# Line defects and tilting bundles

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November 19, 2019

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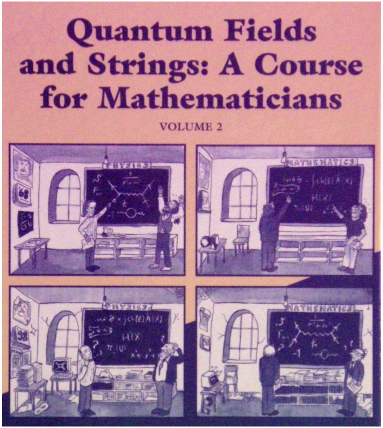
Quantum field theory is a subject which has caused a great deal of confusion for mathematicians (myself included) over the years:



I recently had a perspective shift that helped a great deal with understanding the connection, though:

**quantum field theory = representation theory + topology**

**Topology** because we need to consider how points and higher dimensional defects interact on the world-sheet, and **representation theory** (with an eye on integrability) because we need to consider algebraically how operators collide with each other and with defects.

| | | |
|---|---|---|
| a quantum field theory | = | an algebra |
| boundary condition/defects | = | modules |

This needs to be understood in a very general sense. It's probably better to say that quantum field theory is a massive generalization of representation theory as we currently know it, and a very different perspective from how most representation theorists think.

In particular, any quantum field theory with a twist gives a generalization of the concept of algebra (actually, $A_\infty$-algebra, but that's not so important for me)

- $d = 1$, topological twist: algebra
- $d = 2$, topological twist: Gerstenhaber algebra
- $d = 2$, holomorphic twist: vertex operator algebra
- $d = 3$, topological twist: conic symplectic singularity
- $\vdots$

Ok, that last one contains a lot of simplification: you get a graded algebra with a degree $-2$ Poisson bracket. If you're lucky, Spec of this algebra is a symplectic singularity, with a conic $\mathbb{C}^*$-action.

We're supposed be studying mirror symmetry here, which is a story about the interaction of different twists on one theory, which means we need more supersymmetry.

In a $d = 2$-dimensional theory with $\mathcal{N} = (2, 2)$ supercharges, there are particularly nice choices of twists $Q_A, Q_B$ called the $A$ and $B$ twists.

**Conjecture (2d Mirror symmetry)**

*Kähler manifolds come in pairs such that the associated sigma models are "equivalent" in a way that switches the A and B twists.*

Homological mirror symmetry simply says that the categories of boundary conditions compatible with the two twists are interchanged.

There is much, much more to say about the $d = 2$ case, but I want to think instead about the $d = 3$.

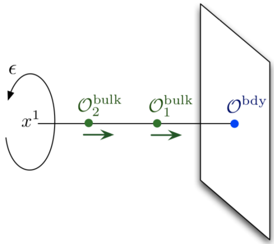
Also, given an interesting $d = 2$ theory, it's natural to look for $d = 3$ theories it can lie on the boundary of.

Because of the structure of the supersymmetry algebras, a $2d, \mathcal{N} = (2, 2)$ theory is most comfortable on the boundary of a $3d, \mathcal{N} = 4$ theory. Inclusion of $R$-symmetry:

$$\mathrm{Spin}(2) \times \mathrm{Spin}(2) \cong U(1) \times U(1) \hookrightarrow SU(2) \times SU(2) \cong \mathrm{Spin}(4)$$

This inclusion turns $Q_A$ and $Q_B$ into twists of the ambient $3d$ theory.

Thus, placing a $2d, \mathcal{N} = (2, 2)$ theory $\mathcal{X}$ on the boundary of a $3d, \mathcal{N} = 4$ theory $\mathcal{Y}$ gives local operators in $\mathcal{X}$ a module structure for local operators in $\mathcal{Y}$, and similarly for the twists by $Q_A$ and $Q_B$.



This suggests an intriguing possibility:

## Conjecture (3d mirror symmetry)

*$3d, \mathcal{N} = 4$ theories come in pairs which are "equivalent" in a way that switches the A and B twists.*

There's a pedantic sense in which this is true (just redefine the supersymmetry action), but like in the 2d case, an interesting 3d theory should also have an interesting origin when you switch the twists.

