Coulomb branches and representation theory

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There are a lot of non-commutative algebras in the world.

Thus, if you're going to study them, you have to have some opinions about which are manageable to study and which are too wild.

1. One natural class is **almost commutative algebras** A: non-commutative algebras with $\operatorname{gr} A$ commutative (for some filtration).
2. In this case, $\mathfrak{M} = \operatorname{Spec}(\operatorname{gr} A)$ is a Poisson variety, and we can ask that this to live in some special class.

My favorite choice is to let $\mathfrak{M}$ be a (singular) symplectic variety (or the universal deformation of one).

1. The most famous of these is the nilcone $\mathcal{N}_{\mathfrak{g}} \subset \mathfrak{g}^*$. Its universal deformation is $\mathfrak{g}^*$.
2. If a finite group G has a symplectic representation (V, Ω), then V/G is singular symplectic.
3. If $\mathfrak{M}$ is 2-dimensional, symplectic form is a volume form, so $\mathbb{C}^2/\Gamma$ for $\Gamma \subset SL_2(\mathbb{C})$ is an example.

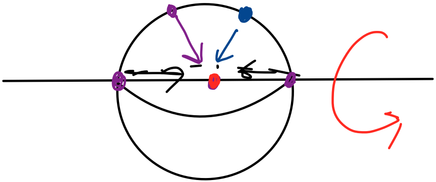
These are essentially the only examples that appear in Beauville's original paper.

Lots more examples come from topologically twisted 3d QFT.

Peter's talk: a 1d TQFT gives an algebra, and boundary conditions give modules.

Local operators in a 3d theory have a 3-d space of multiplications which are all the same up to homotopy ($\mathbb{E}_3$-algebra).

This manifests more concretely as a Poisson structure and quantization (which comes from the S^1 action on $\mathbb{R}^3$).



Ω-background

Many 3d field theories have enough supersymmetry ($\mathcal{N} = 4$) that they automatically come with two different twists, and thus two different Poisson varieties.

In particular, there's one of these theories attached to any symplectic representation V of a reductive Lie group G.

1. The Higgs branch is a symplectic reduction of V by G. We can quantize by considering q-Hamiltonian reduction of differential operators.
2. The Coulomb branch is, well, more complicated. It's birational to $T^*\check{T}/W$, but the connection is surprisingly complicated. It also has a quantization birational to $\mathrm{Diff}(\check{T})^W$.

Mathematical def, by BFN!

The motivation for studying these algebras is two-fold:

Lots of interesting algebras show up this way:

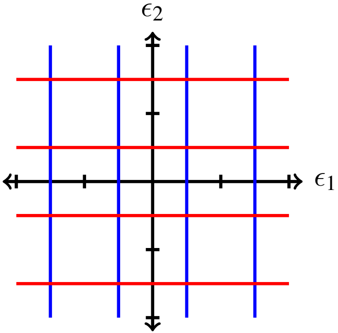
1. UEA and W-algebras in type A — $U(\mathfrak{gl}_n)$
2. cyclotomic Cherednik algebras
3. truncated shifted Yangians
4. hypertoric enveloping algebras — $V = T^*N \circlearrowleft T$
5. ??

Conjecture (Braden-Licata-Proudfoot-W.)

The categories $\mathcal{O}$ attached to the Higgs and Coulomb branches of $d = 3, \mathcal{N} = 4$ theories are Koszul dual.

Let's specialize for now to the case of $G = T$ abelian. Let $\varphi_1, \dots, \varphi_d$ be the weights in N such that $V = T^*N$.

Draw in all the hyperplanes $\varphi_i(X) \in \mathbb{Z} + 1/2$ in $\mathfrak{t}_{\mathbb{R}}$. For $D \subset GL(2)$ acting on $\mathbb{C}^2 \oplus \mathbb{C}^2$, the weights are $\varphi_1 = \varphi_2 = \epsilon_1, \varphi_3 = \varphi_4 = \epsilon_2$.



