Unfurling Khovanov-Lauda-Rouquier algebras

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Abstract. In this paper, we study the behavior of categorical actions of a Lie algebra $\mathfrak{g}$ under the deformation of their spectra. We give conditions under which the general point of a family of categorical actions of $\mathfrak{g}$ carry an action of a larger Lie algebra $\tilde{\mathfrak{g}}$, which we call an **unfurling** of $\mathfrak{g}$. This is closely related to the folding of Dynkin diagrams, but to avoid confusion, we think it is better to use a different term.

Our motivation for studying this topic is the difficulty of proving that explicitly presented algebras and categories in the theory of higher representation theory have the "expected size." Deformation is a powerful technique for showing this because of the upper semicontinuity of dimension under deformation. In particular, we'll use this to show the non-degeneracy (in the sense of Khovanov-Lauda) of the 2-quantum group $\mathcal{U}$ for an arbitrary Cartan datum and any homogeneous choice of parameters.

1. Introduction

The categorification of Lie algebras and their representations has proven to be a rich and durable subject since its introduction roughly a decade ago. This theory produces a 2-category $\mathcal{U}$ depending on the choice of a Cartan datum and a choice of parameters; a representation of this 2-category is called a **categorical Lie algebra action** of the Kac-Moody algebra $\mathfrak{g}$ corresponding to the Cartan datum. Many interesting categories carry a categorical Lie algebra action, though it must be admitted that most of the interesting examples are for a Cartan datum of (affine) type A.

However, since the 2-quantum group $\mathcal{U}$ was first defined by Khovanov-Lauda [KL10] and Rouquier [Rou], it has been haunted by a serious problem: since it is presented by generators and relations, it is hard to check that it is not smaller than expected. Specifically, in [KL10], it's proven that there is a surjective map from the modified quantum group $\dot{\mathbf{U}}$ to the Grothendieck group of $\mathcal{U}$ and that the dimension of 2-morphism spaces between 1-morphisms in $\mathcal{U}$ is bounded above by a variation on Lusztig's bilinear form on the modified quantum group $\dot{\mathbf{U}}$. If equality holds

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in this bound, then $\dot{\mathbf{U}}$ is isomorphic to the Grothendieck group and we call the corresponding categorification **non-degenerate**.

As a general rule, proving non-degeneracy depends on constructing appropriate representations where one can show that no unexpected relations exist between 2-morphisms in $\mathcal{U}$. This is done for $\mathfrak{sl}_n$ in [KL10], using an action on the cohomology of flag varieties. The next major step was independent proofs by Kang-Kashiwara [KK12] and the author [Web17a] that the simple highest weight representations of $\mathfrak{g}$ possess categorifications which are non-degenerate in an appropriate sense; this is sufficient to prove the non-degeneracy for finite type Cartan data.

This allowed significant progress, but along with many other techniques (such as connections to quiver varieties studied in [CKL13, Rou12, Web17b]), it has an unfortunate defect. Recall that the *open Tits cone* of a Cartan datum is the elements in the orbit of a dominant weight under the Weyl group; for example, if $\mathfrak{g}$ is affine, then the open Tits cone is the set of weights of positive level. This set is convex and every weight of a highest integrable representation lies inside the open Tits cone. Similarly, lowest weights of representations lie in the negative of the Tits cone. If the Cartan datum is of infinite type, then no information about a weight λ outside the open Tits cone and its negative (in the affine case, these are level 0 representations) is contained in any of these representations, since the corresponding idempotent $1_\lambda \in \dot{\mathbf{U}}$ kills any integrable highest weight representation.

Thus, if we are to understand non-degeneracy for weights outside the open Tits cone and its negative, we must have access to representations which are not highest or lowest weight. For our purposes, the most promising are those given by a tensor product of highest and lowest weight representations (perhaps many of each type). A construction of such categorifications was given in [Web15], but the non-degeneracy proof given there is only valid for weights inside the Tits cone. Thus, to access these other weights, we must give a new argument for the non-degeneracy of these tensor product categorifications, which will then imply the non-degeneracy of $\mathcal{U}$. In particular, we prove that:

Theorem A (Theorem 4.10) *Fix a field $\Bbbk$ and consider any Cartan datum $(I, \langle -,- \rangle)$, and choice of the polynomials $Q_{ij}(u,v) \in \Bbbk[u,v]$ which is homogeneous (in the sense discussed in Section 2.1). The associated 2-quantum group $\mathcal{U}$ is non-degenerate, and the Grothendieck group of $\mathcal{U}$ is $\dot{\mathbf{U}}$.*

While an interesting theorem in its own right, the techniques introduced here to prove this result have a significance of their own. A simple, but underappreciated, technique for proving these sort of non-degeneracy arguments is the upper semicontinuity of dimension under deformation. Perhaps calling this "underappreciated" is unfair, since it is certainly a well-known and much used trick, but at least this author wishes he had exploited it more systematically earlier.

Thus, much of this paper will be dedicated to an exploration of the behavior of categorical actions under deformation. Let R be the KLR algebra of the Cartan datum $(I, \langle -, -\rangle)$ (defined in Section 2.1). We wish to consider quotients of this algebra where the dots (the elements usually denoted $y_k \in R$) have a fixed spectrum; of course, all of these quotients can be packaged together into a completion $\hat{R}$. Most often, people have studied representations where the elements y_k act nilpotently (all gradeable finite dimensional representations have this property), but we can also have them act with certain fixed non-zero eigenvalues. Given a choice of spectrum for the dots, we have an associated graph with vertex set $\tilde{I}$, with its associated Cartan datum. There's a natural map $\tilde{I} \to I$, which one can informally think of as a "branched cover" of the Cartan datum $(I, \langle -, -\rangle)$.

This is closely related to the phenomenon of **folding** of Dynkin diagrams, but due to some technical differences, we think it would be misleading to use the term "folding" here. Thus, we call $\tilde{I}$ an **unfurling** of I (and I a **furling** of $\tilde{I}$). Note that whereas I is not necessarily symmetric as a Cartan datum (i.e. it has roots of different lengths), we will define $\tilde{I}$ in such a way as to be symmetric. To give the reader a sense of this operation, let us discuss some examples:

- If I is simply-laced, then $\tilde{I}$ will be a topological cover of I, such as an A_∞ graph covering an n-cycle, or the trivial cover $\tilde{I} \cong I \times U$.
- If I is not simply laced, we can arrange for $\tilde{I}$ to be given by a simply-laced Cartan datum with an isomorphism $\mathfrak{g} \cong \tilde{\mathfrak{g}}^\sigma$ for some diagram automorphism σ. Note, this means that I is the Langlands dual of what is usually called a folding of $\tilde{I}$ for the automorphism σ.

At the moment, it is unclear to the author what, if any, is the relationship between this work and that of McNamara [McN] and Elias [Eli] which also combine the ideas of categorification and folding. Obviously, this would be an interesting topic for future consideration.

We always have a map of Lie algebras $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$, and this map has a categorical analogue:

Theorem B (Proposition 3.3, Theorem 3.10)

(1) *The completion $\hat{R}$ is isomorphic to the completion of the KLR algebra R for $\tilde{I}$ with respect to its grading.*
(2) *A categorical action of $\mathfrak{g}$ satisfying certain spectral conditions corresponding to $\tilde{I}$ also carries an action of $\tilde{\mathfrak{g}}$.*

This theorem is particularly useful when we have a family of categorical $\mathfrak{g}$-modules. We will typically have a categorical action of a larger Lie algebra at the generic point of the deformation, but this action will fail to exist at certain special points where the spectrum drops in size. Explicitly, the operators for $\tilde{\mathfrak{g}}$ are defined as the images of idempotents acting on 1-morphisms in $\mathcal{U}$; the formula for these idempotents depend on dividing by certain expressions that the spectral conditions guarantee are invertible. At points where these denominators vanish, the idempotents and thus their images may no longer be well-defined.

That is, we can take a categorical $\mathfrak{g}$-module C which does not have a $\tilde{\mathfrak{g}}$-action, find a deformation $\tilde{C}$ where the generic point satisfies the spectral conditions, and then exploit the categorical $\tilde{\mathfrak{g}}$-action to prove the original categorification C is not smaller than expected.

This is our strategy for studying the categorifications $\mathcal{X}^{\underline{\lambda}}$ of tensor products of highest and lowest weight representations. These live in a natural family $\mathfrak{X}^{\underline{\lambda}}$, where the generic point $\mathfrak{X}^{\underline{\lambda}}_{\mathbb{K}}$ categorifies an irreducible representation of a larger Lie algebra. We can then apply the categorification result for this larger irreducible (from [KK12, Web17a]) to prove the non-degeneracy of $\mathcal{X}^{\underline{\lambda}}$ and $\mathcal{U}$.

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2. Background

2.1. **The KLR algebra.** Throughout, we'll fix a finite set I, and a Cartan datum on this set. We'll consider the root lattice X, the free abelian group generated by the simple roots α_i for $i \in I$, and we let $\langle -, - \rangle$ denote the symmetric bilinear form on this abelian group attached to the Cartan datum.

The coroot lattice $X^\vee$ is the free abelian group generated by the symbols $\alpha_i^\vee$. We have a pairing of $X^\vee \times X \to \mathbb{Z}$ such that $\alpha_i^\vee(\alpha_j) := 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}$. The matrix of this form is the Cartan matrix $C = (c_{ij} = \alpha_i^\vee(\alpha_j))$. Note that $d_i = \langle \alpha_i, \alpha_i \rangle / 2$ are symmetrizing coefficients for this Cartan matrix: $d_i c_{ij} = d_j c_{ji} = \langle \alpha_i, \alpha_j \rangle$ for all i, j. For our purposes a **weight** is an element of the dual lattice to $X^\vee$, so the pairing above lets us consider each element of the root lattice as a weight[2]. Given a weight λ, we let $\lambda^i = \alpha_i^\vee(\lambda)$.

Fix an algebraically closed field $\Bbbk$ of characteristic coprime to all d_i, and choose polynomials $P_{ij}(u, v)$ such that the product $Q_{ij}(u, v) = P_{ij}(u, v) P_{ji}(v, u)$ is homogeneous of degree $-2\langle \alpha_i, \alpha_j \rangle = -2d_i c_{ij} = -2d_j c_{ji}$ when u is given degree $2d_i$ and v degree $2d_j$. We'll assume throughout that $Q_{ij}(1, 0)$ is non-zero for all i, j. Let $p_{ij} = P_{ij}(1, 0)$. Let

[2]In terms of Kac-Moody groups, these are weights of the torus of the derived subgroup of a Kac-Moody group of this type; the rest of the torus will not play an important role for us.

$g_{ij} = \gcd(-c_{ij}, -c_{ji})$ and $h_{ij} = -c_{ij}/g_{ij}$. We must have that

$$P_{ij}(u,v) = p_{ij} \prod_{a_{ij}^{(k)} \in A_{ij}} (u^{1/d_i} - a_{ij}^{(k)} v^{1/d_j})$$

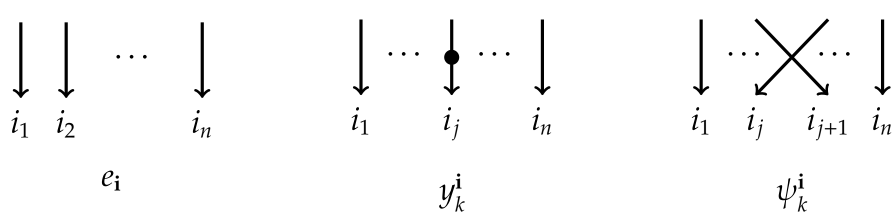
where $A_{ij} = \{a_{ij}^{(k)}\}$ is the multiset of roots of $P_{ij}(x^{d_i}, 1)$, considered with multiplicity. Let $B_{ij} = \{b_{ij}^{(k)} = (a_{ji}^{(k)})^{-1}\}$ be the reciprocals of these numbers. Note that

$$Q_{ij}(u,v) = t_{ij} \prod_{a_{ij}^{(k)} \in A_{ij}} (u^{1/d_i} - a_{ij}^{(k)} v^{1/d_j}) \prod_{b_{ij}^{(k)} \in B_{ij}} (u^{1/d_i} - b_{ij}^{(k)} v^{1/d_j})$$

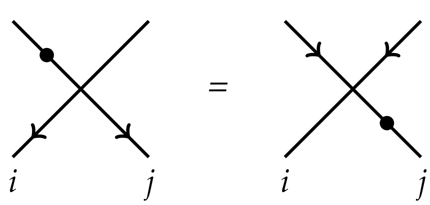
Homogeneity requires that $P_{ij}(x^{1/h_{ij}}, 1)$ be a polynomial. We'll also let $\mathcal{A}_{ij} = \{\alpha_{ij}^{(k)}\}$ be the roots of $P_{ij}(x^{1/h_{ij}}, 1)$, again considered with multiplicity; note that the elements of A_{ij} are the $d_i h_{ij}$th roots of the elements of $\mathcal{A}_{ij}$. Furthermore, we have that $d_i h_{ij} = d_j h_{ji} = \mathrm{lcm}(d_i, d_j)$. We also let $\mathcal{B}_{ij} = \{\beta_{ij}^{(k)} = (\alpha_{ji}^{(k)})^{-1}\}$. As before, the elements of B_{ij} are the $d_i h_{ij}$th roots of the elements of $\mathcal{B}_{ij}$.

