# Representation theory of symplectic singularities

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For a lot of history, it seemed as though commutative rings were maybe the most natural framework in which to view mathematics.

- Obvious context for number theory, algebraic geometry.
- In physics, observable quantities form a commutative ring.

Then quantum mechanics came along, and the picture looked a bit different. The algebra of observables becomes non-commutative, but with a classical limit (it's "**almost commutative**").

On the classical side, physicists had already noticed a hint of the non-commutativity of quantum mechanics: the Poisson bracket (which is often called "**semi-classical**")

Hamilton's equation (for an observable): $$\frac{\partial f}{\partial t} = \{H, f\}$$

Heisenberg's equation (for an operator): $$i\hbar\frac{\partial \hat{f}}{\partial t} = [\hat{H}, \hat{f}]$$

"Almost" commutative algebras

commutative

non-commutative

commutative semi-classical almost commutative non-commutative

$$\overbrace{\text{commutative} \qquad \text{semi-classical}}^{\text{algebraic geometry}} \qquad \overbrace{\text{almost commutative} \qquad \text{non-commutative}}^{\text{representation theory}}$$

algebraic geometry: commutative, semi-classical

X ← where I'd like to be

representation theory: almost commutative, non-commutative

## Definition

*An* **almost commutative ring** *is a ring $A$ with a filtration $A_0 \subset A_1 \subset \cdots$ and an integer $n > 0$ such that*

$$A_iA_j \subset A_{i+j} \qquad [A_i, A_j] \subset A_{i+j-n}$$

In particular, the ring $\mathrm{gr}(A) \cong \oplus_{i=0}^{\infty} A_i/A_{i-1}$ is commutative and $\mathbb{Z}_{\geq 0}$-graded.

The ring $\mathrm{gr}(A)$ inherits a semi-classical structure:

## Definition

*A* **conical Poisson ring** *is a* $\mathbb{Z}_{\geq 0}$*-graded commutative ring* $R$ *with a second operation* $\{-,-\}\colon R\times R\to R$*, homogeneous of degree* $-n$*, that satisfies the relations of a Lie bracket (bilinear, anti-symmetric, Jacobi) such that the Leibnitz rule holds:*

$$\{ab,c\} = a\{b,c\} + b\{a,c\}.$$

There's a **classical limit** functor $A \mapsto (\mathrm{gr}(A), \{-,-\})$ from almost commutative algebras to conical Poisson algebras, with the Poisson bracket given by

$$\{\bar{a},\bar{b}\} = \overline{[a,b]} \in A_{i+j-n}/A_{i+j-n-1}.$$

The most basic case is when $A = \mathbb{C}\langle x, \frac{d}{dx}\rangle$ is the algebra of polynomial differential operators. This is filtered with

$$A_1 = \operatorname{span}\left(1, x, \frac{d}{dx}\right) \qquad A_n = A_1^n$$

This is almost commutative ($n = 2$) with $\operatorname{gr}(A) = \mathbb{C}[x, p]$.

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial x} \qquad \{p, x\} = 1$$

Similarly, $U(\mathfrak{g})$ for any Lie algebra $\mathfrak{g}$ is almost commutative, with classical limit $\mathbb{C}[\mathfrak{g}^*]$ with the KKS Poisson structure.

## Definition

*If $R$ is an conical Poisson algebra, then a* **quantization** *of $R$ is an almost commutative algebra $A$ whose classical limit is $R$.*

You can easily check that $A$ is the unique quantization of $\mathbb{C}[x, p]$. (Hint: $A_1 \cong \mathbb{C} \cdot \{x, p, 1\}$ as 3-dimensional Lie algebras).

What happens when we consider other Poisson varieties?

In general, finding all quantizations is not easy; Kontsevich got a Fields Medal in large part for doing so for a *real* Poisson structure on $\mathbb{R}^n$.

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We call an affine variety $Y$ conical Poisson if its coordinate ring has that structure.