The coordinate ring of the Coulomb branch has an explicit description in these terms:

Theorem

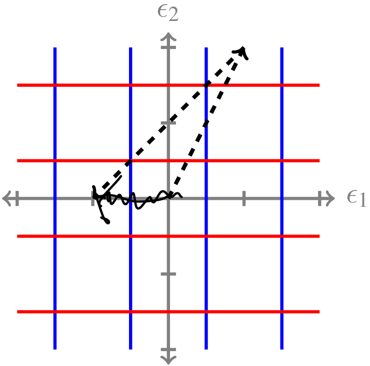
$\mathbb{C}[\mathfrak{M}] := \mathsf{A}$ *is an* $S = \mathrm{Sym}(\mathfrak{t}^*)$ *algebra with free basis* r_ν *for* $\nu \in \mathfrak{t}_{\mathbb{Z}}$*, and multiplication rule*

$$r_\nu r_\mu = \prod \varphi_i^{\rho_i(\mu,\nu)} r_{\mu+\nu}$$

where $\rho_i(\mu,\nu)$ *is the number of hyperplanes* $\varphi_i(X) \in \mathbb{Z} + 1/2$ *crossed twice by the path* $0 \to \mu \to \mu + \nu$*.*

For example:

$$r_{(-2,-2)}r_{(1,2)} = \varphi_1\varphi_2\varphi_3^2\varphi_4^2 \cdot r_{(-1,0)}$$



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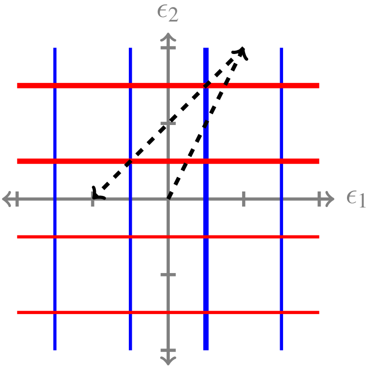


The quantized Coulomb branch has a very similar presentation; the only difference is that we pay attention to where the hyperplanes are. We fix scalars $m_1, \dots, m_d$. These are the **flavors**.

Theorem

$A_{\mathbf{m}}$ is a S-algebra with free basis r_ν for $\nu \in \mathfrak{t}_{\mathbb{Z}}$, and multiplication rules

$$r_\nu r_\mu = r_{\mu+\nu} \prod_i \prod_{n \in D^{(i)}_{\nu,\mu}} (\varphi_i + m_i + n)$$

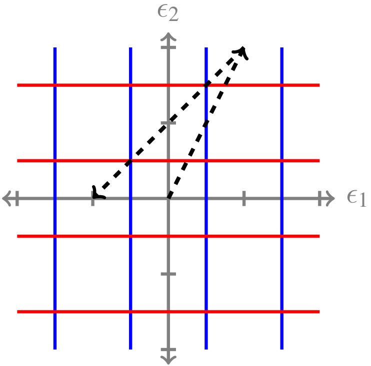
$$(\varphi - \langle \varphi, \nu \rangle) r_\nu = r_\nu \varphi$$

where $D^{(i)}_{\mu,\nu}$ is the set of values of φ_i on hyperplanes $\varphi_i(X) \in \mathbb{Z} + 1/2$ crossed twice by the path $0 \to \mu \to \mu + \nu$.

If you shift $m_i \mapsto m_i + \langle \varphi_i, \nu \rangle$ for some $\nu \in \mathfrak{t}$, the result will be isomorphic.

For example if $m_1 = m_3 = 1/2, m_2 = m_4 = -1/2$, then we have

$$r_{(-2,-2)}r_{(1,2)} = r_{(-1,0)}\varphi_1(\varphi_2 - 1)\varphi_3(\varphi_3 - 1)(\varphi_4 - 1)(\varphi_4 - 2)$$



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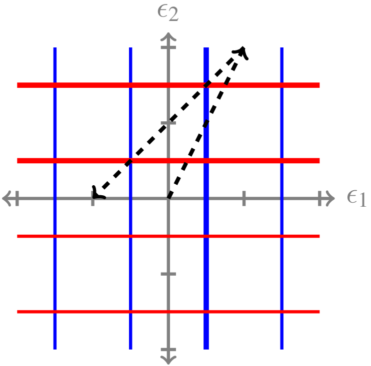


We can think of this ring in two different ways:

= Sym($\mathfrak{t}^*$)

1. We have an action of $A_{\mathbf{m}}$ on S, with $S \subset A_{\mathbf{m}}$ acting by multiplication, and

shift by ν

$$r_\nu = \delta_\nu \prod_{i} \prod_{D^{(i)}_\mu(+)} (\varphi_i + m_i + n).$$

positive values of φ_i passed on $0 \neq \mu$

2. By Mellin transform, we can also think of this as a subring of Diff$(\check{T})$, with $\mathfrak{t}^*$ identified with invariant vector fields, and r_μ with μ as a function on $\check{T}$ times an element of $U(\mathfrak{t}^*)$.