Definition 2.1 *Let R denote the KLR algebra with generators given by:*

- *The idempotent $e_{\mathbf{i}}$ which is straight lines labeled with $(i_1, \dots, i_n) \in I^n$.*
- *The element $y_k^{\mathbf{i}}$ which is just straight lines with a dot on the kth strand.*
- *The element $\psi_k^{\mathbf{i}}$ which is a crossing of the i and $i+1$st strand.*



and relations:

(2.1a) 

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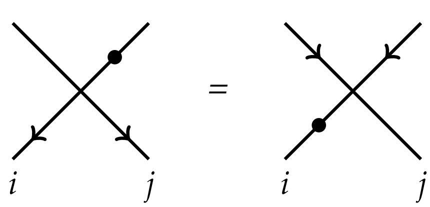
(2.1b) 

unless $i = j$

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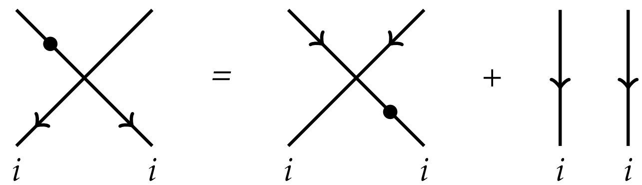

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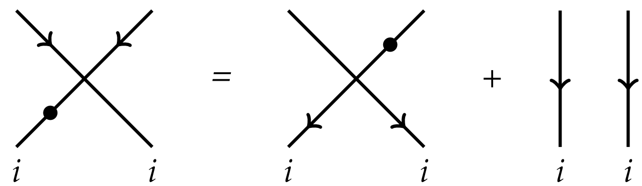

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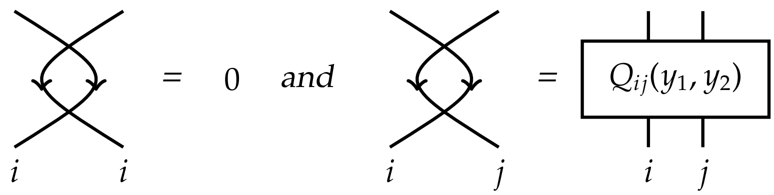

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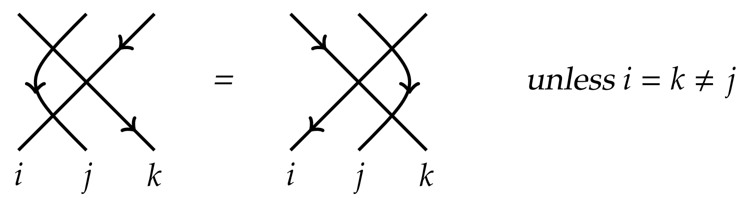

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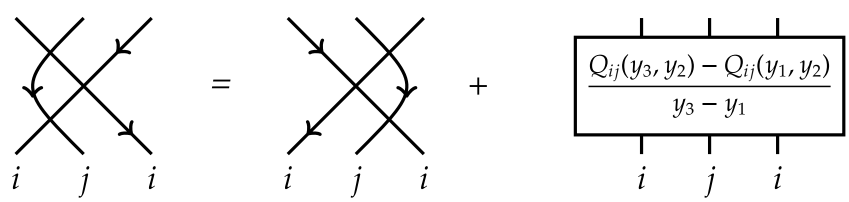


Fix a countable set $U_i \subset \Bbbk \setminus \{0\}$ for each $i \in I$; for each $u \in U_i$, we fix a choice of d_ith root, which we denote u^{1/d_i} (there are d_i choices, since d_i is coprime to the characteristic of $\Bbbk$). Since U_i is countable, we can write it as a union of nested finite sets $U_i^{(N)}$ for $N \in \mathbb{Z}_{\geq 0}$ (this choice is purely for technical reasons, and nothing we do will depend on it).

Definition 2.2 *Let $\hat{R}_n$ be the completion of the KLR algebra R_n by the system of ideals $\mathscr{I}_N$ generated by $e_{\mathbf{i}} \prod_{u \in U_{i_j}^{(N)}} (y_j - u)^N$ for all $j \in [1, n]$ and $\mathbf{i} \in I^n$.*

This is the coarsest completion where we require that the topological spectrum of a dot on a strand with label i lives in the set U_i.

2.2. **Valued graphs.** We'll follow the conventions of Lemay [Lem] in this section. For simplicity, "graph" will always mean a graph without loops.

Definition 2.3 *A* **relatively valued graph** *is an oriented graph with vertex set I with a pair of rational numbers (η_e, ν_e) assigned to each edge such that there exist $d_i \in \mathbb{Q}$ for each $i \in I$ such that $d_i\eta_e = d_j\nu_e$ for $e\colon i \to j$.*

An **absolutely valued graph** *is an oriented graph as above with a choice of rational numbers d_i for each vertex i and m_e for each edge e.*

Each absolutely valued graph has an associated relatively valued graph with

$$\eta_e = \frac{m_e}{d_i} \qquad \nu_e = \frac{m_e}{d_j}$$

for an edge $e\colon i \to j$, and every relatively valued graph has this form. The values η_e and ν_e will be integers if d_i and m_e are and $\mathrm{lcm}(d_i, d_j)$ divides m_e for an edge $e\colon i \to j$. Note that relatively valued graphs have a natural notion of **Langlands duality**, given by switching η_e and ν_e. We attach a Cartan matrix to each such graph without loops, with $c_{ii} = 2$ and

$$c_{ij} = -\sum_{e\colon i\to j} \eta_e - \sum_{e\colon j\to i} \nu_e = -\frac{1}{d_i}\sum_{e\colon i\to j} m_e - \frac{1}{d_j}\sum_{e\colon j\to i} m_e.$$

Note that Langlands duality transposes this Cartan matrix.

Having chosen $P_{ij}(u, v)$ for each pair $i, j \in I^2$, we can canonically associate an absolutely valued graph with vertex set I where we add an edge $i \to j$ whenever $P_{ij}(u, v)$ is non-constant. The values d_i are as before, and $m_e = \deg P_{ij}(x^{d_i}, 0) = \deg P_{ij}(0, x^{d_j})$. In the associated relatively valued quiver, the values we add to this edge are $(\deg P_{ij}(x, 0), \deg P_{ij}(0, x))$. The Cartan matrix of the result is our original Cartan matrix C.

Given a graph homomorphism between two valued graphs, we can consider various forms of compatibility between the valuings on the two graphs. One notion considered by Lemay [Lem] is a morphism of valued graphs: this is a homomorphism of graphs where the appropriate statistics $(\eta_*, \nu_*, m_*, d_*)$ are preserved; this is too inflexible for our purposes. Instead, we'll consider a set of maps which are more analogous to topological covers.

Definition 2.4 *We call a map $f\colon X \to Y$ of relatively valued graphs a* **furling** *if given any $y, y' \in Y$, and $x \in f^{-1}(y)$, we have that each edge $d\colon y \to y'$ and each edge $e\colon y' \to y$,*

$$\nu_d = \sum_{x'\in f^{-1}(y')} \sum_{\substack{d'\colon x\to x' \\ f(d')=d}} \nu_{d'} \qquad \eta_e = \sum_{x'\in f^{-1}(y')} \sum_{\substack{e'\colon x'\to x \\ f(e')=e}} \eta_{e'}.$$

The notion of a furling is very closely related to a "folding," but we won't use this term, since it usually applies to the Langlands dual of the operation above, and implies the existence of a group action. We'll call Y a **furling** of X and X an **unfurling** of Y. A morphism of relatively valued graphs which is also a topological cover is a furling.

Note that if X has an compatible absolute valued structure such that d_x and m_e is constant on the fibers of f and these fibers are finite, then for an edge $e\colon y' \to y$, we have that

$$\eta_e = \sum_{x' \in f^{-1}(y')} \frac{1}{d_{x'}} \sum_{\substack{e'\colon x' \to x \\ f(e')=e}} m_{e'} \qquad \nu_e = \sum_{x \in f^{-1}(y)} \frac{1}{d_x} \sum_{\substack{e'\colon x' \to x \\ f(e')=e}} m_{e'}.$$

Thus, we can choose an absolute valued structure on Y such that

$$d_y = \frac{d_x}{|f^{-1}(y)|} \qquad m_e = \frac{\sum_{f(e')=e} m_{e'}}{|f^{-1}(y)| \cdot |f^{-1}(y')|}.$$

As defined here, d_y and m_e may not be integers, but if Y is finite, then we can always just multiply every d_y and m_e by $\mathrm{lcm}(|f^{-1}(y)|)$ to clear denominators.

One special case of particular interest is when X is given the trivial valuation $d_x = m_e = \nu_e = \eta_e = 1$ and is equipped with an admissible automorphism σ; recall that we call an automorphism of a graph **admissible** if no edges connect two vertices in the same orbit under the action. We let Y be the quotient graph X/σ and $f\colon X \to Y$ the obvious projection map. In this case, we have

$$d_y = \frac{1}{|f^{-1}(y)|} \qquad m_e = \frac{|f^{-1}(e)|}{|f^{-1}(y)| \cdot |f^{-1}(y')|}. \tag{2.2}$$

This is the Langlands dual of the "folding" discussed in [Lem, §1] (which is the more common way of associating a Cartan matrix to a graph with automorphism).

Lemma 2.5 *Given a furling $f\colon X \to Y$, for any fixed $y, y' \in Y$ and $x' \in f^{-1}(y')$ we have that:*

$$c_{yy'} = \sum_{x \in f^{-1}(y)} c_{xx'}.$$

Proof.

$$\begin{aligned} c_{yy'} &= -\sum_{e\colon y \to y'} \eta_e - \sum_{e\colon y' \to y} \nu_e \\ &= -\sum_{x \in f^{-1}(y)} \left(\sum_{e\colon x \to x'} \eta_e - \sum_{e\colon x' \to x} \nu_e \right) \\ &= \sum_{x \in f^{-1}(y)} c_{xx'} \end{aligned}$$

□

We assume from now on for all valued graphs appearing that the matrix C is a generalized Cartan matrix (in particular, all off-diagonal entries are negative integers).

Definition 2.6 *Given a valued graph X, let $\mathfrak{g}_X$ be the associated derived Kac-Moody algebra generated by E_i, F_i, H_i with the relations*

$$[H_i, E_j] = c_{ij}E_j \qquad [H_i, F_j] = -c_{ij}F_j \qquad [E_i, F_j] = \delta_{ij}H_i \tag{2.3}$$

$$\mathrm{ad}_{E_i}^{1-c_{ij}} E_j = \mathrm{ad}_{F_i}^{1-c_{ij}} F_j = 0. \tag{2.4}$$

A straightforward extension of [Kac90, 7.9] shows that:

Proposition 2.7 *If $f\colon X \to Y$ is a furling of valued graphs, there is an induced homomorphism of Kac-Moody algebras $\mathfrak{g}_Y \to \mathfrak{g}_X$ given by the formulas:*

$$F_y \mapsto \sum_{x\in f^{-1}(y)} F_x \qquad E_y \mapsto \sum_{x\in f^{-1}(y)} E_x \qquad H_y \mapsto \sum_{x\in f^{-1}(y)} H_x.$$

Proof. The relations (2.3) are straightforward computations using Lemma 2.5. We have that:

$$\Big[\sum_{x\in f^{-1}(y)} H_x, \sum_{x'\in f^{-1}(y')} E_{x'}\Big] = \sum_{x'\in f^{-1}(y')} \Big(\sum_{x\in f^{-1}(y)} c_{xx'}\Big)E_{x'} = \sum_{x'\in f^{-1}(y')} c_{yy'}E_{x'}$$

$$\Big[\sum_{x\in f^{-1}(y)} H_x, \sum_{x'\in f^{-1}(y')} F_{x'}\Big] = \sum_{x'\in f^{-1}(y')} \Big(\sum_{x\in f^{-1}(y)} -c_{xx'}\Big)F_{x'} = \sum_{x'\in f^{-1}(y')} -c_{yy'}F_{x'}$$

$$\Big[\sum_{x\in f^{-1}(y)} E_x, \sum_{x'\in f^{-1}(y')} F_{x'}\Big] = \delta_{y,y'} \sum_{x\in f^{-1}(y)} H_x$$

The remaining relations (2.4) follow from the Gabber-Kac theorem [GK81]. □

We wish to consider the "order of vanishing" of $Q_{ij}(x, y)$ at $x = u, y = u'$; of course, this is not well-defined for a 2-variable polynomial, but because of the homogeneity, we can make sense of it in this case as the vanishing order of $Q_{ij}(u, y)$ at $y = u'$ or of $Q_{ij}(x, u')$ at $x = u$. For a general 2-variable polynomial, these will not be the same, but in our case, it will be the number of elements of A_{ij} such that $u^{1/d_i} = a_{ii'}^{(k)}(u')^{1/d_{i'}}$. These solutions are also in bijection with solutions to $u^{h_{ij}} = \alpha_{ii'}^{(k)}(u')^{h_{ij}}$ for $\alpha_{ii'}^{(k)} \in \mathcal{A}_{ii'}$ (which tend to be slightly easier to count).

