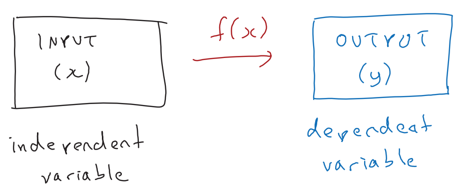
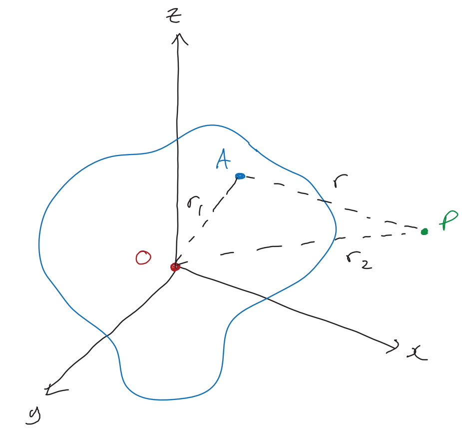
1.2 Functions, Variables, and Constants

Function: a rule that associates a y-value for every x-value



For future calculations, we must realize there are three points to consider:

1. Origin "O" is the origin of the coordinate system
2. An arbitrary point "A" on the physical object
3. A point "P" in space where we make our observations



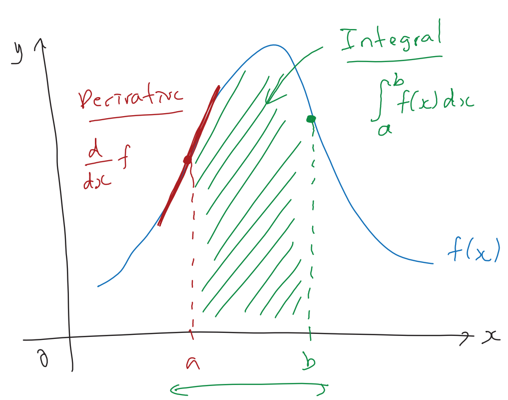
We can use "r" to represent "distance" between points:

↳ if AP is r, then OA and OP can be r_1 and r_2 respectively

What if we had a function $f(r_1)$?

↳ we know it will be dependent upon the location of Point A w.r.t. Origin O

Difference Between Derivatives & Integrals



Derivative: calculates the slope of a curve at a specific point, such as at $x = a$

Integral: determines the area underneath the curve between two points, such as $a \le x \le b$

Looking at Constants

→ always look to see what variable you are working with respect to

Example: $\int_a^b \frac{1}{r}\,dr = \ln r \Big|_a^b = \ln(b) - \ln(a) = \ln\left(\frac{b}{a}\right)$

→ what about: $\int_a^b \frac{1}{R}\,dr$ where R is a constant

$$\int_a^b \frac{1}{R}\,dr = \frac{1}{R}\int_a^b 1\,dr = \frac{1}{R}\left[r\right]_a^b = \frac{1}{R}(b-a)$$

Diagnostics Quiz Question

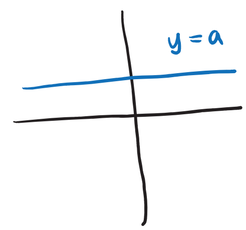
Diagnostics Quiz Question

$$\int_{-\pi}^{\pi} \sin(\alpha)\, d\theta = \sin\alpha \int_{-\pi}^{\pi} d\theta = \sin\alpha\, (\theta)\Big|_{-\pi}^{\pi} = \sin\alpha\,(\pi - (-\pi)) = \underline{\underline{2\pi \sin\alpha}}$$

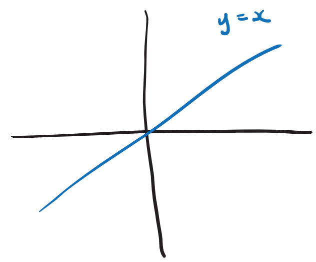
A note on boundary conditions:

↳ if $f(r)$ is bounded by $a \leq r \leq b$, then we can get the magnitude of the function at any point between a and b