## Question for mathematicians:

How does this duality of theories manifest? How are mathematically comprehensible objects on both sides related?

The ring of local operators for these twisted theories carries:

1. a commutative algebra structure (from colliding operators)
2. a Poisson bracket compatible with this product (from integrating over the $S^2$ of choices of how to collide 2 points in $\mathbb{R}^3$).
3. a grading such that product has degree 0 and $\{-.-\}$ has degree $-2(= 1 - 3)$.

This is the same information as an affine algebraic variety $\mathfrak{M}_{A/B}/\mathbb{C}$ with Poisson bivector $\Pi$, and a $\mathbb{C}^*$ action with $t \cdot \Pi = t^{-2}\Pi$.

Physics seems to suggest that this should have an underlying hyperkähler metric.

We should get **two** of these for each theory, for the $A$ and $B$ twists. You'll often hear these called the Coulomb and Higgs branches.

The best understood theories are gauge theories, constructed from a compact connected Lie group $G$ and a complex representation $V$.

For those who like physics terminology, we couple a hypermultiplet valued in $V$ to a vectormultiplet for the group $G$.

1. The Higgs branch is the usual hyperkähler quotient of $T^*V$ by $G$.
2. The Coulomb branch is much more mysterious. Can't be calculated "classically," and nature of "quantum corrections" is hard to describe precisely.

   However, Braverman-Finkelberg-Nakajima have given a precise description of this ring, which is a bit complicated and geometric, but can be represented algebraically.

Examples:

| $(V, G)$ | Higgs | Coulomb |
|---|---|---|
| $(\mathbb{C}^{n+1}, U(1))$ | $T^*\mathbb{P}^n$ | $\mathbb{C}^2/\mathbb{Z}_{n+1}$ |
| $(\mathbb{C}^{n+1}, U(1)^n)$ | $\mathbb{C}^2/\mathbb{Z}_{n+1}$ | $T^*\mathbb{P}^n$ |
| $(\mathrm{Mat}_{n\times n} \times \mathbb{C}^n, GL_n)$ | $\mathrm{Sym}^n(\mathbb{C}^2)$ | $\mathrm{Sym}^n(\mathbb{C}^2)$ |
| $(\mathrm{Mat}_{n\times n-1} \times \mathrm{Mat}_{n-1\times n-2} \times \cdots, GL_{n-1} \times GL_{n-2} \times \cdots)$ | $\mathrm{Nil}_{n\times n}$ | $\mathrm{Nil}_{n\times n}$ |
| $(0, G)$ | $pt/G$ | Toda phase space |
| quiver gauge theory | Nakajima quiver variety | affine Grassmannian slice |

For mathematicians, the obvious thing to study is coherent sheaves $\mathfrak{M}_{A/B}$ or its resolutions.

For Higgs branches, we have lots of experience studying the structure of these categories: variation of GIT for comparing resolutions, associated bundles for $G$-modules ("Wilson lines" in physics) for constructing objects.

For Coulomb branches, much more mysterious. That's what I'll try to explain in this talk.

The best way to think about the Coulomb branch is in the context of line operators. For mathematicians, it's best to think of this as the category of D-modules on the loop space $V((t))/G((t))$.

To connect to actual field theory, our gauge theory has hypermultiplet fields valued in $T^*V$, and our line operator is defined by letting/forcing these fields to take on values in the singular support of our D-module.

Of course, the trivial line operator $\mathbb{1}$ corresponds to not imposing any zeros or allowing any poles: it corresponds to the pushforward of the functions by $V[[t]]/G[[t]] \to V((t))/G((t))$.

## Definition (Braverman-Finkelberg-Nakajima)

The local operators $\mathbb{C}[\mathfrak{M}_A]$ of the $A$ twist is

$$\operatorname{Ext}^\bullet(\mathbb{1}, \mathbb{1}) = H_*^{BM}\left(\frac{V[[t]]}{G[[t]]} \times_{\frac{V((t))}{G((t))}} \frac{V[[t]]}{G[[t]]}\right).$$

This was the first way mathematicians or physicists were able to calculate this ring, but it also tells us much more, since there are many other objects in this category.

