.

Conjecture 1. *There exists a cluster category $\mathcal{C}_{G,S}$ for $\mathcal{X}_{G,S}$, that is, an additive category such that clusters of $\mathcal{X}_{G,S}$ are in bijection with tilting objects of $\mathcal{C}_{G,S}$, cluster variables are in bijection with rigid indecomposable objects, and the coordinate ring of $\mathcal{X}_{G,S}$ can be recovered using the dual Hall algebra of $\mathcal{C}_{G,S}$.*

Such a categorification would be useful for a number of reasons. Firstly, it could be used in understanding the structure of the TQFT conjecturally attached to $\mathcal{X}_{G,S}$, and perhaps even categorifying it. It could also allow us to better understand the action of Γ_S, and particularly the extended mapping class group $\Gamma_{G,S}$, defined by Fock and Goncharov, by defining a $\Gamma_{G,S}$ invariant cell structure on $\mathcal{X}^+_{G,S}$.

Conjecture 2. *The varieties $\mathcal{X}^+_{G,S}$ have cell structures analogous to that defined by Penner in* [Pen87]*. In particular, this cell structure is invariant for the action of the extended mapping class group $\Gamma_{G,S}$, and cells are indexed by the simplicial complex whose maximal faces are clusters.*

Such a cell structure would provide a geometric model with which to study the group cohomology of $\Gamma_{G,S}$, which should be thought of as the cohomology of Fock and Goncharov's "moduli space" $\mathcal{X}^+_{G,S}/\Gamma_{G,S}$ (as yet, the objects this should be the moduli space of have yet to be defined, but are expected to be connected with W-algebras), and perhaps investigate analogues of Mumford's conjecture on the stable group cohomology of the extended mapping class group.

I am currently preparing a preprint [Weba], which describes an analogous invariant cell structure on the $\mathcal{X}$-space of a finite-type cluster algebra, using the cluster category of [BMR+].

Project 2: *Khovanov-Rozansky homology and categorification of TQFT*

I intend to work on both theoretical and computational aspects of Khovanov-Rozansky homology, and more generally the categorification of Witten-Reshetikhin-Turaev TQFT's.

Khovanov-Rozansky homology is a system of link homology theories which categorify the quantum invariants of knots corresponding to the vector representation of $\mathfrak{sl}_n$. Its original definition was combinatorial in flavor, using explicit complexes of matrix factorizations built using a diagram of the link in question, but has since been tied to categorifications of tensor products of fundamental representations of $\mathfrak{sl}_n$ constructed using category $\mathcal{O}$ for the Lie algebra $\mathfrak{gl}_m$ for various values of m by work of Khovanov, Frenkel, Stroppel and Sussan [FKS, Kho05, Str06a, Str06b], and to convolution varieties coming from the affine Grassmannian by Kamnitzer.

I believe the correct analogue of these construction is provided by the geometry of tensor product quiver varieties, as defined by Nakajima [Nak01]. I have studied the geometry of quiver varieties previously, and their geometry played a key role in my proof of stabilization result for the tensor product multiplicities of wild Kac-Moody algebras [Web06].

As Nakajima showed, the equivariant K-theory of the tensor product quiver varieties of a Dynkin quiver categorifies certain tensor products of representations for the quantum affinization of the Kac-Moody algebra corresponding to that quiver, with the tensor product of finite dimensional representations being a natural submodule.

Conjecture 3. *For each Dynkin diagram, and each m-tuple of weights for the corresponding simple Lie algebra $\mathfrak{g}$, there is a subcategory $\mathcal{C}$ of category of coherent sheaves on the tensor product quiver variety $\mathcal{M}$ whose Grothendieck group is naturally isomorphic to a full lattice in the tensor product of the corresponding representations, and the $\mathfrak{g}$-action and* *action of the R-matrix are geometrically realized by convolution with coherent sheaves on Hecke correspondences between quiver varieties.*

This categorification can be used to construct a categorification of Witten-Reshetikhin-Turaev invariants of knots and 3-manifolds for all types.

Another approach toward generalizing Khovanov-Rozansky I plan to investigate is that using the Lie superalgebras $\mathfrak{gl}(n|m)$. The quantum invariant corresponding to the vector representation of $\mathfrak{gl}(n|m)$ is simply the same as that of $\mathfrak{gl}(n-m)$.