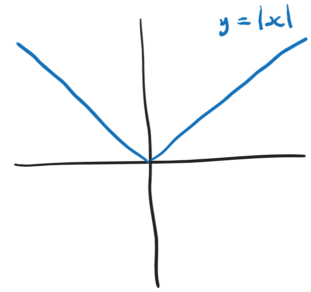
BASIC SHAPES OF FUNCTIONS



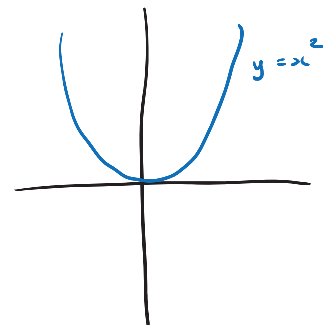
CONSTANT



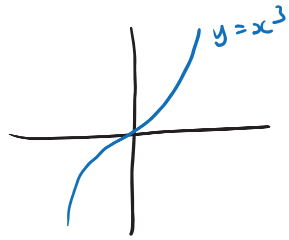
LINEAR



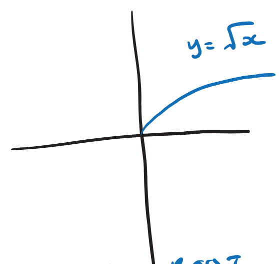
ABSOLUTE VALUE



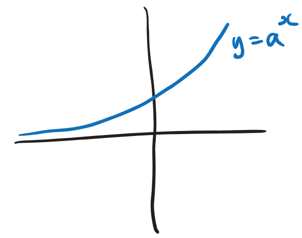
QUADRATIC



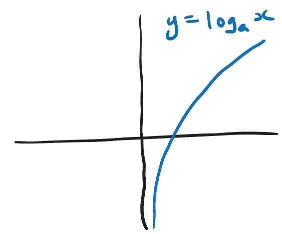
CUBIC



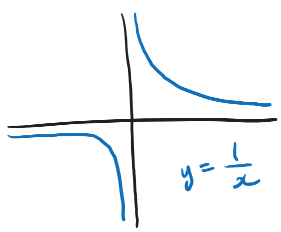
SQUARE ROOT



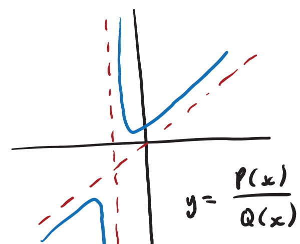
EXPONENTIAL



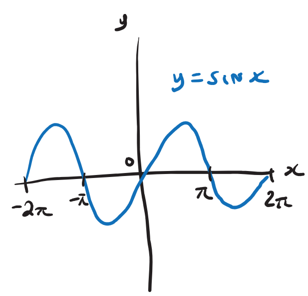
LOGARITHMIC



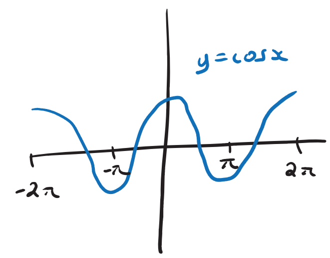
RECIPROCAL



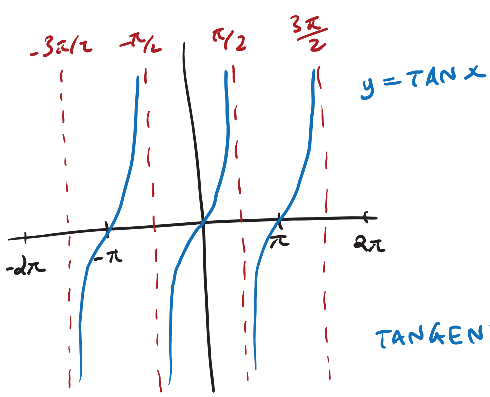
RATIONAL



SINE FUNCTION



COSINE FUNCTION



TANGENT FUNCTION

LOOKING AT THE LINEAR FUNCTION

→ generic form of a linear function is $y = ax + b$, where a and b are constants

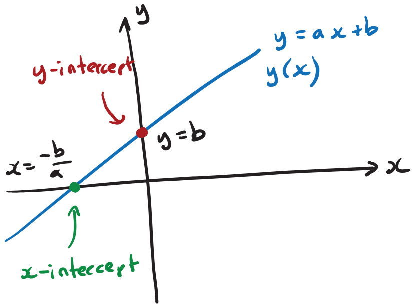
$$y(x) = ax + b$$

dependent variable independent variable

→ to plot this, we have:

y-intercept	x-intercept
when $x = 0$	when $y = 0$

→ to plot this, we have:



y-intercept

when $x = 0$

$y(0) = a(0) + b$

$y = 0 + b$

$\underline{y = b}$

x-intercept

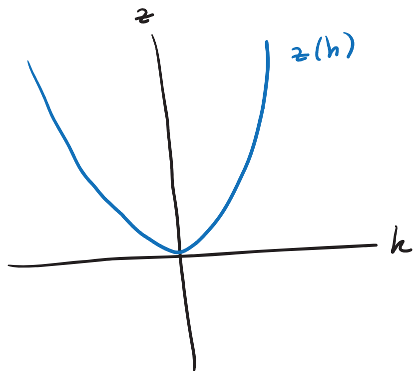
when $y = 0$

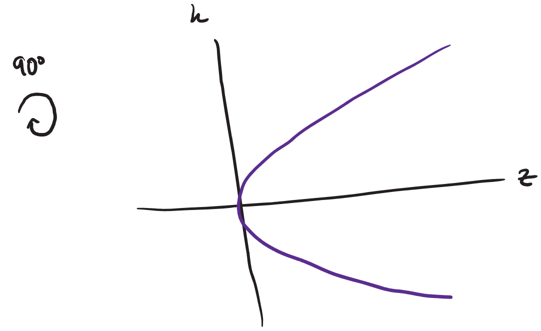
$0 = ax + b$

$$x = \frac{-b}{a}$$

Looking at a Quadratic Function

→ use example function $z(h) = rk^2$ means that r is a constant and z, k are the variables





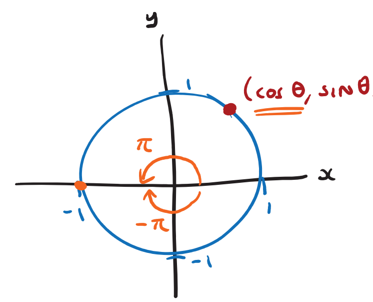
Example

→ Solve for $\int_{-\pi}^{\pi} \sin\theta \, d\theta = \left[-\cos\theta\right]_{-\pi}^{\pi} = -\cos\pi + \cos(-\pi)$

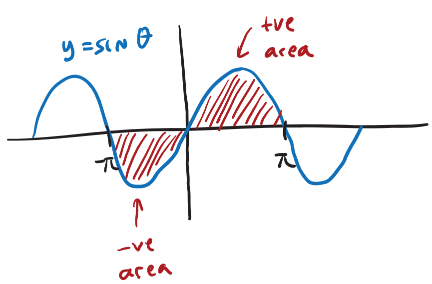
$= -(-1) + (-1)$

$= 1 - 1 = \underline{0}$

Aside:



→ an alternative is to solve visually



→ same amount of −ve and +ve area

→ $\therefore \int_{-\pi}^{\pi} \sin\theta \, d\theta = \underline{0}$

1.4 Vectors and Unit Vectors

VECTORS VS. SCALARS

Vectors → have both magnitude and direction

Scalars → only have magnitude

If we have a vector $\vec{A}$, its magnitude is either $|\vec{A}|$ or A
(scalar)

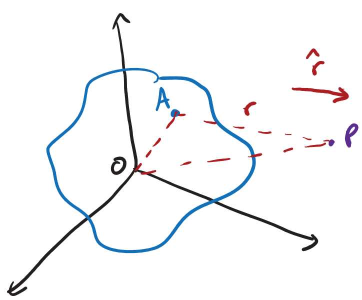
↳ we use the unit vector $\hat{a}$ to give us the direction of vector $\vec{A}$

↳ have a length of 1

↳ be in the same direction as its "parent" $\vec{A}$

$$\hat{a} = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A}$$

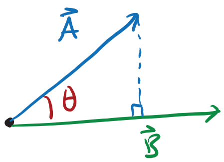
A common vector is called position vector $\vec{r}$



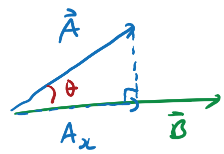
Dot Product

Let's define two vectors $\vec{A}$ and $\vec{B}$

There is an angle θ between these:



Let's look at $\underline{\vec{A}\cdot\vec{B}}$



A_x

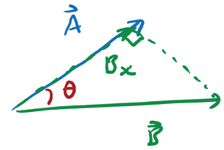
$A_x = |\vec{A}|\cos\theta$

$= A\cos\theta$

$\vec{A}\cdot\vec{B} = (A\cos\theta)B$

$= \underline{AB\cos\theta}$

and $\underline{\vec{B}\cdot\vec{A}}$



$B_x = |\vec{B}|\cos\theta$

$\vec{B}\cdot\vec{A} = (B\cos\theta)A$

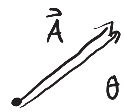
$= \underline{AB\cos\theta}$

Because $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$, the dot product follows the <u>commutative rule</u>

The dot product is also <u>distributive</u>:

$$\vec{A}\cdot(\vec{B}+\vec{C}) = (\vec{A}\cdot\vec{B}) + (\vec{A}\cdot\vec{C})$$

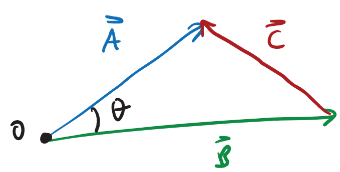
Example #1: Determine the magnitude of vector $\vec{A}$



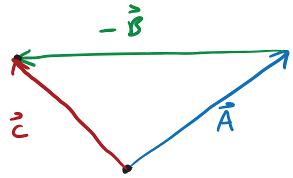
$\theta = 0 \Rightarrow \cos 0 = 1$

$$\vec{A}\cdot\vec{A} = |\vec{A}||\vec{A}|\cos\theta = A^2(1) = A^2$$

Example #2: Determine the magnitude of vector $\vec{C}$ in the following:



reorganize →



$\vec{A} + (-\vec{B}) = \vec{C}$

$\underline{\vec{C} = \vec{A} - \vec{B}}$

→ we can use $\vec{C}\cdot\vec{C} = C^2\cos 0 = C^2$

$$C^2 = \vec{C}\cdot\vec{C} = (\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})$$

$$C^2 = \vec{A}\cdot\vec{A} - 2\vec{B}\cdot\vec{A} + \vec{B}\cdot\vec{B}$$

$$C^2 = A^2 - 2(BA\cos\theta) + B^2$$

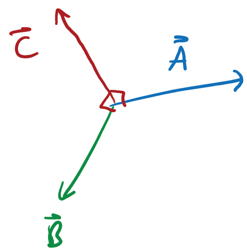
$$\underline{C^2 = A^2 + B^2 - 2AB\cos\theta}$$ ← this is the <u>cosine law</u>

Cross Product

→ we want a third vector that is orthogonal to both $\vec{A}$ and $\vec{B}$

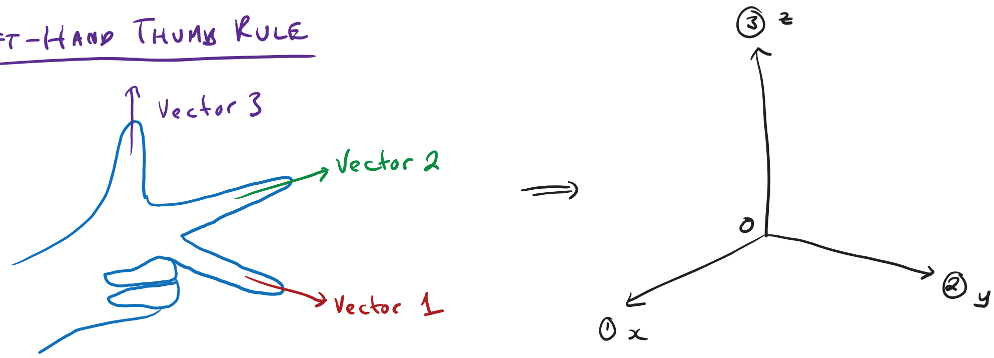
vector quantity

→ we want a third vector that is orthogonal to both $\vec{A}$ and $\vec{B}$



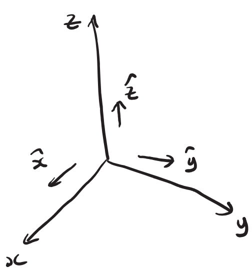
∴ a cross product produces a vector quantity