## Definition

*We call $Y$ a* **conical symplectic variety** *(i.e. conical variety w/ symplectic singularities) if the Poisson bracket induces a symplectic structure on the smooth locus (+silly technical conditions).*

- If $\mathbb{C}^{2n}$ has the usual symplectic structure, and $\Gamma$ is a finite group preserving $\omega$, then $Y \cong \mathbb{C}^{2n}/\Gamma$ is an example.
- The variety of nilpotent matrices (and more generally, nilpotent cone of a semi-simple Lie algebra $\mathfrak{g}$) has a natural symplectic structure.

The correspondence between almost commutative and semi-classical is particularly nice in this case.

**Theorem (Namikawa)**

*Every conic symplectic variety $Y$ has a universal deformation $\mathcal{Y}$ that smoothes it out as much as possible while staying symplectic (which is thus rigid). The base of this deformation is a vector space $H$.*

For example, $\mathfrak{g}^*$ comes from the nilcone $\mathcal{N}$. The simplest example is:

$$Y = \mathbb{C}^2/(\mathbb{Z}/\ell\mathbb{Z}) \cong \{(u = x^\ell, v = y^\ell, w = xy) \mid uv = w^\ell\}$$

The universal deformation is given by adding formal coordinates $a_i$ on $H$ of degree $2i$.

For a general $\mathbb{C}^{2n}/\Gamma$, there's a similar deformation direction attached to every conjugacy class of symplectic reflection (an element that fixes a codimension 2 subspace).

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$$\mathcal{Y} \cong \{(u, v, w, a_1, \dots a_\ell) \mid uv = w^\ell + a_1 w^{\ell-1} + \cdots + a_\ell\}$$

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So, how do we get an analogue of the universal enveloping algebra?

**Theorem (Bezrukavnikov-Kaledin, Braden-Proudfoot-W., Losev)**

*Every conical symplectic variety $Y$ has a unique quantization $A$ of its universal Poisson deformation $\mathcal{Y}$; this is an almost commutative algebra such that $\operatorname{gr} A \cong \mathbb{C}[\mathcal{Y}]$.*

*The center $Z(A)$ is the polynomial ring $\mathbb{C}[H]$; the quotient $A_\lambda$ for a maximal ideal $\lambda \in H$ has an isomorphism $\operatorname{gr} A_\lambda \cong \mathbb{C}[Y]$. This gives a complete irredundant list of quantizations of $\mathbb{C}[Y]$.*

- If $Y$ is a nilcone, then $A$ is the universal enveloping algebra.
- If $Y \cong \mathbb{C}^{2n}/\Gamma$, then $A$ is a spherical symplectic reflection algebra.

*Symplectic singularities are the Lie algebras of the 21st century.* - Okounkov

There's a very interesting interplay between the geometry of $\mathcal{Y}$ and the representation theory of $A$, modeled on that of $\mathfrak{g}^*$ and $U(\mathfrak{g})$:

Geometry

- orbit method
- geometric quantization
- flag variety and Schubert varieties
- localization, D-modules, intersection cohomology
- support varieties

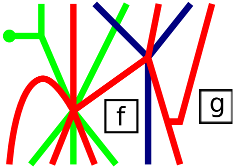
Algebra

- primitive ideals
- Harish-Chandra (bi)modules
- category $\mathcal{O}$
- character formulae
- translation/projective functors

We should really have a third column here:

Combinatorics: Coxeter groups, tableaux, cells, KL polynomials

Culmination is the Soergel calculus of Elias and Williamson.




To algebraists, describes category $\mathcal{O}$ and HC bimodules. To geometers, describes the $B \times B$ equivariant D-modules on $G$.

The first description is of the endomorphisms of a projective generator, the second is of Ext of a simple generator. Thus, these are Koszul dual to each other.

How do we generalize this picture? Unfortunately, the case of a UEA has a lot of special structure we can't expect in other cases.

I don't have a good general answer; I'm pretty skeptical about one existing at all. I do know a very interesting set of examples, though:

Beginning with a connected reductive complex group $G$, and a representation $V$, there's a 3-d $\mathcal{N} = 4$ supersymmetric field theory you can build from this.

This field theory has a moduli space of vacua (lowest energy states) which is a big reducible algebraic variety with two distinguished components: the **Higgs** and **Coulomb** branches.