Definition 2.8 *Let $\tilde{I}$ be the oriented graph whose vertex set is the pairs $\{(i, u) \in I \times \Bbbk \mid u \in U_i\}$ with the number of edges oriented from (i, u) to (i', u') being the order of vanishing of $Q_{ij}(x, y)$ at $x = u, y = u'$ as defined above. We will consider this as an absolutely/relatively valued graph with trivial valuation $\eta_e = \nu_e = d_i = m_e = 1$.*

Let $\tilde{\mathfrak{g}}$ be the associated Kac-Moody algebra to this graph.

Definition 2.9 *We call a choice of spectra U_i* **complete** *if whenever $Q_{ij}(u, u') = 0$ for $u \in U_i$, then $u' \in U_j$.*

Note that if all $a_{ij}^{(k)}$'s are mth roots of unity for some m, then any finite choice of U_i's can be made complete by adding finitely many elements (for example, all products of elements of U_i and mth roots of unity will suffice). Any countable choice of spectra can be completed to a countable complete choice of spectra; we let $V_i^{(0)} = U_i$ and define $V_i^{(k)} = \{v \in \Bbbk \mid Q_{ij}(v, v') = 0 \text{ for } v' \in V_j^{(k-1)}\}$. The union $V_i^{(\infty)} = \cup_{i=0}^{\infty} V_i^{(k)}$ is a complete choice of spectra.

Proposition 2.10 *If U_i is a complete choice of spectra, then the map $\tilde{I} \to I$ is a furling of valued graphs.*

Proof. The unique edge $e\colon i \to i'$ in I has preimages corresponding to each element $u \in U_i$ and each root of $P_{ii'}(u, x)$ as a polynomial in x. Thus the number of preimages e' is the degree of this polynomial in x, and each has $\nu_{e'} = 1$, so this agrees with $\nu_e = \deg P_{ij}(0, x)$. Similarly, if we consider the edge $d\colon i' \to i$, the edges are in bijection with the roots of $P_{i'i}(x, u)$, and $\eta_d = \deg P_{i'i}(x, 0)$. This completes the proof. □

The most important example is the so-called "geometric" parameters for the symmetric Cartan matrix for an oriented graph, where $\mathsf{P}_{ii'}(u, v) = (u - v)^{\# i \to i'}$. In this case, $a_{ii'}^{(k)} = 1$ for all k and (i, u) is connected to (i', u') by the same number of edges as i and i' if $u = u'$ and none otherwise. Thus, if we choose $U_i = U$ for some fixed set $U \subset \Bbbk$, this is a complete choice of spectra and $\tilde{I} = I \times U$ with the obvious graph structure.

If I is simply laced, but not simply-connected, then we can obtain non-trivial covers as $\tilde{I}$. For example, if I in an n-cycle with its vertex set identified with $\mathbb{Z}/n\mathbb{Z}$ with edges $i \to i + 1$. Fix some $q \in \Bbbk$ and choose $Q_{i,i+1}(u, v) = qu - v$. If we fix $U_0 \subset \Bbbk$ to be any subset closed under multiplication by q^n, then we have a complete choice of spectra with $U_i = q^i U_0$. The components of the graph $\tilde{I}$ correspond to the orbits of multiplication by q; these will be cycles if q is a root of unity, or A_∞ graphs if q is not.

On the other hand, if we have a nonsymmetric Cartan matrix, then we may find a more interesting result. If $Q_{12}(u, v) = u^2 - v$ and $d_1 = 1, d_2 = 2$ (so $\mathfrak{g} = \mathfrak{sp}_4$), then the factorization shown earlier is that $Q_{12}(u, v) = (u + v^{1/2})(u - v^{1/2})$, and so $a_{12}^{(1)} = 1$ and $a_{12}^{(2)} = -1$. Thus, the number of edges joining $(1, x)$ to $(2, y)$ is given by the number of solutions to $x = \pm\sqrt{y}$. Thus, every component of $\tilde{I}$ is a subgraph of an A_3 formed by $(1, x) \to (2, x^2) \leftarrow (1, -x)$ (assuming $1 \neq -1$).

More generally, let $d = \operatorname{lcm}(d_i)_{i \in I}$. Let U_i be the d/d_ith roots of unity; there are d/d_i distinct roots of unity since d is coprime to the characteristic of $\Bbbk$. Assume that each $a_{ij}^{(k)}$ is a dth root of unity for all i, j, k. For example, we can assume that

$$Q_{ij}(u, v) = \pm(u^{h_{ij}} - v^{h_{ji}})^{g_{ij}} \tag{2.5}$$

in which case, $Q_{ij}(x^{d_i}, 1) = \pm(x^{d_i h_{ij}} - 1)^{g_{ij}}$, so the multiset of $a_{ij}^{(k)}$ and $b_{ij}^{(k)}$ is given by the $d_i h_{ij} = d_j h_{ji} = \operatorname{lcm}(d_i, d_j)$ roots of unity each with multiplicity g_{ij}. Alternatively, we have $\alpha_{ij}^{(k)} = \beta_{ij}^{(k)} = 1$ for any k.

We have that each $u \in U_i$ is connected by c_{ji} edges to elements of U_j, given by the h_{ji}th roots of $\alpha_{ij}^{(k)} u^{h_{ij}}$. If Q_{ij} is as in (2.5), then for each $d/d_i h_{ij}$th root of unity ξ, we connect each h_{ij}th root of ξ in U_i to each h_{ji}th root of ξ in U_j with g_{ij} edges (with orientation depending on P_{ij}). Let ζ be a primitive dth root of unity.

Proposition 2.11 *Assuming $a_{ij}^{(k)}$ is a dth root of unity for all i, j, k and U_i is the d/d_ith roots of unity, the map $\sigma\colon (i, u) \mapsto (i, \zeta^{d_i} u)$ is an admissible automorphism of the graph $\tilde{I}$; the map $\tilde{I} \to I = \tilde{I}/\sigma$ induces the relative valued structure on I associated to the polynomials P_{ij}.*

Note that the absolute weighting of (2.2) is the symmetrization we have chosen for our Cartan matrix divided by d, since $|f^{-1}(i)| = d/d_i$.

3. Completed KLR algebras

3.1. **An isomorphism of completed KLR algebras.** In this section we'll show how the completions of KLR algebras for I we discussed earlier are related to the KLR algebras of $\tilde{I}$.

Let $\mathbf{j} = (j_1 = (i_1, u_1), \dots, j_n = (i_n, u_n))$. By abstract Jordan decomposition, for any element $x \in \hat{R}$ and $u \in \mathbb{k}$, there is unique idempotent e in the quotient $\hat{R}/I_N$ whose image is the generalized u-eigenspace of x acting by left multiplication; by uniqueness, these are compatible under the quotient maps, and thus give an idempotent in $\hat{R}$. Performing this construction inductively, we can consider any commuting set of elements $\{x_j\}$ and construct an idempotent projecting to their simultaneous generalized eigenspace in $\hat{R}/I_N$ for any choice of scalars $\{u_j\}$.

Definition 3.1 *Let $\epsilon_{\mathbf{j}}$ be the resulting idempotent where we perform this construction with $e_{\mathbf{i}}$ having eigenvalue 1 and y_j eigenvalue u_j for all j.*

In particular, $e_{\mathbf{i}}\epsilon_{\mathbf{j}} = \epsilon_{\mathbf{j}} e_{\mathbf{i}} = \epsilon_{\mathbf{j}}$, and $(y_j - u_j)\epsilon_{\mathbf{j}}$ is topologically nilpotent.

There is a unique d_{i_k}th root of $y_k\epsilon_{\mathbf{j}}$ such that $(y_k^{1/d_{i_k}} - u_k^{1/d_{i_k}})\epsilon_{\mathbf{j}}$ is topologically nilpotent. This is given by

$$y_k^{1/d_{i_k}} = u_k^{1/d_{i_k}}(u_k^{-1}y_k)^{1/d_{i_k}} = u_k^{1/d_{i_k}}\Big(1 + \frac{1}{d_{i_k}}(u_k^{-1}y_k - 1) + \binom{1/d_{i_k}}{2}(u_k^{-1}y_k - 1)^2 + \binom{1/d_{i_k}}{3}(u_k^{-1}y_k - 1)^3 + \cdots\Big).$$

Here the symbol $\binom{1/d_{i_k}}{n}$ denotes the image of this binomial coefficient under the canonical map $\mathbb{Z}[1/d_{i_k}] \to \mathbb{k}$.

Consider the KLR algebra R of $\tilde{I}$ with the symmetric Cartan datum associated to its graph structure: to avoid confusion, we'll denote the elements $\psi_k, \mathsf{y}_k, \mathsf{e}_{\mathbf{j}}$ of this algebra with sans serif letters. We use the geometric coefficients given by $\mathsf{P}_{jj'}(u, v) =$

$(u-v)^{\#j\to j'}$. We let A_k be the unique invertible element of the completion $\hat{R}$ such that

$$A_k\epsilon_{\mathbf{j}} = p_{i_ki_{k+1}}\epsilon_{\mathbf{j}} \prod_{u_k^{1/d_{i_k}} \neq a^{(m)}_{i_ki_{k+1}}(u')^{1/d_{i_{k+1}}}} (y_k^{1/d_{i_k}} - a^{(m)}_{i_ki_{k+1}} y_{k+1}^{1/d_{i_{k+1}}}) \prod_{u_k^{1/d_{i_k}} = a^{(m)}_{i_ki_{k+1}}(u')^{1/d_{i_{k+1}}}} u_k^{1/d_{i_k}}$$

The multiplicative inverse A_k^{-1} makes sense because each non-scalar factor of $A_k\epsilon_{\mathbf{j}}$ is of the form $(u_k^{1/d_{i_k}} - b^{(m)}_{i_ki_{k+1}} u_{k+1}^{1/d_{i_{k+1}}})\epsilon_{\mathbf{j}}$, which is a non-zero multiple of the idempotent $\epsilon_{\mathbf{j}}$ plus a topologically nilpotent element. Note that:

$$P_{i_ki_k+1}(y_k,y_{k+1})\epsilon_{\mathbf{j}} = A_k \prod_{u_k^{1/d_{i_k}} = a^{(m)}_{i_ki_{k+1}} u_{k+1}^{1/d_{i_{k+1}}}} \Big((u_k^{-1}y_k)^{1/d_{i_k}} - (u_{k+1}^{-1}y_{k+1})^{1/d_{i_{k+1}}}\Big)\epsilon_{\mathbf{j}} \tag{3.1}$$

Proposition 3.2 *There is a homomorphism $\nu\colon \mathsf{R} \to \hat{R}$ sending*

$$\mathsf{e}_{\mathbf{j}} \mapsto \epsilon_{\mathbf{j}} \qquad \mathsf{y}_k\mathsf{e}_{\mathbf{j}} \mapsto \Big((u_k^{-1}y_k)^{1/d_{i_k}} - 1\Big)\epsilon_{\mathbf{j}}$$

$$\uppsi_k\mathsf{e}_{\mathbf{j}} \mapsto \begin{cases} A_k^{-1}\psi_k\epsilon_{\mathbf{j}} & i_k \neq i_{k+1} \\ ((y_{k+1}-y_k)\psi_k + 1)\epsilon_{\mathbf{j}} & i_k = i_{k+1}, u_k \neq u_{k+1} \\ \psi_k\epsilon_{\mathbf{j}} & j_k = j_{k+1} \end{cases}$$

Proof. One can see that this is a homomorphism by comparing polynomial representations. By [Rou, 3.12], there is a polynomial representation of R_n on $Z = \oplus_{\mathbf{i}\in I^n} \Bbbk[z_1,\dots,z_n]e_{\mathbf{i}}$ by the rule:

$$y_k \cdot fe_{\mathbf{i}} = z_kfe_{\mathbf{i}} \qquad e_{\mathbf{i}'} \cdot fe_{\mathbf{i}} = \delta_{\mathbf{i},\mathbf{i}'}fe_{\mathbf{i}}$$

$$\psi_k \cdot fe_{\mathbf{i}} = \begin{cases} \dfrac{f^{(k,k+1)} - f}{z_{k+1} - z_k} e_{\mathbf{i}} & i_k = i_{k+1} \\ P_{i_{k+1}i_k}(z_k,z_{k+1}) \cdot f^{(k,k+1)} e_{\mathbf{i}^{(k,k+1)}} & i_k \neq i_{k+1} \end{cases}$$

and similarly R_n has a representation of $\mathsf{Z} = \oplus_{\mathbf{j}\in\tilde{I}^n} \Bbbk[\mathsf{z}_1,\dots,\mathsf{z}_n]e_{\mathbf{j}}$ by the same formulas. The same formula as ν gives a homomorphism $\nu_Z\colon \mathsf{Z} \to \hat{Z}$:

$$\mathsf{z}_k\mathsf{e}_{\mathbf{j}} \mapsto \Big((u_k^{-1}z_k)^{1/d_{i_k}} - 1\Big)\epsilon_{\mathbf{j}}$$

Note that this map becomes an isomorphism after completion, since $(u_k^{-1}z_k)^{1/d_{i_k}} - 1 = \frac{1}{d_{i_k}}u_k^{-1}z_k + \cdots$. The homomorphism ν is induced by transport of structure via ν_Z. The only interesting calculation needed to confirm this is the image of $\uppsi_k\mathsf{e}_{\mathbf{j}}$ if $j_k \neq j_{k+1}$. From the definition, we have that

$$\uppsi_k\mathsf{e}_{\mathbf{j}} \cdot f\epsilon_{\mathbf{j}} = \prod_{u_{k+1}^{1/d_{i_{k+1}}} = a^{(m)}_{i_{k+1}i_k} u_k^{1/d_{i_k}}} \Big((u_{k+1}^{-1}z_{k+1})^{1/d_{i_{k+1}}} - (u_k^{-1}z_k)^{1/d_{i_k}}\Big) f^{(k,k+1)}\epsilon_{\mathbf{j}^{(k,k+1)}}$$

Equation (3.1) shows that the RHS is the same as $A_k^{-1}\psi_k \cdot f\epsilon_{\mathbf{j}}$. □

Let $\hat{\mathsf{R}}$ be the completion of R with respect to the grading.