Conjecture 4. *There exists a categorification of the representation $V^{\otimes k}$ of $\mathfrak{gl}(n|m)$, where V is the vector representation. This categorification has an associated link homology theory $L_{n,m}$, and there is a natural spectral sequence from K-R HOMFLY homology converging to this homology theory.*

Rasmussen [Rasb] has shown that such spectral sequences exist for the $\mathfrak{sl}(n)$ Khovanov-Rozansky invariants.

The most interesting case would be that of $\mathfrak{gl}(n|n)$, whose quantum invariant is the Alexander polynomial. One categorification of the Alexander polynomial, knot Floer homology, has already been constructed by Ozsvath, Szabo and Rasmussen, but its remains difficult to calculate and is only conjecturally related to Khovanov-Rozansky homology. If $L_{n,n}$ were to coincide with knot Floer homology, this would construct the spectral sequence from HOMFLY homology to knot Floer homology conjectured in [DGR], and given the power of knot Floer homology, could useful in calculating the genus and slice genus of knots (somewhat analogously with Rasmussen's combinatorial proof of the Milnor conjecture [Rasa]).

Another problem which interests me along these lines is constructing the analogue for $\mathfrak{gl}(1|1)$ of Rouquier and Chuang's universal description of $\mathfrak{sl}(2)$ categorifications [CR]. This would hopefully help guide me to the categorifications discussed above.

Project 2a: *Computation in Khovanov-Rozansky homology*

Much of my previous work on KR homology has concentrated on computation. In 2005-2006, I wrote a program to compute Khovanov-Rozansky homology using Macaulay2. I would continue to refine this program, since I believe it could be made much more efficient, and can only take braid representatives of a knot as input. Ultimately, I would like to implement simplifications as described in my preprint [Webb] for a general knot diagram, which follow the divide and conquer strategy of Bar-Natan. This philosophy has been used to compute $\mathfrak{sl}_2$ KR homology very quickly, but will be much harder to implement for $\mathfrak{sl}_2$. This would probably require programming a new Macaulay2 package for dealing with matrix factorizations and bimodules.

Project 3: *Derived equivalences for Poisson varieties.*

Associated to each hyperplane arrangement is a hyperkähler orbifold called a **hypertoric variety**. This is a quaternionic analogue of a toric variety, and can be constructed from a similar symplectic reduction procedure. A hypertoric variety can be equipped with a complex structure in which it is a quasi-projective variety, and a resolution of singularities of its affinization. Hypertoric varieties are one of the most accessible examples of *hyperkähler analogues*, a class which also includes the hyperkähler quiver varieties described by Nakajima [Nak98].

In previous work with Proudfoot [PW06], I studied the cohomology of these varieties, and the intersection cohomology of their affinizations by reduction to characteristic p.

In recent years, Kaledin and others have proved by non-constructive methods that symplectic resolutions such as hypertoric varieties have a number of good properties (assuming technical restrictions are satisfied) [BK, BK04, Kala, Kalb]. Most notably, they are equipped with tilting generators: vector bundles E such that $\mathrm{RHom}(-, E)$ is an equivalence of derived categories from coherent sheaves on X to modules over $\mathrm{End}(E)$. In most cases, including hypertoric varieties and quiver varieties, these bundles and their endomorphism algebras have not been described.

I intend to seek an explicit description of Kaledin and Bezrukavnikov's quantizations of symplectic resolutions of singularities and the tilting generators attached to them in concrete cases, particularly hypertoric and quiver varieties. Since Grothendieck groups are preserved by derived equivalences, this will hopefully provide a new approach to the K-theory of quiver varieties and hypertoric varieties.

I would also like to study Poisson varieties, such as Poisson-Lie groups, their Poisson homogeneous spaces and cluster varieties from this perspective. This approach has recent lead to a proof of a long standing conjecture of de Concini-Kac-Procesi by Kremnizer [Kre], and I think it will be a fruitful line of research. One directions of particular interest to me is a finding a geometric description of the braiding of quantum groups at roots of unity, which has been used by Reshetikhin and Kashaev to define knot invariants [KR], in hopes of understanding these invariants better.

Also, the moduli spaces described in Problem 1 are examples of Poisson varieties (in fact, of deformations of symplectic varieties), and so studying them from this perspective may well be fruitful.

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