In many important cases, there are commutative and non-commutative resolutions of $\mathfrak{M}_A$ inside it.

From now on, we'll specialize to two nice cases:

1. $G$ is commutative ($\mathfrak{M}_B$ is a hypertoric variety)
2. $(V, G)$ corresponds to a linear or cyclic quiver for a dominant weight ($\mathfrak{M}_B$ is the corresponding Nakajima quiver variety)

**Theorem**

*In the cases above:*

1. $\mathfrak{M}_A$ *is the Gale dual hypertoric variety,*
2. $\mathfrak{M}_A$ *is the rank-level dual quiver variety.*

A **flavor** is a cocharacter $\varphi\colon \mathbb{G}_m \to \mathrm{Aut}_G(V)$.

Multiplication by $\varphi(t)$ gives an automorphism of $V((t))/G((t))$, so we can apply this to any D-module on this category.

**Theorem (BFN)**

*For each $\varphi$, we have a graded commutative algebra*

$$R_\varphi = \bigoplus_{k=0}^{\infty} \mathrm{Ext}^\bullet(\varphi(t^{-k})\mathbb{1}, \mathbb{1}).$$

*For $\varphi$ generic, $\tilde{\mathfrak{M}}_A = \mathrm{Proj}(R_\varphi)$ is a symplectic resolution of $\mathfrak{M}_A$, and every symplectic resolution is of this form.*

This means that every D-module $\mathcal{M}$ gives us a coherent sheaf $\mathscr{F}_{\mathcal{M}}$ on $\tilde{\mathfrak{M}}_A$ by sheafifying the $R_\varphi$-module

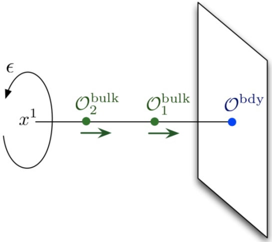
$$\bigoplus_{k=0}^{\infty} \operatorname{Ext}^\bullet(\varphi(t^{-k})\mathbb{1}, \mathcal{M}).$$

Which coherent sheaves can we construct this way? What can this tell us about the category of coherent sheaves?

**Theorem (Aseel-Gomis)**

*Wilson lines can be constructed using 1d/3d quiver diagrams.*

The chiral rings $\mathbb{C}[\mathfrak{M}_{A/B}]$ have a quantization which comes from an $\Omega$ background/working equivariantly for the rotation of $\mathbb{R}^3$ around an axis. Actually they are many, because you have to choose a flavor $\lambda$ which determines how $S^1$ acts on $V$.




This is non-commutative because you can't swap two points equivariantly for the rotation, and

$$a \star b - b \star a = \hbar\{a, b\} + \hbar^2 \cdots$$

Quantization results in some well-known algebras with quantum integrable systems (i.e. commutative subalgebras):

$$
\begin{array}{rcl}
T^*\mathbb{P}^n & \to & \mathrm{Diff}(\mathbb{P}^n) \supset U(\mathfrak{t}) \\
\mathrm{Sym}^n(\mathbb{C}^2/\mathbb{Z}_\ell) & \to & \text{rational Cherednik algebra} \\
 & & \quad \supset \text{Dunkl-Opdam operators} \\
\mathrm{Nil}_{n\times n} & \to & U(\mathfrak{sl}_n) \\
 & & \quad \supset \text{Gelfand-Tsetlin subalgebra} \\
\text{affine Grassmannian slice} & \to & \text{shifted Yangian} \\
\text{Nakajima quiver variety} & \to & \text{quantum Hamiltonian reduction}
\end{array}
$$

Sidenote: considering this quantization over $\mathbb{C}$ gives a new perspective on $\mathfrak{sl}_n$-modules locally finite for the GT subalgebra.