These descriptions realize $A_{\mathbf{m}}$ as a **principal Galois order** (for $W = 1$).

This presentation is particularly well suited to understanding the action of this algebra on Gelfand-Tsetlin modules.

Definition

An A module is called a **Gelfand-Tsetlin (GT) module** *if* $\mathrm{Sym}(\mathfrak{t}^*)$ *acts locally finitely, with finite dimensional (generalized) eigenspaces.*

For $\in \mathfrak{t}$, the (generalized) ν-weight space for A is

$$M_\nu = \{m \in M \mid \text{for all } \mu \in \mathfrak{t}^*, \text{ we have } (\mu - \langle \nu, \mu \rangle)^N m = 0 \text{ for } N \gg 0\}.$$

Proposition

The operator $\mu - \beta$ *for* $\mu \in \mathfrak{t}^*$ *and* $\beta \in \mathbb{C}$ *is invertible on* M_ν *if and only if* $\langle \nu, \mu \rangle \neq \beta$.

We'll consider not just on the weight spaces, but the functor $M \mapsto M_\nu$.

These functors are represented by the inverse limits $P_\nu := \varprojlim A/A(\mu - \langle \nu, \mu \rangle)^N$:

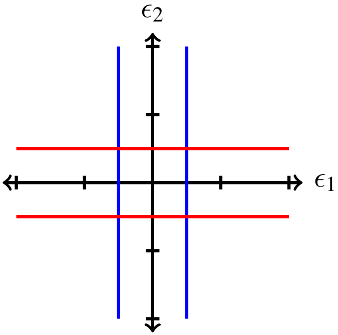
$$M_\nu \cong \mathrm{Hom}(P_\nu, M).$$

Now consider the (finite) arrangement $\mathcal{A}$ where you only keep hyperplanes of the form $\varphi_i = -m_i$.

Theorem (Musson-van der Bergh)

The projectives $P_{\nu'}$ for $\nu' \in \mathfrak{t}_\mathbb{Z}$ are classified up to isomorphism by the chambers in $\mathcal{A}$ which contain elements of $\mathfrak{t}_\mathbb{Z}$.

In our running example, the hyperplane arrangement $\mathcal{A}$ is:



so there are 9 different projectives.

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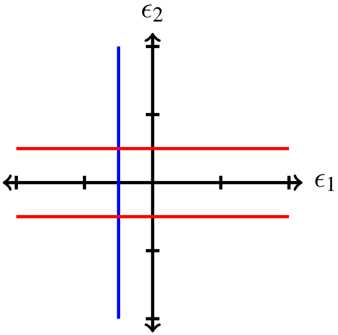


so there are 9 different projectives.

If instead $m_1 = \pi$, one of these hyperplanes will vanish, and there will only be 6.

This leads to a presentation of the endomorphisms of these projectives:

Theorem (Musson-van der Bergh)

The category of GT modules with weights in $\nu' \in \mathfrak{t}_{\mathbb{Z}}$ is equivalent to the category of modules over the $\mathrm{Sym}(\mathfrak{t}^*)$-algebra Y generated by $r(C, C')$ for C, C' chambers in $\mathcal{A}$, with the multiplication rule

$$r(C, C')r(C', C'') = r(C, C'') \prod_i \varphi_i^{\rho_i(C,C',C'')}.$$

where $\rho_i(C, C', C'')$ is the number of hyperplanes labeled by φ_i crossed twice by the path $C \to C' \to C''$.

This gives a direct algebraic proof of Koszul duality in this case.

Theorem (Braden-Proudfoot-Licata-W.)

The Koszul dual of category $\mathcal{O}$ for one abelian theory is again the category $\mathcal{O}$ of another abelian theory.

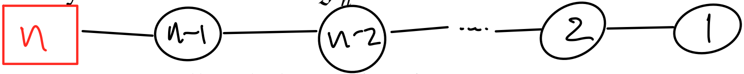
This case is much harder, but let's generalize to the case of G nonabelian with T its maximal torus.

Most important example ("quiver gauge theory"): Let Γ be a quiver, and $\mathbf{v}$, $\mathbf{w}$ dimension vectors.