Proposition 3.3 *The map ν induces an isomorphism $\hat{\mathsf{R}} \to \hat{R}$.*

Proof. Let A_k be the element of $\hat{\mathsf{R}}$ mapping to A_k under ν. That is:

$$\mathsf{A}_k\mathsf{e}_\mathbf{j} = p_{i_k i_{k+1}}\mathsf{e}_\mathbf{j} \prod_{u_k^{1/d_{i_k}} \neq a^{(m)}_{i_k i_{k+1}}(u')^{1/d_{i_{k+1}}}} \left(u_k^{1/d_{i_k}}(1+\mathsf{y}_k) - a^{(m)}_{i_k i_{k+1}} u_{k+1}^{1/d_{i_{k+1}}}(1+\mathsf{y}_{k+1})\right) \prod_{u_k^{1/d_{i_k}} = a^{(m)}_{i_k i_{k+1}}(u')^{1/d_{i_{k+1}}}} u_k^{1/d_{i_k}}$$

Note that A_k is invertible in $\hat{\mathsf{R}}$, since it has non-zero constant term.

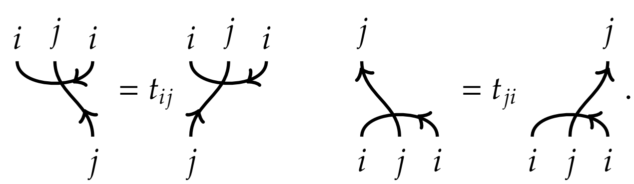
The homomorphism ν is inverted by the map:

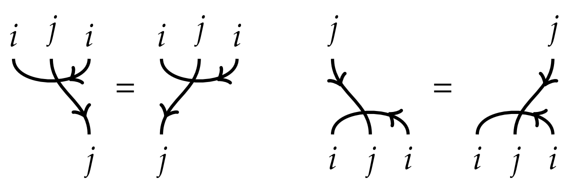
$$\epsilon_\mathbf{j} \mapsto \mathsf{e}_\mathbf{j} \qquad y_k\epsilon_\mathbf{j} \mapsto u_k(1+\mathsf{y}_k)^{d_{i_k}}$$

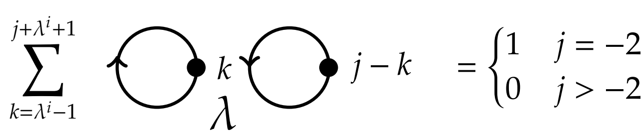
$$\psi_k\epsilon_\mathbf{j} \mapsto \begin{cases} \mathsf{A}_k\uppsi_k\mathsf{e}_\mathbf{j} & i_k \neq i_{k+1} \\ \dfrac{1}{u_{k+1}(1+\mathsf{y}_{k+1})^{d_{i_{k+1}}} - u_k(1+\mathsf{y}_k)^{d_{i_k}}}(\psi_k - 1)\mathsf{e}_\mathbf{j} & i_k = i_{k+1}, u_k \neq u_{k+1} \\ \uppsi_k\mathsf{e}_\mathbf{j} & j_k = j_{k+1} \end{cases} \tag{3.2}$$

Thus the map ν is an isomorphism after completion. □

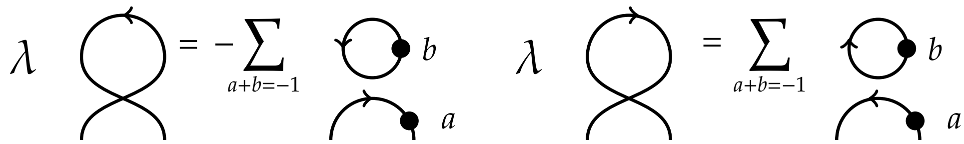
3.2. **Application to categorical actions.** We've considered the KLR algebra R with an eye toward studying categorical actions of Lie algebras. By "a categorical action of a Lie algebra" we mean a representation of a specific 2-category $\mathcal{U}$ defined by Khovanov-Lauda [KL10] and Rouquier [Rou] (the equivalence of these 2-categories is proven in [Bru16]). We'll follow the conventions of [Web17a, 2.4], where this 2-category is presented by taking the quotient of a 2-category $\tilde{\tilde{\mathcal{U}}}$ of KL diagrams (see [Web17a, 2.3]) by the KLR relations (2.1a–2.1g) and the following relations on 2-morphisms:

(3.3a)


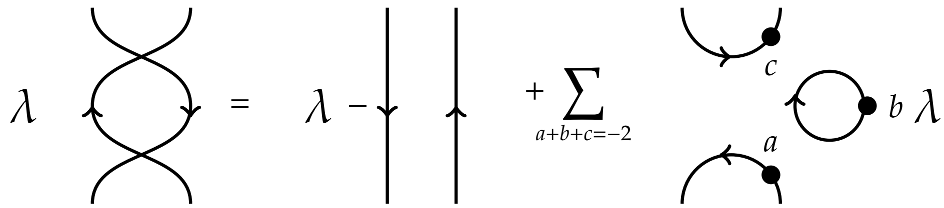
(3.3b)


(3.3c)


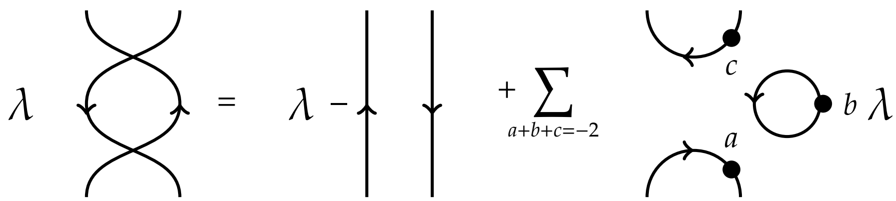
(3.3d)



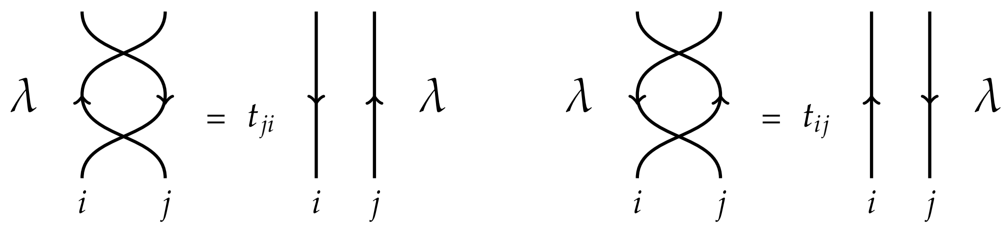
(3.3e)



(3.3f)



(3.3g)



In much of the literature, the representations of $\mathcal{U}$ considered have had the action of the dots and bubbles be nilpotent; this is necessary in a graded 2-representation in the 2-category of Schurian categories over a field $\Bbbk$. However, it can be a very powerful technique to deform these representations in such a way as to break this assumption. Let $\Bbbk$ be a field. Consider a representation of the 2-category $\mathcal{U}$, sending $\lambda \mapsto C_\lambda$ such that C_λ is $\Bbbk$-linear Schurian; that is, all objects are of finite length, there are enough projectives and injectives, and the endomorphism algebras of the irreducible objects are one dimensional.

We'll first want a preparatory lemma about the structure of these sort of actions. The conditions above guarantee that for any object in C_λ, any endomorphism satisfies a minimal polynomial of degree bounded above by the length of the object. Let S be a simple object in C_λ

Definition 3.4 *Let $p_S \in \Bbbk[x]$ be the minimal polynomials of y acting on $\mathcal{E}_iS$ and similarly, q_S the minimal polynomial of y acting on $\mathcal{F}_iS$.*

We'll assume for the sake of simplicity that the polynomials p_S and q_S split completely in $\mathbb{k}$ (of course, we can always assure this as the cost of passing to the algebraic closure of $\mathbb{k}$), so they are the products of the form

$$p_S(x) = \prod_{u\in\mathbb{k}}(x-u)^{a_{S,u}} \qquad q_S(x) = \prod_{u\in\mathbb{k}}(x-u)^{b_{S,u}}. \tag{3.4}$$

This ensures that on any object X in C, the endomorphism y acts on $\mathcal{E}_iX$ or $\mathcal{F}_iX$ by a minimal polynomial that splits completely, since it must divide the product of the minimal polynomials of the composition factors. Thus, projection to the generalized eigenspace is a natural transformation of $\mathcal{E}_i$ or $\mathcal{F}_I$.

Definition 3.5 *Let $\mathcal{E}_{i,u}$ be the u-generalized eigenspace of y acting on $\mathcal{E}_i$, and similarly, $\mathcal{F}_{i,u}$ the analogous eigenspace for $\mathcal{F}_i$.*

The complete splitting on minimal polynomials guarantees that $\mathcal{E}_i \cong \oplus_{u\in\mathbb{k}}\mathcal{E}_{i,u}$ and $\mathcal{F}_i \cong \oplus_{u\in\mathbb{k}}\mathcal{F}_{i,u}$. Note that the functor $\mathcal{E}_{i,u}$ can be non-zero for infinitely many u, but any given object will be killed by almost all such functors by the finite length hypothesis. Note that:

Lemma 3.6 *We have that $\mathcal{E}_i^{(\deg p_S+1)}S = \mathcal{F}_i^{(\deg q_S+1)}S = 0$. In particular, the categorical action on C is locally nilpotent, in the sense of* [Rou].

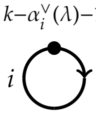
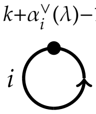
Note, this is nilpotence on the level of 1-morphisms and unrelated to whether the natural transformation y is nilpotent, which we are *not* assuming.

Proof. Recall that if we let $\psi^{(n)}$ denote the half-twist of strands in the nilHecke algebra acting on $\mathcal{E}_i^nS$, then we have $\psi^{(n)} = \psi^{(n)}\cdot(y^{n-1}\otimes y^{n-2}\otimes\cdots\otimes 1)\cdot\psi^{(n)}$, and if we put in any lower degree polynomial in the y's we get 0. Thus, we have

$$\psi^{(\deg p_S+1)} = \psi^{(\deg p_S+1)}\cdot(p_s(y)\otimes y^{\deg p_S-1}\otimes\cdots\otimes 1)\cdot\psi^{(\deg p_S+1)} = 0.$$

Since $\mathcal{E}_i^{(\deg p_S+1)}S$ is the image of $\psi^{(\deg p_S+1)}$, this is 0 as well. A symmetric argument shows the result for $\mathcal{F}_i$ as well. □

Let

(3.5) $\circlearrowright_i(w) = \sum_{k=0}^{\infty}$ i  w^k $\qquad \circlearrowleft_i(w) = \sum_{k=0}^{\infty}$ i  w^k

Note that these series satisfy $\circlearrowright_i(w)\cdot\circlearrowleft_i(w) = 1$, by (3.3c).

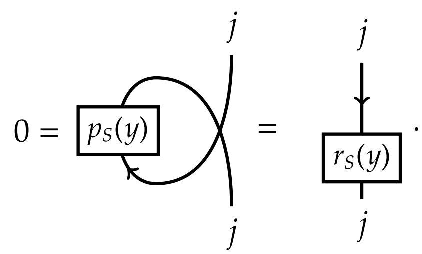
Lemma 3.7 *The action of these series on S satisfy*

$$\circlearrowright_i(w) = w^{\alpha_i^\vee(\lambda)}\frac{q_S(w^{-1})}{p_S(w^{-1})} \qquad \circlearrowleft_i(w) = w^{-\alpha_i^\vee(\lambda)}\frac{p_S(w^{-1})}{q_S(w^{-1})}.$$

Of course, the right hand side in the equalities above should be interpreted as the Taylor expanstion of these rational functions at $w = 0$.