- The Higgs branch is the hyperkähler quotient

$$\{(v,\xi) \in T^*V \mid \mathfrak{g} \cdot v = T_vG \cdot v \perp \xi\}/\!\!/G$$

  Examples of such varieties are Nakajima quiver varieties and hypertoric varieties (when $G$ is abelian).

- The Coulomb branch is given by starting with the variety $T^*\check{T}/W$ (for $T \subset G$ a maximal torus), and applying "quantum corrections."

  By Braverman-Finkelberg-Nakajima, I know what these corrections are, but there is no room in this margin, etc. Examples include slices between $\mathrm{Gr}_\lambda$ and $\mathrm{Gr}_\mu$ in the affine Grassmannian, $\widehat{A}$ quiver varieties, hypertoric varieties, and (conjecturally) $G$-instantons on $\mathbb{C}^2$.

- For Higgs branches, a quantization can be constructed by replacing $T^*V$ with differential operators, and performing non-commutative Hamiltonian reduction.
- For Coulomb branches, $\mathbb{C}[Y]$ is itself the homology of a space with a convolution giving multiplication, and the quantization is $\mathbb{C}^*$-equivariant homology.

A whole zoo of algebras appear in both Higgs and Coulomb presentations: hypertoric enveloping algebras, parabolic W-algebras, rational Cherednik algebras. I'll focus on the last of these.

How are we supposed to study these algebras? With Lie theory, we're starting from having a century of experience, and a rather special set up.

- On the Higgs side, the geometric column works pretty well. The replacement for D-modules is "quantum coherent sheaves" on a resolution of the Higgs branch. These are hard to work with (no six functor formalism) but close enough to $G$-equivariant D-modules on $V$ to make things work.
- With Coulomb branches, the algebraic column is more successful. There's a natural "torus" in the quantization $A$ and you can analyze its weight spaces, with the structure of the representation $V$ influencing how they are related.

Luckily, the combinatorial column can tie them together.

We call $\xi \in A_\lambda$ a grading element if $[\xi, -] \colon A_\lambda \to A_\lambda$ is semi-simple with integral eigenvalues.

## Definition

*Category $\mathscr{O}^\xi_\lambda$ for $\xi$ over $A_\lambda$ is the subcategory of modules where $\xi$ acts with finite length Jordan blocks, and f.d. eigenspaces, and eigenvalues bounded above.*

Note this category depends on $\xi$ and $\lambda$; it's richest if $\lambda$ is "integral" in some appropriate sense.

These switch roles between Higgs and Coulomb: a grading element on one side corresponds to an integral quantization parameter on the other.

We can define a graded algebra $T$ (purely based on the combinatorics of $G, V, \xi, \lambda$) such that:

### Theorem (W.)

*For $\xi$ integral:*

1. *$T$ is isomorphic to the endomorphisms of a projective generator in $\mathcal{O}_\lambda^\xi$ for $A_{\mathrm{Coulomb}}$.*
2. *$T$ is isomorphic to the Ext-algebra of a semi-simple generator in $\mathcal{O}_\xi^\lambda$ for $A_{\mathrm{Higgs}}$ (assuming certain hypotheses on $V$ and $G$).*
3. *The category $\mathcal{O}_\xi^\lambda$ is Koszul dual to $\mathcal{O}_\lambda^\xi$.*

You can think of the algebra $T$ as a replacement of Soergel calculus in the Higgs/Coulomb context.

This should reflect some underlying geometric connection between the varieties (there are many other coincidences of underlying geometric information).

**Conjecture (Braden-Licata-Proudfoot-W.)**

*There is a duality on the set of symplectic singularities which switches Higgs and Coulomb branches.*

The algebra $T$ is graded, whereas the Coulomb category $\mathscr{O}$ isn't. This provides a graded lift, which has good positivity properties.

**Theorem (W.)**

*There are "Verma" modules in $\mathscr{O}^{\xi}_{\lambda}$, and the multiplicities of simples in them are given by a version of Kazhdan-Lusztig polynomials/canonical bases.*

*In particular, we get a new proof of Rouquier's conjecture that decomposition numbers for rational Cherednik algebras are given by affine parabolic KL polynomials.*

In general, lots of "canonical bases" (for reps of Lie algebras, for the Hecke algebra, etc.) show up this way.