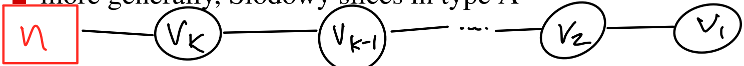
$$N_{\mathbf{v}} = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \oplus \bigoplus_{i} \mathrm{Hom}(\mathbb{C}^{w_i}, \mathbb{C}^{v_i}) \qquad G_{\mathbf{v}} = \prod_i GL(\mathbb{C}^{v_i})$$

Interesting examples of Coulomb branches for quiver gauge theories:

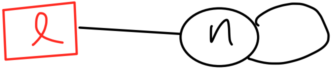
- my favorite: nilcone of $\mathfrak{gl}_n$



- more generally, Slodowy slices in type A



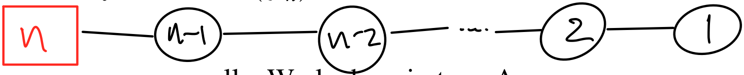
- symmetric power $\mathrm{Sym}^n(\mathbb{C}^2/\mathbb{Z}_\ell)$



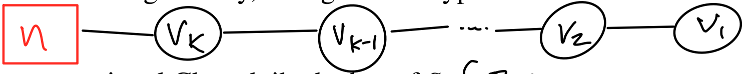
All of these varieties are Nakajima quiver varieties, but for potentially different quivers and dimension vectors. The quiver variety and Coulomb branch for a given quiver should be related by “3d mirror symmetry”/“symplectic duality.”

Interesting examples of Coulomb branches for quiver gauge theories:

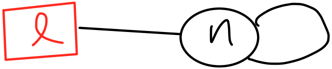
- my favorite: $U(\mathfrak{gl}_n)$



- more generally, W-algebras in type A



- rational Cherednik algebra of S_n $\wr \mathbb{Z}/\ell\mathbb{Z}$



All of these varieties are Nakajima quiver varieties, but for potentially different quivers and dimension vectors. The quiver variety and Coulomb branch for a given quiver should be related by "3d mirror symmetry"/"symplectic duality."

Presenting these as Coulomb branches shines a spotlight on a Q which looked a bit less motivated before:

Can you describe the category of GT modules for these algebras? What are the simple modules? What are the weight multiplicities of simple modules?

Up until ~2018, none of these questions where resolved in any interesting cases. Doing this "by hand" in the case of $U(\mathfrak{sl}_3)$ is a 97 page paper of Futorny-Grantcharov-Ramirez.

I can now systematically answer all these questions (though large cases are computationally difficult).

Usually, we can think of the non-abelian case as abelian + Weyl group, but usually with a few extra kinks.

For the Coulomb branch, we have to account for the fact that the affine Grassmannian of T is a discrete set of points in the affine Grassmanian of G; we have to incorporate the tangent spaces to these points.

We'll construct the non-abelian Coulomb branch by first constructing a larger algebra. Take the picture from the abelian case, and add in the hyperplanes $\alpha \in \mathbb{Z}$. We'll now change our running example to $GL(2)$ acting on $\mathbb{C}^2 \oplus \mathbb{C}^2$:

Add the elements of the Weyl group W to the algebra, with the commutation relation $wr_\nu w^{-1} = r_{w\cdot\nu}$. That is, consider the smash product $W\#\mathsf{A}_{\mathrm{ab}}$.

We still have elements r_ν, but we need to change the relations between them.

Each time we hit a hyperplane $\alpha = n$, we "insert" an action of the Demazure operator $\partial_\alpha \colon \operatorname{Sym}(\mathfrak{t}^*) \to \operatorname{Sym}(\mathfrak{t}^*)$ by

$$\partial_\alpha(f) = \frac{s_\alpha f - f}{\alpha}.$$

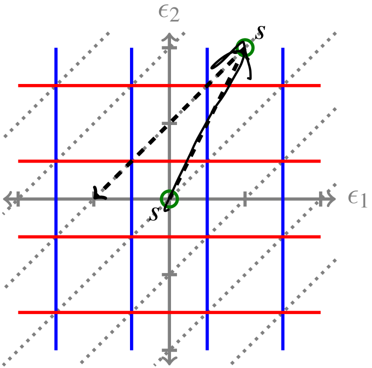
Let E be the resulting algebra; this is no longer a subalgebra of $W\#\mathsf{A}_{\mathrm{ab}}$, but is birational to it.