Proof. Note that the w^k term of the series $\circlearrowright_i(w)p_S(w^{-1})$ is given by closing the Laurent polynomial $y^{k-\alpha_i^\vee-1}(\lambda)p_S(y)$ in the dot y on a clockwise bubble, where we interpret this expression using fake bubbles if the Laurent polynomial has negative powers of y. Thus, the fact that dots on $\mathcal{E}_i$ satisfy the polynomial relation $p_S(y) = 0$ shows that $\circlearrowright_i(w)p_S(w^{-1})$ vanishes in all degrees where no fake bubbles are used in this expression, that is, when $k > \alpha_i^\vee(\lambda)$ (this includes negative degrees). In particular, this shows that $\circlearrowright_i(w)$ is of the form $w^{-\alpha_i^\vee(\lambda)}r_S(w^{-1})/p_S(w^{-1})$ for some monic polynomial $r_S(w)$ of degree $\deg p_S + \alpha_i^\vee(\lambda)$.

Furthermore, if we consider the loop-de-loop diagram with p_S applied to the dot inside the loop, we see that applied to S, we have

(3.6) 

Thus, since q_S is the minimal polynomial of y acting on $\mathcal{F}_iS$, we have that q_S divides r_S, with these being equal if and only if $\deg q_S = \deg p_S + \alpha_i^\vee(\lambda)$.

Symmetrically, we have that $\circlearrowleft_i(w)p_S(w^{-1})$ vanishes in degrees satisfying $k > -\alpha_i^\vee(\lambda)$, so $\circlearrowleft_i(w) = w^{\alpha_i^\vee(\lambda)}t_S(w^{-1})/q_S(w^{-1})$ for some monic polynomial $t_S(w)$ of degree $\deg q_S - \alpha_i^\vee(\lambda)$. Reversing orientations in (3.6) shows that p_S divides t_S. Since we already know that $\deg p_S \geq \deg t_S$, this is only possible if $p_S = t_S$ and $q_s = r_S$, yielding the result. ☐

Thus, we see that the action of $\circlearrowleft_i(w)$ and $\circlearrowright_i(w)$ control the difference between the action of y on $\mathcal{E}_i$ and $\mathcal{F}_i$. More generally, if X is an indecomposable object, the endmorphism ring $\mathrm{End}(X)$ is local, and we can let p_X, q_X be the minimal degree monic polynomials such that $p_X(y)$ acting on $\mathcal{E}_iX$ lies in the 2-sided ideal generated by the maximal ideal of $\mathrm{End}(X)$. The same argument will show that the expressions

$$\circlearrowright_i(w) - w^{\alpha_i^\vee(\lambda)}\frac{q_X(w^{-1})}{p_X(w^{-1})} \qquad \circlearrowleft_i(w) - w^{-\alpha_i^\vee(\lambda)}\frac{p_X(w^{-1})}{q_X(w^{-1})}$$

have all coefficients in the maximal ideal of $\mathrm{End}(X)$.

Given S simple, let m_S be the order to which $\circlearrowright_i(w)$ (thought of as a rational function) vanishes at $w = u^{-1}$; in terms of (3.4), this is the difference $a_{S,u} - b_{S,u}$. We can similarly assign m_X to any indecomposable object X by considering the vanishing order of this series in $\mathrm{End}(X)$ modulo its maximal ideal; this is the same as m_S for any simple in its composition series in the abelian envelope.

Let $U_i = \{u \in \Bbbk | \mathcal{E}_{i,u} \neq 0\} \subset \Bbbk$; note that since the corresponding functors are adjoint, $\mathcal{E}_{i,u}$ or $\mathcal{F}_{i,u}$ can be used symmetrically in this definition. Consider the locally finite graph $\tilde{I}$ with vertices given by pairs (i, u) with $u \in U_i$ constructed from the polynomials P_{ij} as in Defintion 2.8.

We assign to each indecomposable object X the unique weight μ_X in the weight lattice for $\tilde{I}$ such that $\alpha^\vee_{i,u}(\mu) = m_X$.

Definition 3.8 *Let $C_{(\mu)}$ be the subcategory of sums of indecomposable objects with weight μ as assigned by this rule.*

Lemma 3.9 *On $\mathcal{E}_{j,u}S$, the differences*

$$\circlearrowleft_i (w) - w^{\alpha_i^\vee(\lambda)-c_{ij}} \frac{t_{ij}^{-1} \cdot Q_{ji}(u, w^{-1}) q_S(w^{-1})}{p_S(w^{-1})} \qquad \circlearrowleft_j (w) - w^{\alpha_j^\vee(\lambda)} \frac{q_S(w^{-1})}{(1-uw)^2 p_S(w^{-1})}$$

acts nilpotently. Similarly on $\mathcal{F}_{j,u}S$, the differences

$$\circlearrowleft_i (w) - w^{\alpha_i^\vee(\lambda)-c_{ij}} \frac{q_S(w^{-1})}{t_{ij}^{-1} \cdot Q_{ji}(u, w^{-1}) p_S(w^{-1})} \qquad \circlearrowleft_j (w) - w^{\alpha_j^\vee(\lambda)} \frac{(1-uw)^2 q_S(w^{-1})}{p_S(w^{-1})}$$

Proof. The first equation on each line above follows from the bubble slide [Web17a, Prop 2.8], and the fact that $y - u$ acts nilpotently. The second equation follows from [Lau10, Prop. 5.6]. □

Corollary 3.10 *The functors $\mathcal{E}_{i,u}, \mathcal{F}_{i,u}$ send objects in $C_{(\mu)}$ to $C_{(\mu \pm \alpha_{i,u})}$.*

Thus, the category C has a direct sum decomposition into $\oplus C_{(\mu)}$ indexed by weights of $\tilde{\mathfrak{g}}$, with the eigenspace functors $\mathcal{E}_{i,u}, \mathcal{F}_{i,u}$ defined as in Definition 3.5 act as expected on weights.

Theorem 3.11 *The functors $\mathcal{E}_{i,u}$ and $\mathcal{F}_{i,u}$ and the weight space categories $C_{(\mu)}$ define a categorical action of $\tilde{\mathfrak{g}}$.*

Proof. We'll use [Rou, Thm. 5.25], which shows that we have a categorical $\tilde{\mathfrak{g}}$-action if we confirm that:

(1) Let $\mathcal{E} = \oplus_{i \in I} \mathcal{E}_i$. We must show that there is an appropriate R_n-action on $\mathcal{E}^n$. As part of the structure of a categorical action, R_n acts on the nth power $\mathcal{E}^n$. Since the action of any dot on $\mathcal{E}_i$ satisfies a polynomial relation with roots in U_i, this extends to an action of $\hat{R}_n$. By transport of structure using the isomorphism ν of Proposition 3.3, we have an induced action of R_n such that y is nilpotent. we already discussed this above, based on Proposition 3.3.
(2) The functors $\mathcal{E}_{i,u}$ and $\mathcal{F}_{i,u}$ are adjoint and locally nilpotent. This follows immediately from the adjointness of $\mathcal{E}_i$ and $\mathcal{F}_i$ and Lemma 3.6.

(3) The morphism

$$\rho_{i,u,\lambda}\colon \mathcal{E}_{i,u}\mathcal{F}_{i,u}M \oplus M^{\oplus \max(0,-\alpha_{i,u}^\vee(\mu))} \cong \mathcal{F}_{i,u}\mathcal{E}_{i,u}M \oplus M^{\oplus \max(0,\alpha_{i,u}^\vee(\mu))}$$

defined in [Rou, §4.1.3] using the action of R_n, is an isomorphism.

We need only establish item (3).

It's enough to prove this for a simple object S in $\hat{\mathcal{C}}$, and by symmetry, we may assume that $m = \alpha_{i,u}^\vee(\mu) \geq 0$. Applying (3.2), we see that the map $\mathcal{E}_{i,u}\mathcal{F}_{i,u}S \to S^{\oplus m} \oplus \mathcal{F}_{i,u}\mathcal{E}_{i,u}S$ defined by the direct sum of the counit times $1, y \otimes 1, \dots, y^{m-1} \otimes 1$ and the rotated crossing times projection to $\mathcal{F}_{i,u}\mathcal{E}_{i,u}S$.

Consider the subspace of $\mathcal{F}_i\mathcal{E}_iS$ which is killed by the counit times $y^p \otimes 1$ for all p. This is invariant under the action of $y \otimes 1$ and $1 \otimes y$, and we can easily calculate that the rotation of a crossing ψ gives an isomorphism between this submodule, and the corresponding one of $\mathcal{E}_i\mathcal{F}_iS$. Furthermore, this isomorphism intertwines $1 \otimes y$ with $y \otimes 1$ and vice versa by the rotation of (2.1c–2.1d). That is, it induces isomorphisms between the intersection of these subspaces with $\mathcal{F}_{i,u}\mathcal{E}_{i,u}S$ and $\mathcal{E}_{i,u}\mathcal{F}_{i,u}S$. Thus, we need only show that $\rho_{i,u,\lambda}$ induces an isomorphism modulo these subspaces.

A dual argument shows the rotated crossing induces an isomorphism commuting with $y \otimes 1$ and $1 \otimes y$ between the quotients by the submodules of $\mathcal{F}_i\mathcal{E}_iS$ and $\mathcal{E}_i\mathcal{F}_iS$ generated by the image of the unit.

Let S', S'' be the subquotients of $\mathcal{F}_{i,u}\mathcal{E}_{i,u}S$ and $\mathcal{E}_{i,u}\mathcal{F}_{i,u}S$ respectively generated by the image of the unit, modulo the elements killed by the counit times $y^p \otimes 1$ for all p. It now suffices to show $\rho_{i,u,\lambda}$ induces an isomorphism between $S'' \cong S' \oplus S^{\oplus m}$.

Since the action of $y \otimes 1$ and $1 \otimes y$ on the subquotient coincide, let us denote the induced endomorphism y. Note that for each $t \geq 0$, we have a map from $S^{\oplus t}$ to S', sending the summands to the unit ι times $1, y, \dots, y^{t-1}$. This map is surjective for t sufficiently large, since by adjunction y satisfies the relation $p_s(y) = 0$. This shows that under the action of y, we have isomorphisms $S' \cong S \otimes_{\mathbb{k}} \mathbb{k}[y]/(p(y))$ for some polynomial p and similarly that subquotient of S'' is isomorphic to $S \otimes_{\mathbb{k}} \mathbb{k}[y]/(q(y))$ for some polynomial q. Obviously, these polynomials only depend on the value of the bubbles. Computations as in the proof of Lemma 3.7 show that these polynomials are characterized by being coprime and satisfying

$$\circlearrowright_i (w) = w^{-\alpha_i^\vee(\lambda)}\frac{q(w^{-1})}{p(w^{-1})} \qquad \circlearrowleft_i (w) = w^{\alpha_i^\vee(\lambda)}\frac{p(w^{-1})}{q(w^{-1})}.$$

Since $m \geq 0$, we have that u is a root of order m of q, and not a root of p (if m where negative, the roles of the polynomials would switch.

By the Chinese remainder theorem, the u-generalized eigenspace of y on S'' is isomorphic to $S^{\oplus m}$, and that of S' is trivial. Note that every element of this generalized egienspace is killed by $(y-u)^m$, and must pair nontrivially with some power of $y-u$ times the counit (by definition), so there must be such a power which is $< m$. This shows that $\rho_{i,u,\lambda}$ induces an injective map $S'' \to S^{\oplus m}$ which is thus an isomorphism. □

This result also has an obvious converse. If we begin with a categorical action of $\tilde{g}$ on a category $\mathcal{D}$ such that y act nilpotently, then we obtain an action of the algebra R_n on the nth power of the functor $\mathcal{E} \cong \bigoplus_{i\in I, u\in U_i} \mathcal{E}_{i,u}$ with e_{j} being projection to the corresponding summand.

By transport of structure using the isomorphism ν of Proposition 3.3, we also have an action of $\hat{R}_n$ (and thus R_n) on $\mathcal{E}^n$. If we let

$$\mathcal{F}_i \cong \bigoplus_{u\in U_i} \mathcal{F}_{i,u} \qquad \mathcal{E}_i \cong \bigoplus_{u\in U_i} \mathcal{E}_{i,u}, \tag{3.7}$$

then we have that under this action, the image of the idempotent $e_{\mathbf{i}}$ on $\mathcal{E}^n$ is $\mathcal{E}_{i_1} \cdots \mathcal{E}_{i_n}$.