**Theorem (W.)**

*The simple GT modules over $\mathfrak{gl}_n$ are classified by combinatorial data (the indexing set isn't bad, but takes a few minutes to explain). The dimensions of weight spaces for the GT subalgebra can be computed by a Kazhdan-Lusztig type algorithm.*

In fact, this works for all Coulomb branches, essentially equally well.

$U(\mathfrak{sl}_n)$ also arises from quantizing a Higgs branch, but it's essential to think about it as a Coulomb branch to prove this theorem; the category of line defects in the theory makes things much easier.

Another natural construction to try to understand: Kaledin's tilting generators.

## Theorem (Kaledin)

*The resolution $\tilde{\mathfrak{M}}_A$ carries a tilting generator, i.e. a vector bundle $\mathcal{T}$ such that*

1. $\mathrm{Ext}^{>0}(\mathcal{T}, \mathcal{T}) = 0$
2. *We have an equivalence of derived categories to $A = \mathrm{End}(\mathcal{T})^{\mathrm{op}}$-modules*

$$\mathrm{Ext}^{\bullet}(\mathcal{T}, -) \colon D^b(\mathsf{Coh}(\tilde{\mathfrak{M}}_A)) \to D^b(A\text{-mod}).$$

Unfortunately, the construction of this tilting generator is rather complicated; it passes through methods of quantization in characteristic $p$.

Since our quantization has geometric origin, Kaledin's procedure inherits geometric meaning. Let $C_p^\lambda \subset S^1$ be the $p$-torsion points, acting on $V((t))$ by loop rotation twisted by $\lambda$.

Note that the fixed points of $C_p^\lambda$ on $V((t))$ are exactly $t^{-\lambda}V((t^p))$, and the obvious isomorphism $\mathrm{Fr}\colon V((t)) \cong t^{-\lambda}V((t^p))$ intertwines the action of $G((t))$ with that of $G((t^p))$.

### Theorem (W.)

*Kaledin's tilting generator for a quantization parameter $\lambda$ is given by $\mathscr{F}_{\mathcal{M}_\lambda}$ where $\mathcal{M}_\lambda$ is the pushforward by the map*

$$X_\lambda = \left(\frac{G((t)) \times V[[t]]}{G[[t]]}\right)^{C_p} \to t^{-\lambda}V((t^p)) \cong V((t)).$$

Note that we chose $\lambda$ (the quantization parameter) and $\varphi$ (the choice of symplectic resolution) independently.

**Theorem**

*The algebra $A_\lambda = \mathrm{End}(\mathscr{F}_{\mathcal{M}_\lambda})$ defines a non-commutative crepant resolution of singularities for $\mathfrak{M}_A$ (independent of $\varphi$).*

An important motivation for Kaledin was to prove that the derived categories for different symplectic resolutions are equivalent (since they are all equivalent to $D^b(A_\lambda\text{-mod})$); these equivalences depend on $\lambda$, though, and generate an action of **wall-crossing functors** on these derived categories.

Every component of $X_\lambda$ is of the form $(G((t^p)) \times U)/P$ for

- $P$ a parahoric in $G((t^p))$ and
- $U \subset t^{-\lambda}V((t^p))$ a $P$-invariant subspace.

Let me not torture you with the actual combinatorics of describing these, but this perspective shows that $A_\lambda$ has a combinatorial construction.

Thus, in the language of Dimofte-Garner-Geracie-Hilburn, this is the vortex line operator attached to the Lagrangian conormal to $\mathrm{Fr}^{-1} U \subset V((t))$ with the action of the group $\mathrm{Fr}^{-1} P \subset G((t))$.

Applications:

- Explicit presentations of endomorphisms: for quiver varieties, get "KLR algebras on cylinders."
- Explicit presentations of wall-crossing functors and construction of a Schober, can verify Bezrukavnikov-Okounkov conjecture "by hand."
- Explicit stability conditions coming from assigning slopes to simple $A$-modules.
- Same line operators define tilting generators on K-theoretic Coulomb branch where Kaledin's trick doesn't work to build a global tilting generator.
- This was key in work of Gammage-McBreen-W. on mirror symmetry for multiplicative hypertoric varieties.

Thanks for listening.