Definition

The coordinate ring $\mathbb{C}[\mathfrak{M}] := Z(\mathsf{E}) \cong \mathsf{eEe}$ *of the Coulomb branch is the centralizer of the idempotent* e *projecting to* $\operatorname{Sym}(\mathfrak{t}^*)^W$.

Nonabelian Coulomb branches

$$r_{(-2,-2)}sr_{(1,2)}s = r_{(-2,-2)}r_{(2,1)}$$



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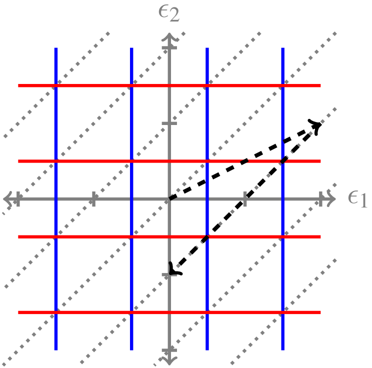

Definition

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With similar Demazure insertions, we can construct a quantization $E_{\mathbf{c}} \supset A_{\mathbf{c},\mathrm{ab}}$ where $\mathbf{c}$ is compatible with G.

The algebra $E_{\mathbf{c}}$ contains a "projection" idempotent e.

Theorem

The algebra $A_{\mathbf{c}} = eE_{\mathbf{c}}e$ *is a quantization of the Coulomb branch.*

Rather than working with $eE_{\mathbf{c}}e$, I'd rather stick with $E_{\mathbf{c}}$. This will be Morita equivalent via the functor $M \mapsto eM$.

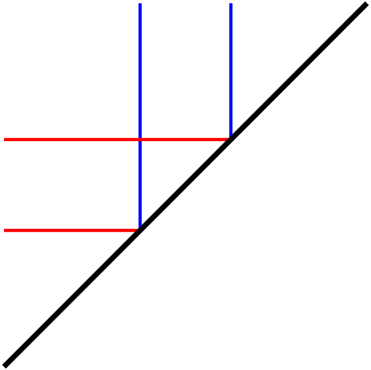
One reason this is better is that $A_{\mathbf{c},\mathrm{ab}} \subset E_{\mathbf{c}}$. In particular, I still have a copy of $\mathrm{Sym}(\mathfrak{t}^*)$ to serve as my "torus."

Definition

We call a $E_{\mathbf{c}}$-module M a **GT module**, *if it is a GT module restricted to $A_{\mathbf{c},\mathrm{ab}}$. Similarly, the (generalized) weight space M_ν is as before.*

As before, the functor $M \mapsto M_\nu$ is proprensentatable by a projective P_ν, and $P_\nu \cong P_{w\cdot\nu}$. Also, if an element of $A_{\mathbf{c},\mathrm{ab}}$ gives an isomorphism $P_{\nu,\mathrm{ab}} \cong P_{\nu',\mathrm{ab}}$, then $P_\nu \cong P_{\nu'}$ by the same element.

So now, the projectives P_ν are isomorphic if they lie in the same chamber of $\mathcal{A}$ up to the action of the Weyl group. Thus we only need one Weyl chamber:



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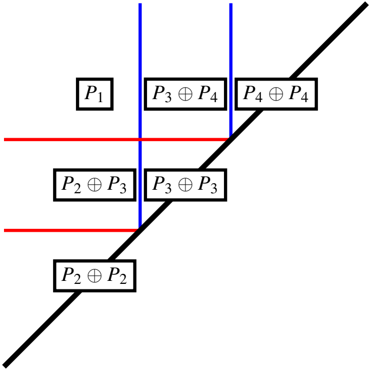


These projectives can be decomposable, though. There are only 4 indecomposable projectives.

Having fixed $\nu + \mathfrak{t}_{\mathbb{Z}}$, we let Δ_ν^+ be the positive roots of G such that $\alpha(\nu) \in \mathbb{Z}$, and let W be the Weyl group generated by reflection in these roots.

Theorem

The endomorphism algebra Y of the projectives is given by the smash product algebra $Y_{\mathrm{ab}}\#W$ with the Demazure operator $\partial_\alpha(C)$ acting on the chamber C adjoined for any $\alpha \in \Delta_\nu^+$ such that $s_\alpha \cdot C = C$.