Theorem 3.12 *The functors $\mathcal{F}_i$ and $\mathcal{E}_i$ defined in* (3.7) *with the algebra R acting as above define an integrable categorical action of $\mathfrak{g}$ such that $\mathcal{F}_{i,u}$ is the u-generalized eigenspace of y acting on $\mathcal{F}_i$.*

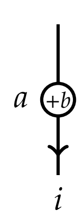
Proof. We'll use [Rou, Thm. 5.27], which shows that we have a categorical $\mathfrak{g}$-action if we confirm that:

(1) There is an appropriate R_n-action on $\mathcal{E}^n$; we already discussed this above, based on Proposition 3.3.
(2) The functors $\mathcal{E}_i$ and $\mathcal{F}_i$ are adjoint and locally nilpotent[3]. This follows immediately from the adjointness of $\mathcal{E}_{i,u}$ and $\mathcal{F}_{i,u}$.
(3) The action of $\mathcal{E}_i$ and $\mathcal{F}_i$ on the Grothendieck group satisfy the relations of $\mathfrak{g}$. This follows immediately from the fact that $\mathcal{E}_{i,u}$ and $\mathcal{F}_{i,u}$ satisfy the relations of $\tilde{\mathfrak{g}}$, so the corresponding result for $\mathfrak{g}$ follows from Proposition 2.7. □

4. Deformed tensor product algebras

4.1. **The definition.** In [Web15, §5], we introduced a natural categorifications $\mathcal{X}^{\underline{\lambda}}$ for tensor products of highest and lowest weight representations. These categorifications have natural deformations, which we wish to study in the context of the previous section.

We will use the notation

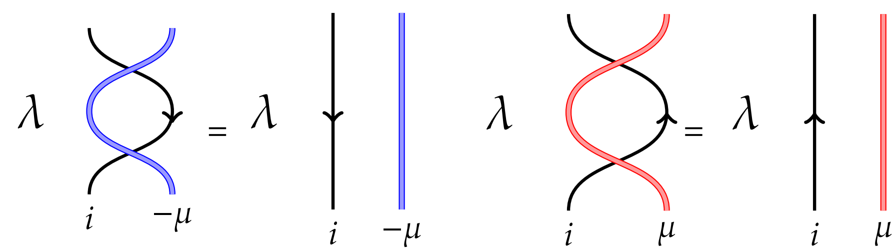


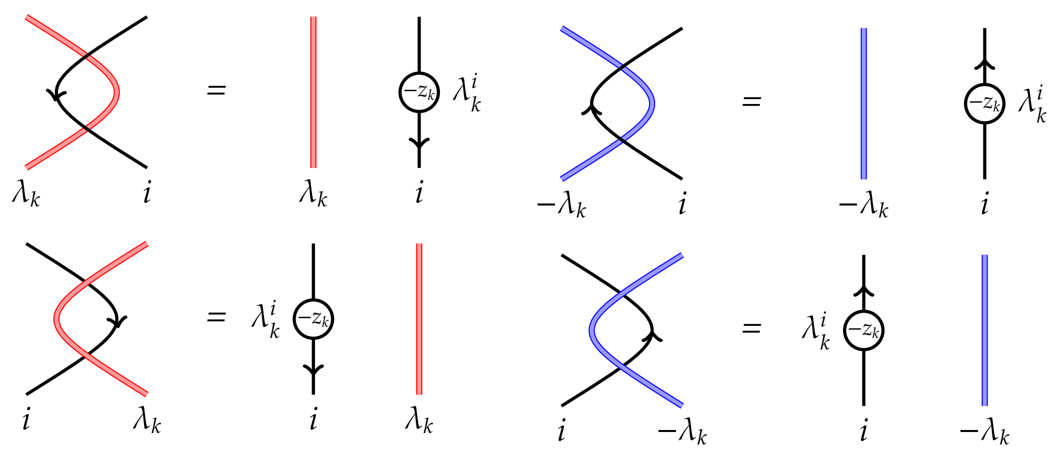
to denote the endomorphism $(y+b)^a$ of $\mathcal{F}_i$, and similarly for $\mathcal{E}_i$.

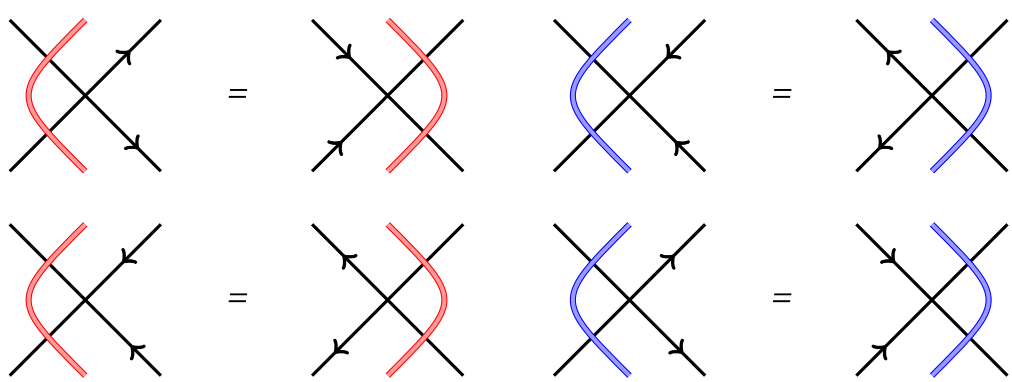
[3]Note, this is on the level of 1-morphisms and unrelated to whether the natural transformation y is nilpotent.

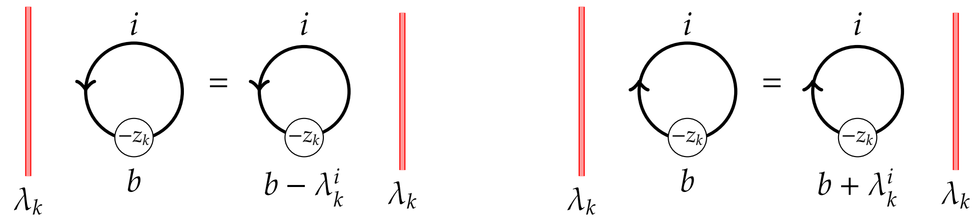
In [Web15, §4], we introduced a notion of **tricolore diagrams**, which naturally form the 2-morphisms of a 2-category $\mathfrak{T}$. The categorifications $\mathcal{X}^{\underline{\lambda}}$ are natural subquotients of this category, and our deformed categorifications arise from a straightforward deformation of the relations from [Web15], which we present below:

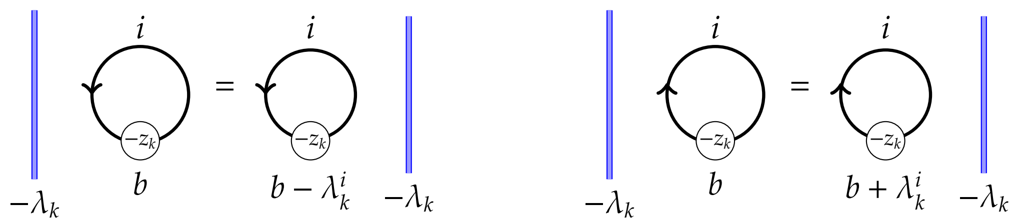
Definition 4.1 *Let $\mathfrak{T}$ be the quotient of $\tilde{\tilde{\mathfrak{T}}}\otimes\Bbbk[z_1,\dots,z_\ell]$ by the relations (2.1a–2.1g,3.3a–3.3g) on black strands and (4.1a–4.1i) below relating red and blue strands to black. Note that the relations (4.1a–4.1i) are deformations of the relations of $\mathcal{T}$ in* [Web15, 4.3]*; we will thus recover the category $\mathcal{T}$ if we specialize $z_i = 0$.*

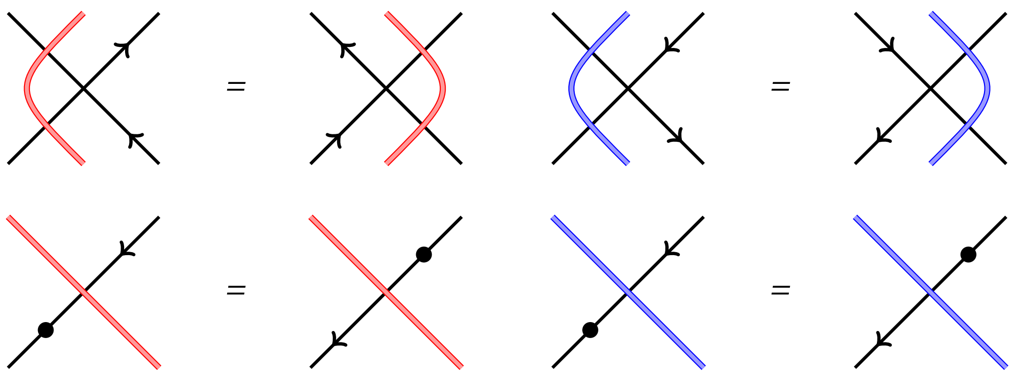
(4.1a)


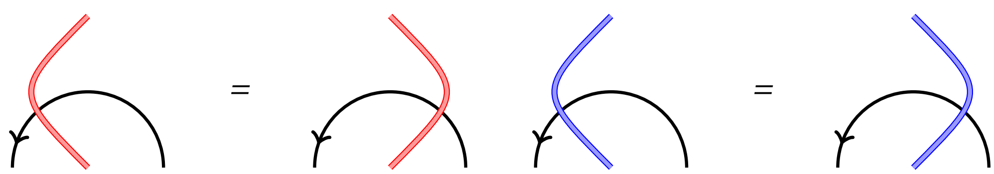
(4.1b)


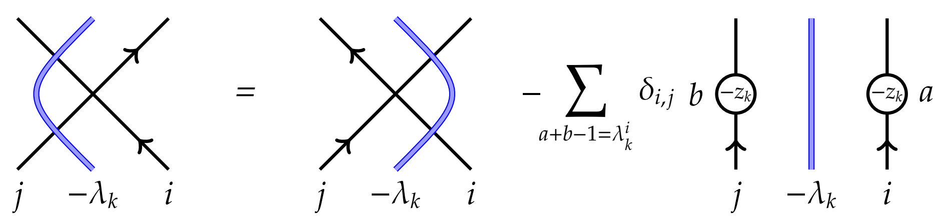
(4.1c)


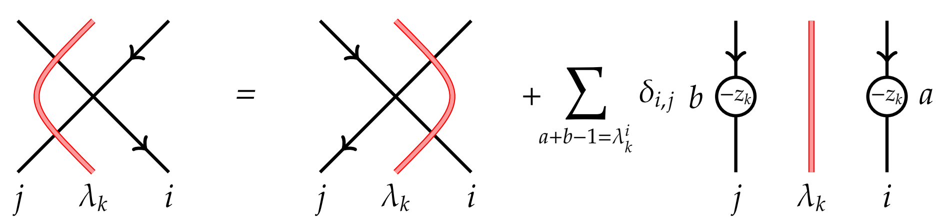
(4.1d)


(4.1e)


(4.1f)


(4.1g)


(4.1h)


(4.1i)


The reader should read the label λ_k in this diagram to indicate that the strand shown is the kth of the red and blue strands from the left. In particular, z_k is connected to this

kth strand, and could be thought of as a new endomorphism of the tricolore triple with a single red or blue strand and $\mathbf{i} = \emptyset$.

We let $\mathcal{X}^{\underline{\lambda}}$ be the idempotent completion of the quotient of the category of tricolore quadruples $(\underline{\lambda}, \mathbf{i}, \kappa, \mathcal{L})$ in $\mathcal{T}$ by the tricolore quadruples where $\kappa(1) > 0$. That is, we consider 1-morphisms with label 0 at the left, where we fix the labels of the red and blue strands as well as their order to match $\underline{\lambda}$, but allow arbitrary black strands. We then take the quotient of this category of 1-morphisms by killing the diagrams with a black line at the far left.

The definition of $\mathcal{X}^{\underline{\lambda}}$ has precisely the same form as that of $\mathcal{X}^{\underline{\lambda}}$, with the only difference being the relations (4.1a–4.1i) in place of the relations in [Web15, 4.3].

From the definition, it's clear that there is a 2-functor $\mathcal{U} \to \mathcal{T}$, since (3.3a– 3.3g) are simply the relations of $\mathcal{U}$. Thus, composition on the right induces a $\mathcal{U}$ action on $\mathcal{X}^{\underline{\lambda}}$.

Definition 4.2 *Given a $\Bbbk[z_1, \cdots z_\ell]$-algebra K, we let $\mathcal{X}^{\underline{\lambda}}_K$ be the idempotent completion of the extension of scalars $\mathcal{X}^{\underline{\lambda}} \otimes_{\Bbbk[z_1,\cdots z_\ell]} K$.*

The main examples we'll want to consider are $\mathbb{K} = \Bbbk(z_1, \dots, z_\ell)$ and the algebraic closure $\bar{\mathbb{K}}$.

4.2. Spectral analysis.

Definition 4.3 *Define sets $U_i \subset \bar{\mathbb{K}}$ as follows: if for some k, we have $\alpha_i^\vee(\lambda_k) \neq 0$, then $z_k \in U_i^{(0)}$, and all elements of $U_i^{(0)}$ are of this type. Now we inductively define $U_i^{(N)}$ to be the union of $U_i^{(N-1)}$ with the elements u of $\bar{\mathbb{K}}$ that satisfy $Q_{ij}(u, u') = 0$ for $u' \in U_j^{(N-1)}$, and $U_i = \cup_{N \in \mathbb{Z}} U_i^{(N)}$.*

Let U'_i be the union of U_i with the set of elements in $\bar{\mathbb{K}}$ that appear in the spectrum of the elements $y_k e_\mathbf{i}$ with $i_k = i$ acting on objects in the category $\mathcal{X}^{\underline{\lambda}}_{\bar{\mathbb{K}}}$; that, is the eigenvalues that appear when dots on strands with label i act.