If we start with a quiver $\Gamma$, and dimension vectors $\mathbf{v}, \mathbf{w}$, then quiver varieties are the Higgs branches in the case

$$V = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \oplus \bigoplus_{i} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i}) \qquad G = \prod GL(v_i, \mathbb{C}).$$

## Theorem (W.)

*The category $T$-gmod categorifies a tensor product of simple representations (depending on* $\mathbf{w}$ *and $\xi$) for the quantum group attached to $\Gamma$ ($q =$grading shift).*

*There are natural functors categorifying all natural maps in quantum group theory. This allows us to construct a categorified Reshetkhin-Turaev knot invariant for any representation.*

Even the Lie algebras not attached to graphs (types BDFG) have an associated algebra $T$, and this construction can be carried through purely combinatorially.

The algebra $T$ has a purely combinatorial definition in terms of the hyperplane arrangement of $\mathfrak{t}$ induced by $\varphi_i = n_i$ for the weights $\varphi_i$ of $V$ and certain scalars $n_i$, (together with the action of $W$).

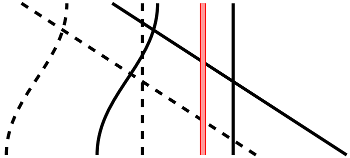
In the case of a quiver variety, the family of algebras which show up already has a rich theory: they are (weighted) Khovanov-Lauda-Rouquier algebras. Like the Soergel calculus, we can define this algebra in terms of diagrams modulo local relations.

Let's consider the special case of

$$G = GL(n, \mathbb{C}) \qquad V \cong \mathfrak{gl}(n) \oplus (\mathbb{C}^n)^{\oplus \ell}.$$

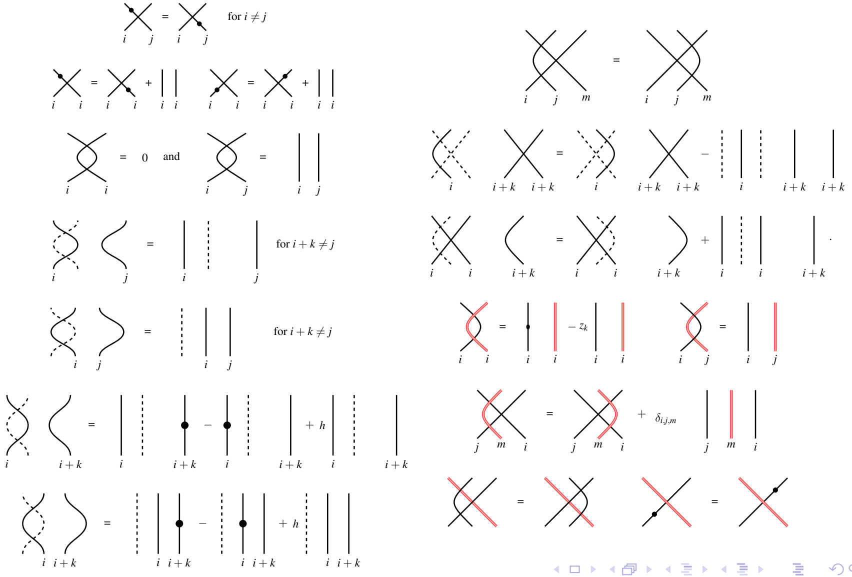
The corresponding Coulomb quantization is the rational Cherednik algebra of $S_n \operatorname{wr} \mathbb{Z}/\ell\mathbb{Z}$.

In this case, the diagrams of $T$ look like:



- The $x$-values of the strands in a horizontal slice give an element of $\mathfrak{t}$.
- The weights of $\mathbb{C}^n$ correspond to the red lines: you cross a corresponding hyperplane when a red and black line cross.
- the weights of $\mathfrak{gl}(n)$ are the distance between points, so you cross one of their hyperplanes when two strands cross, or reach a fixed distance from each other.

## KLR algebras



Thanks for listening.