This endomorphism algebra also has a geometric realization as the convolution algebra of a finite dimensional variety.

Theorem

The category of GT modules has a graded lift, and we can calculate dimensions of graded weight spaces of simples via the "Kazhdan-Lusztig method": the simples give a "dual canonical basis."

In all the cases we've discussed, this description also proves the Koszul duality: the convolution algebra that appears computes the Ext's between semi-simple D-modules you can q-Hamiltonian reduce to get simples in category $\mathscr{O}$.

In the quiver case, this algebra is already well-known (to me, at least).

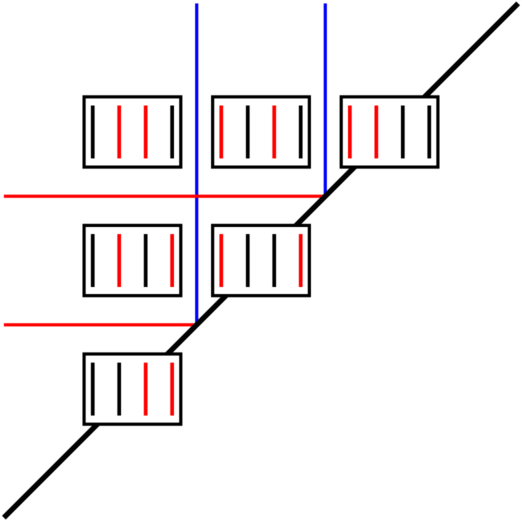
You represent a chamber by using the coordinates in $\mathfrak{t} \subset \bigoplus_i \mathfrak{gl}(d_i)$ as positions of black dots on the line, with labels corresponding to the factor. The values of m_i are represented by red dots.

You change your chamber when you pass a black dot past a red, or two black dots with adjacent labels. You cross a root hyperplane when two points with the same label pass.

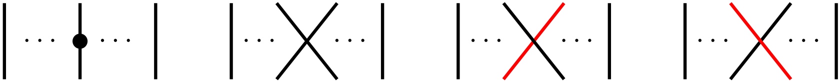
Theorem

The algebra which appears is a KLRW algebra for Γ depending on the choice of weights m_i.

GT modules

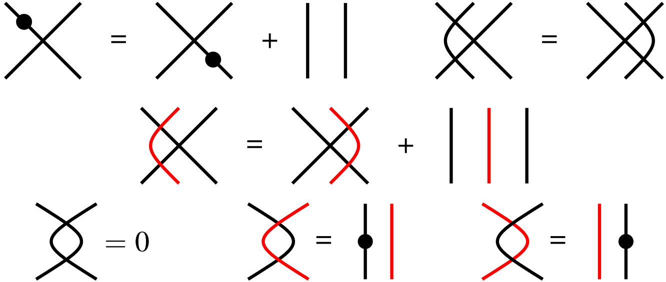


generators:



ϵ_i · ∂_i · $r(C, C')$ · $r(C', C)$

relations:



This tells us the first classification of simple GT modules over:

- $U(\mathfrak{gl}_n)$ and type A W-algebras (W.)
- truncated shifted Yangians (Kamnitzer-W.-Weekes-Yacobi)
- rational Cherednik algebras of $G(\ell, p, n)$ (LePage-W.)

In all cases, questions reduce to questions about KLRW algebras, which we know how to answer.

However, these questions are still hard to answer computationally. With Silverthorne, we found all the simples and all weight multiplicities for $\mathfrak{sl}_4$. For $\mathfrak{sl}_5$, too hard for our computer. In general, number of simples in regular integral block grows very fast.

n	2	3	4	5	6	7	8
# of simples	3	20	259	6005	235,546	14,981,789	1.494×10^9

FGR	L_{32}	L_{21}	L_{24}	L_{26}	L_{16}	L_8	L_{19}	L_{10}	L_{17}	L_{13}	L_6	L_3	L_4	L_{28}	L_{11}	L_{14}	L_5	L_2	L_7	L_1
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FIGURE 3. The table of dimensions of GT weight spaces in the integral regular case for $\mathfrak{gl}_3$

IWS	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F
GK dim	6	5	5	5	5	3	5	4	5	5	4	3	5	4	3	3	4	3	5	3	3	3	3	0	6	5	5	5	5	3
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4444332132	$\boxed{4}$																													
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Thanks for listening.