It might seem strange that we add the elements of U_i to U'_i by definition, but this simplifies matters for us, since we have not yet established that $\mathcal{X}^{\underline{\lambda}}_{\bar{\mathbb{K}}}$ is non-zero. Thus, we have not yet established that there are any elements of this spectrum. We will ultimately see that $U_i = U'_i$, and these both coincide with the union of spectra discussed above.

We let $\tilde{I}', \tilde{I}$ be the graphs constructed from these sets as before and $\tilde{\mathfrak{g}}'$ and $\tilde{\mathfrak{g}}$ the corresponding Kac-Moody algebras. We let $\tilde{\lambda}$ be a weight for $\tilde{\mathfrak{g}}'$ such that $\alpha_{(i,u)}^\vee(\tilde{\lambda}) = \delta_{u,z_m}\alpha_i^\vee(\lambda_m)$.

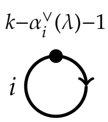
Since the elements z_m are algebraically independent from each other, every element of $u \in U_i$ is algebraically dependent on exactly one z_m. We denote this index $m(u)$. In many cases that interest us, there is exactly one component of $\tilde{I}$ for each of these indices, but if $\alpha_i^\vee(\lambda_m) \neq 0$ for several elements i, the pairs (i, z_m) can lie in different components for different i.

We can define formal power series valued in the center of the category $\mathcal{X}^{\underline{\lambda}}$ which act on the object $(\underline{\lambda}, \mathbf{i}, \kappa)$ as

$$\mathbf{y}_i(w) := \prod_{i_k=\pm i}(1 - wy_k)^{\pm 1} \qquad \mathbf{Q}_{ji}(w) = \prod_{i_k=\pm j}\left(t_{ij}^{-1} \cdot w^{-c_{ij}} Q_{ji}(y_k, w^{-1})\right)^{\pm 1},$$

where y_k is the dot acting on the kth strand from the left.

These are supersymmetric polynomials (in the sense of [Ste85]) in the pair of alphabets given by dots on upward oriented i-strands and dots on downward oriented i-strands. Any such polynomial commutes with all upward or downward oriented diagrams by [KL09, 2.9], since each coefficient is symmetric in the corresponding variables. It commutes with a cup or cap joining the kth strand to the $k+1$st since multiplying by $(1+uy_k)^{\pm 1}$ at one end of the cup or cap cancels with $(1+uy_{k+1})^{\mp 1}$ at the other (this is a restatement of the supersymmetric property). Note that the bubble slides and triviality of bubbles at the far left show that

(4.2) $$\sum_{k=0}^{\infty}$$  $$w^k = \mathbf{y}_i(w)^2 \prod_{j \neq i} \mathbf{Q}_{ji}(w)^{-1} \prod_{m=1}^{\ell}(w - z_m)^{\lambda_m^i}.$$

Let $\tilde{\mu} = \tilde{\lambda} - \sum a_{i,u}\alpha_{i,u}$ be a weight of $\tilde{I}'$. We can define subcategories $\mathcal{V}_{(\tilde{\mu})}$ as in Definition **??**. By Theorem 3.10, the functors $\mathcal{F}_{i,u}$ and their adjoints $\mathcal{E}_{i,u}$ induce a categorical action of $\tilde{\mathfrak{g}}'$ on $\mathcal{X}^{\underline{\lambda}}_{\mathbb{K}}$, with weight decomposition given by $\mathcal{X}^{\underline{\lambda}}_{\mathbb{K}} \cong \oplus_{\tilde{\mu}} \mathcal{V}_{\tilde{\mu}}$.

Given a triple $(\underline{\lambda}, \mathbf{i}, \kappa)$ with $\mathbf{i} = (i_1, \dots, i_n)$ considered as an object in in $\mathcal{X}^{\underline{\lambda}}_{\mathbb{K}}$ (recall that we will often exclude $\underline{\lambda}$ from the notation when it is unlikely to be confused), we can thus decompose it according to the spectrum of the dots y_k. For a sequence $j_k = (i_k, u_k) \in \tilde{I}'$ for $k = 1, \dots, n$, we let $(\mathbf{i}, \kappa)_{\mathbf{u}}$ be the simultaneous stable kernel of $y_k - u_k$ for all $1 \leq k \leq n$.

Lemma 4.4 *We have that*

(4.3) $$(\mathbf{i}, \kappa)_{\mathbf{u}} \cong \mathcal{E}_{i_n,u_n} \cdots \mathcal{E}_{i_1,u_1}(\emptyset, 0)$$

if $u_k \in U_{i_k}$ and $k > \kappa(m(u_k))$ for each k, and $(\mathbf{i}, \kappa)_{\mathbf{u}} = 0$ otherwise.

In particular, we have $U_i = U_i'$ for all i and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}'$, and the category $\mathcal{X}^{\underline{\lambda}}_{\mathbb{K}}$ is generated by the tricolore triple $(\emptyset, 0)$ as a categorical module over $\tilde{\mathfrak{g}}$.

Proof. First, we note that if $u \in U_i$ and $u' \notin U_j$, then $Q_{ij}(y_1, y_2)$ acts on $\mathcal{E}_{i,u}\mathcal{E}_{j,u'}$ with its only eigenvalue $Q_{ij}(u, u') \neq 0$ (by the definition of U_i). Thus the crossing ψ induces an isomorphism $\mathcal{E}_{i,u}\mathcal{E}_{j,u'} \cong \mathcal{E}_{j,u'}\mathcal{E}_{i,u}$. Similarly, $\mathcal{F}_{j,u'}$ commutes past all red and blue strands since $y - z_k$ is invertible, with its only eigenvalue $u' - z_k$; in fact, this still follows for the kth red/blue strand if $u' \neq z_k$ (in particular, if $k \neq m(u')$).

We establish the result by induction on ℓ and n (i.e. on the total number of strands). If $n = 0$, the result is tautological. Otherwise, the rightmost strand in the idempotent for the object $(\mathbf{i}, \kappa)$ is either black, blue or red. If it is black, then $(\mathbf{i}, \kappa) = \mathcal{E}_i(\mathbf{i}^-, \kappa)$ for some $i \in \pm I$, and decomposing with respect to the eigenvalues of y, we have $\mathcal{E}_i(\mathbf{i}^-, \kappa) \cong \oplus \mathcal{E}_{i,u}(\mathbf{i}^-, \kappa)$ where u ranges over the roots of the minimal polynomial of y acting on $\mathcal{E}_i(\mathbf{i}^-, \kappa)$. By induction $(\mathbf{i}^-, \kappa)$ is a sum of modules obtained from $(\emptyset, 0)$ by the functors $\mathcal{E}_{j,u'}$ and $\mathcal{F}_{j,u'}$ for $u' \in U_j$. If u is not in U_i, then all these functors commute with $\mathcal{E}_{i,u}$ (as argued above), and $\mathcal{E}_{i,u}(\emptyset, 0) = 0$, so $\mathcal{E}_{i,u}(\mathbf{i}^-, \kappa) = 0$, and $\mathcal{F}_i(\mathbf{i}^-, \kappa) \cong \oplus_{u \in U_i} \mathcal{E}_{i,u}(\mathbf{i}^-, \kappa)$. By induction, this establishes the result.

On the other hand, if the rightmost strand is blue or red, we simply apply induction with the tricolore triple $(\underline{\boldsymbol{\lambda}}^-, \mathbf{i}, \kappa^-)$ with this strand removed. By induction, $(\underline{\boldsymbol{\lambda}}^-, \mathbf{i}, \kappa^-) \cong \oplus(\underline{\boldsymbol{\lambda}}^-, \mathbf{i}, \kappa^-)_{\mathbf{u}}$ with $k > \kappa(m(u_k))$ and $m(u_k) < \ell$. Since adding in the ℓth blue or red strand does not change the eigenvalues of the dots, we also have $(\underline{\boldsymbol{\lambda}}, \mathbf{i}, \kappa) \cong \oplus(\underline{\boldsymbol{\lambda}}, \mathbf{i}, \kappa)_{\mathbf{u}}$ with $\mathbf{u}$ ranging over the same set. This shows equation (4.3), and that $U_i' = U_i$. □

Thus $\mathcal{X}^{\underline{\boldsymbol{\lambda}}}_{\bar{\mathbb{K}}}$ is generated by a single object, which is highest weight for the components of $\tilde{I}$ with $\lambda_{m(u)}$ dominant and lowest weight for those with $\lambda_{m(u)}$ anti-dominant. Alternatively, we can easily choose a Borel for which this representation is straightforwardly highest weight. To distinguish objects which are highest weight for this Borel, we call them **signed highest weight**. We can write each weight $\tilde{\lambda}$ uniquely as a sum $\tilde{\lambda} = \tilde{\lambda}_1 + \cdots + \tilde{\lambda}_\ell$ where $\tilde{\lambda}_m$ is supported on components with $m(u) = m$.

We can apply the classification of highest weight representations in [Web17a, 3.25], which shows that any idempotent complete representation additively generated by a highest weight object is equivalent to the projective modules over a base change of the deformed cyclotomic quotient $\check{R}^{\tilde{\lambda}}$ for $\tilde{\mathfrak{g}}$ of highest weight $\tilde{\lambda}$. This base change is by the endomorphism ring of the highest weight object over the ring $\check{R}^{\tilde{\lambda}}_{\tilde{\lambda}}$, the polynomial ring on fake bubbles in weight $\tilde{\lambda}$.

We can justify the change of Borel by applying the Cartan involution in the factors where $\lambda_{m(u)}$ is anti-dominant, which is essentially relabeling $\mathcal{F}_{i,u}$ as $\mathcal{E}_{i,u}$ instead. The ring $\check{R}^{\tilde{\lambda}}$ is Morita equivalent to the tensor product $\check{R}^{\tilde{\lambda}_1} \otimes \cdots \otimes \check{R}^{\tilde{\lambda}_\ell}$. The different choice of Borel means that it is more convenient to think of deformed cyclotomic quotient $\check{R}^{\tilde{\lambda}_k}$ as written with a red strand and downward black strands if λ_k is dominant, and with a blue strand and upward black strands if λ_k is anti-dominant.

Thus, we find that $\mathcal{X}^{\underline{\boldsymbol{\lambda}}}_{\bar{\mathbb{K}}}$ is the base change of $\check{R}^{\tilde{\lambda}}$ -pmod via the natural map $\check{R}^{\tilde{\lambda}}_{\tilde{\lambda}} \to \operatorname{End}(\emptyset, 0)$ to the endomorphisms of the signed highest weight object. Since $\mathsf{y}\colon \mathcal{F}_{i,u} \to \mathcal{F}_{i,u}$ is nilpotent and the endomorphisms of $\operatorname{End}(\emptyset, 0)$ are just the scalars, this resulting base change is simply the tensor product of undeformed cyclotomic quotients $\mathcal{X}^{\tilde{\lambda}} \otimes_{\Bbbk} \bar{\mathbb{K}} \cong \mathcal{X}^{\tilde{\lambda}_1} \otimes \mathcal{X}^{\tilde{\lambda}_2} \otimes \cdots \otimes \mathcal{X}^{\tilde{\lambda}_\ell} \otimes_{\Bbbk} \bar{\mathbb{K}}$. Thus, we have an induced strongly equivariant functor

$$\Phi\colon \mathcal{X}^{\tilde{\lambda}} \otimes_{\Bbbk} \bar{\mathbb{K}} \to \mathcal{X}^{\underline{\boldsymbol{\lambda}}}_{\bar{\mathbb{K}}}. \tag{4.4}$$

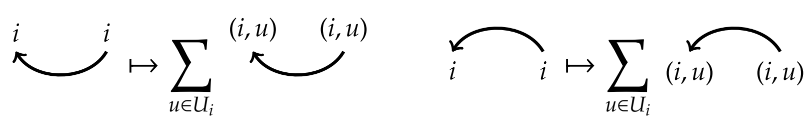
Since this functor is strongly equivariant, and both source and target are generated by a signed highest weight object with only scalar endomorphisms, this functor is either an equivalence or $\mathcal{X}^{\underline{\lambda}}_{\bar{\mathbb{K}}}$ is a trivial category. In particular, $\mathcal{X}^{\underline{\lambda}}_{\bar{\mathbb{K}}}$ satisfies any hypotheses that $\mathcal{X}^{\tilde{\lambda}} \otimes_{\Bbbk} \bar{\mathbb{K}}$ does. For example, each weight space is equivalent to the category of projective modules over a finite dimensional $\bar{\mathbb{K}}$-algebra.

It will actually be more convenient for us to think of the irreducible module with signed highest weight $\tilde{\lambda}$ as a tensor product of simples corresponding to the different components. That is, we use the equivalence of $\mathcal{X}^{\tilde{\lambda}}$ with the tensor product category $\mathcal{X}^{\tilde{\underline{\lambda}}}$ where $\tilde{\underline{\lambda}} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_\ell)$. We can define a functor

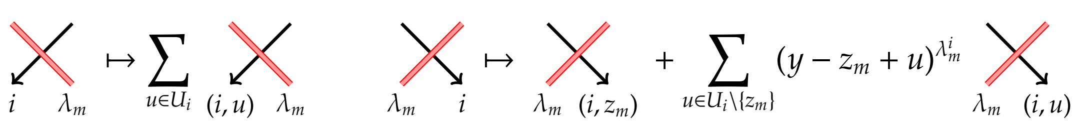
$$\Xi\colon \mathcal{X}^{\underline{\lambda}}_{\bar{\mathbb{K}}} \to \mathcal{X}^{\tilde{\underline{\lambda}}} \otimes_{\Bbbk} \bar{\mathbb{K}}$$

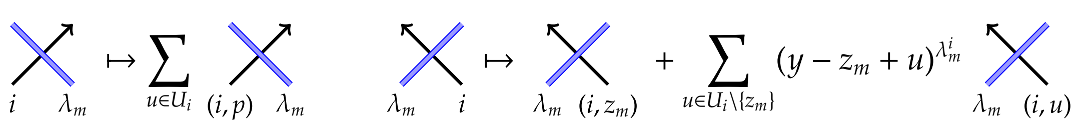
which sends $(\mathbf{i}, \kappa)$ to the sum $\bigoplus(\mathbf{j}, \kappa)$ where $\mathbf{j}$ ranges over sequences with $j_k = (i_k, u_k)$. If $\mathbf{j}$ satisfies $k > \kappa(m(u_k))$, then we have $(\mathbf{j}, \kappa) \cong (\mathbf{j}, 0)$, and otherwise the corresponding object is 0, so Lemma 4.6 shows that this is quasi-inverse to Φ on the level of 1-morphisms.

Now, we must define how Ξ acts on 2-morphisms. First, note that by Theorem 3.10, we have a categorical action of $\mathfrak{g}$ on $\mathcal{X}^{\tilde{\underline{\lambda}}} \otimes_{\Bbbk} \bar{\mathbb{K}}$. On purely black diagrams, Ξ simply employs this action; that is, on upward oriented diagrams, it follows the formula (3.2). Since left (or right) adjunctions are unique up to isomorphism, we can send the leftward cup and cap in $\mathcal{U}_{\mathfrak{g}}$ to any adjunction we choose. For simplicity, we simply match leftward oriented cups and caps as below:

(4.5)


For rightward oriented cups, the formula is quite complicated, but is fixed by the choices we have made thus far, and the existence of a consistent choice follows from the existence of the $\mathfrak{g}$-action. Thus, we need only show this action on diagrams with red/blue strands, with formulas given below (4.6–4.7).

(4.6)


(4.7)


Lemma 4.5 *The functor Ξ is well-defined.*

Proof. In order to check this, we have to verify all the relations on 2-morphisms. The equations that can be stated purely using upward or down diagrams, that is, (2.1a–2.1g, 4.1b, 4.1h–4.1f) all follow by straightforward calculations as in the proof of Proposition 3.3.

Thus we only need to argue for the relations involving right cups and caps. The way we have defined the right cup/cap means that relations (3.3a–3.3g) are automatic. The remaining relations (4.1g, 4.1a) are actually redundant when the right cap and cup are defined in terms of the left cup and cap. The relation (4.1g) is the definition of the upward red/black or downward blue/black crossings, in this perspective. For the relation (4.1a), assume we are considering the red version; the blue one follows similarly. We must consider two different cases. Let μ be the label of the region at the left of the picture.

- If $\mu^i \geq 0$, then we have we have make a loop at the left with μ^i dots. Pulling this through and applying (4.1b), then undoing this bubble, we obtain the desired relation.
- if $\mu^i \leq 0$, then we start with the diagram with a leftward cup at the bottom and rightward cup at top, and compare the result of applying [Web15, 3.6d] to these two strands to the left and right of the red strand. Using the relations (4.1f,4.1i), we can move the bigon to the right side of the red line, and using (4.1b) to remove bigons between red and downward strands. The left- and right-hand sides now have the same pattern of black strands, but in one the upward strands make a bigon with the red strand and in the other, they don't. This can only hold if (4.1a) is true. □

Lemma 4.6 *The category $\mathcal{X}^{\underline{\lambda}}_{\bar{\mathbb{K}}}$ is equivalent to $\mathcal{X}^{\tilde{\lambda}} \otimes_{\Bbbk} \bar{\mathbb{K}}$ via the functor Φ defined in* (4.4).

Proof. By [Web17a, 3.25], this functor is an equivalence if $\mathrm{End}(\emptyset, 0) \cong \Bbbk$. Thus, the only issue is that the object $(\emptyset, 0)$ might simply 0 (in which case the entire category $\mathcal{X}^{\underline{\lambda}}_{\bar{\mathbb{K}}}$ is 0).

The functor Ξ sends $(\emptyset, 0)$ to $(\emptyset, 0)$. The existence of this functor establishes that $\mathcal{X}^{\underline{\lambda}}_{\bar{\mathbb{K}}}$ is not 0, so Φ must be an equivalence. In fact, we can easily see that Ξ is strongly equivariant for $\tilde{\mathfrak{g}}$, so it must be quasi-inverse to Ξ when composed with the equivalence $\mathcal{X}^{\tilde{\underline{\lambda}}} \otimes_{\Bbbk} \bar{\mathbb{K}} \cong \mathcal{X}^{\tilde{\lambda}} \otimes_{\Bbbk} \bar{\mathbb{K}}$. □

4.3. **Applications.** Now, we turn to the application of our results, culminating in the proof of Theorem A.

Theorem 4.7 *For two tricolore triples $(\underline{\lambda}, \mathbf{i}, \kappa)$ and $(\underline{\lambda}, \mathbf{i}', \kappa')$, the set D is a basis over $\mathbb{C}[\mathbf{z}]$ for the morphism space $\mathrm{HOM}_{\mathcal{T}}\left((\underline{\lambda}, \mathbf{i}, \kappa), (\underline{\lambda}, \mathbf{i}', \kappa')\right)$.*

Proof. The proof that these are a spanning set is essentially equivalent to that of [KL10, Prop. 3.11]. First, note that any two minimal diagrams for the same matching are equivalent modulo those with fewer crossings (using the relations (2.1f, 2.1g, 4.1h, 4.1i)). Similarly, moving dots to the chosen positions only introduces diagrams with fewer crossings.

Thus, we only need show that all minimal diagrams span. Of course, if a diagram is non-minimal then it can be rewritten in terms of the relations in terms of ones with fewer crossings, using the relations to clear all strands out from a bigon, and then the relations (2.1e, 3.3e–3.3f, 3.3g, 4.1b) to remove it. Thus, by induction, this process must terminate at a expression in terms of minimal diagrams. Thus, these elements span, and it suffices to show that these elements are linearly independent when $\mathbf{z}$ are generic, that is, after base change to $\bar{\mathbb{K}}$.

Assume $\mathcal{L} = \mu$. We consider how the elements in D act on the quadruple with $\mathbf{i} = \emptyset$ in the deformed category $\mathcal{X}^{\boldsymbol{\lambda}}$ with $\boldsymbol{\lambda}$ chosen so that $\sum \lambda_i = \mu$.

It suffices to check that these elements act linearly independently on $\mathcal{X}^{\underline{\boldsymbol{\lambda}}}_{\bar{\mathbb{K}}}$ for some $\underline{\boldsymbol{\lambda}}$; in the course of the proof we'll modify $\underline{\boldsymbol{\lambda}}$ as necessary to achieve this. Note the enormous advantage obtained by having both dominant and anti-dominant weights, as we can add cancelling pairs of these without changing the total sum.

As in [Web17a, 4.17], we can compose with the diagram η_κ pulling all black strands to the left and $\dot{\eta}_\kappa$, its vertical reflection[4]. This will send a non-trivial relation between the diagrams in D to a non-trivial relation between diagrams where $\kappa(i) = n$ for all i.

We can now project this relation to the subspace where we fix the eigenvalue of each dot acting at the top and bottom. The formulas (3.2, 4.5–4.7) defining the functor Ξ show that this projection is the image under Φ of a diagram with an equal or smaller number of crossings, and we can only have equality if we choose eigenvalues so that they coincide at opposite ends of a strand. Now, fix a matching D such that an associated basis vector appears in our relation, and the corresponding diagram has a maximal number of crossings among those that appear.

We can adjust our weights in the list $\boldsymbol{\lambda}$ so that they include weights λ_a in bijection with the matching pairs in D (which correspond to arcs in the diagram), with λ_a dominant if the corresponding arc a is downward oriented at its lefthand edge and λ_a anti-dominant if the arc a is upward oriented at its lefthand edge. Now, let us take the projection to the subspace where the eigenvalue of the dot at each end of the arc a is the variable z_a. Let $\mathbf{j}$ and $\mathbf{j}'$ be the associated sequences in $\tilde{I}$ at the bottom and top of the diagram.

Note that D gives the only way of matching the terminals in $\mathbf{j}$ and $\mathbf{j}'$ to produce a legal tricolore diagram. Thus all diagrams in our relation that give a different matching from D project to 0, since there is no matching which has the same eigenvalue at both ends of each strand and fewer crossings than D.

Therefore, this must be the projection of a relation in $\mathcal{X}^{\underline{\boldsymbol{\lambda}}}_{\bar{\mathbb{K}}}$ where all terms have the underlying matching D with some number d_a of dots on the arc a, times some monomial M in the bubbles at the left of the diagram. If we show that no such

[4]We use η instead of θ here since we are pulling left rather than right.

relation exists, then for each choice of d_a and M, the corresponding term must have had coefficient 0 in the original relation. It follows that the original relation must have been trivial.

First, we consider bubbles. Any bubble at the left of the diagram evaluates to a scalar, using the relations (4.1a) and (4.1b). The clockwise bubbles with label i evaluate to the coefficients of the power series $\prod_i (u - z_k)^{\lambda_k^i}$ and counterclockwise bubbles to the coefficients of its formal inverse $\prod_i (u - z_k)^{-\lambda_k^i}$. By adding new pairs of red and blue strands with labels ν and $-\nu$ for a strictly dominant weight ν, we can assure that any finite set of monomials in clockwise bubbles are sent to elements of $\bar{\mathbb{K}}$ which are algebraically independent over $\Bbbk$.

Furthermore, we can explicitly describe the space $\mathrm{HOM}_{\mathcal{T}}\left((\underline{\boldsymbol{\lambda}}, \mathbf{j}, \kappa), (\underline{\boldsymbol{\lambda}}, \mathbf{j}', \kappa')\right)$; it has a basis over $\bar{\mathbb{K}}$ given diagrams with matching D and with $d_a < |\lambda_a^{i_a}|$. Thus, for any finite number of ways of choosing d_a and M, we can choose λ_a's so the corresponding diagrams lie in this basis. Since these diagrams remain linearly independent after acting in $\mathcal{X}_{\bar{\mathbb{K}}}^{\underline{\boldsymbol{\lambda}}}$, they must be linearly independent, so we must have that the coefficient of this diagram in $\bar{\mathbb{K}}$ is 0. This in turn supplies a polynomial relation between the values of the clockwise bubbles. We can rule out this possibility by choosing $\underline{\boldsymbol{\lambda}}$ so that the bubbles which appear in the relation are algebraically independent. Thus, we see that the relation we chose is trivial. This establishes the linear independence of our prospective basis and establishes the result. □

Since $\mathcal{U}$ has a natural functor to $\mathcal{T}$ just not using any red or blue strands, this shows:

Theorem 4.8 *The 2-category $\mathcal{U}$ is non-degenerate for any field $\Bbbk$ and choice of $Q_{*,*}$. In particular, the Grothendieck group $K_q(\mathcal{U})$ is isomorphic to $\dot{\mathbf{U}}$.*

Proof. The spanning set $B_{\mathbf{i},\mathbf{j},\lambda}$ of Khovanov and Lauda is still linearly independent after applying the functor to $\mathcal{T}$, and thus must be linearly independent. This shows the non-degeneracy, and the isomorphism of Grothendieck groups follows from [KL10, 1.2]. □

Remark 4.9. If our goal was only to prove Theorem 4.10, we could have avoided some of the difficulties of the proof above: as observed above, we can define a categorical action of $\mathcal{U}$ on $\mathcal{X}^{\lambda_1} \otimes \mathcal{X}^{-\lambda_2}$ which categorifies the tensor product of a highest and lowest weight simple, using Theorem 3.10. Note, however, that this requires y to not be nilpotent, but to have at least two elements in its spectrum. This shows that the map $\dot{\mathbf{U}} \to K(\mathcal{U})$ must be injective, since no element of $\dot{\mathbf{U}}$ kills all highest tensor lowest modules; we can also see that we get the correct inner product, since the Euler form on $\mathcal{X}^{\lambda_1} \otimes \mathcal{X}^{-\lambda_2}$ matches the tensor product of Shapovalov forms.

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