LEFT-HAND THUMB RULE



CARTESIAN COORDINATE SYSTEM

→ we have three unit vectors: $\hat{x}$, $\hat{y}$, $\hat{z}$

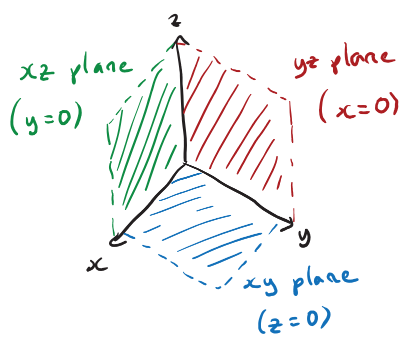


→ we can say that a vector $\vec{A}$ in 3D space can be represented by the path taken in each direction:

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

Aside: A_x, A_y, A_z are the distances moved in the x, y, and z axis respectively

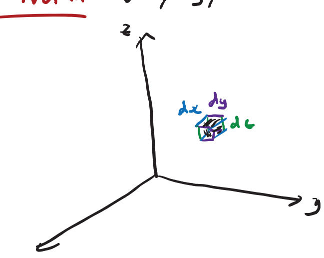
→ we must also recognize different planes:



Differential Elements

2D World → dx, dy for the xy plane

3D World → dx, dy, dz



→ Differential Length Element is called $\vec{dl}$ and defined as:

$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

→ Differential Surface Element is called $\vec{dS}$ and can be represented in one of 3 ways:

③ dydz

→ Differential Surface Element is called $\vec{dS}$ and can be represented in one of 3 ways:

① $dx\,dy$ ② $dx\,dz$ ③ $dy\,dz$

→ If we look at $dx\,dy$, we know that we are not moving in the z-direction, but rather focusing on the xy-plane

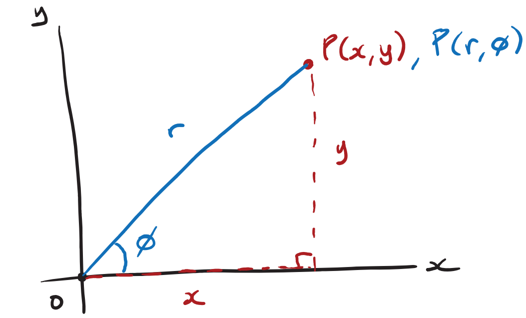
→ We conveniently choose $\vec{dS}$ to be a vector and can define surface elements as follows:

$$\vec{dS}_z = dx\,dy\,\hat{z} \qquad \vec{dS}_y = dx\,dz\,\hat{y} \qquad \vec{dS}_x = dy\,dz\,\hat{x}$$

→ Differential Volume Element is called dv and defined as:

$$dv = dx\,dy\,dz$$

Overview of Polar Coordinates



$x^2 + y^2 = r^2$

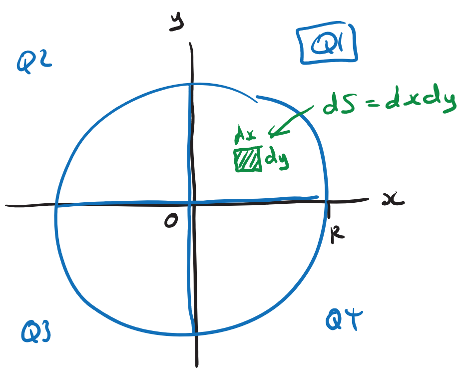
$\sin\phi = \frac{y}{r}$ | $\cos\phi = \frac{x}{r}$ | $\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{y/r}{x/r}$

$y = r\sin\phi$ | $x = r\cos\phi$ | $\tan\phi = \frac{y}{x}$

Calculate Area of a Circle

→ compare 2 methods: ① Cartesian coordinates ② Polar coordinates

Method #1: Cartesian Coordinates

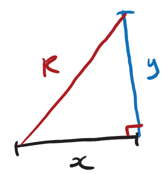


$A_{circle} = 4A_{Q1}$ → how to find A_{Q1}?

→ If $dS = dxdy$, we must find boundaries for each:

x-values: $0 \to R$

y-values: $0 \to \sqrt{R^2 - x^2}$



$x^2 + y^2 = R^2$

$y = \sqrt{R^2 - x^2}$

→ now we have to integrate with respect to $dS = dxdy$

↳ 2 differential elements, so we use double integrals

↳ $\int_c^d \left[\int_a^b dx\right] dy = \int_c^d \underbrace{\int_a^b dx}_{1st}\, dy$

→ let's set up our integral for A_{circle}

$$A_{circle} = 4A_{Q1} = 4\int_0^R \int_0^{\sqrt{R^2 - x^2}} dy\, dx$$

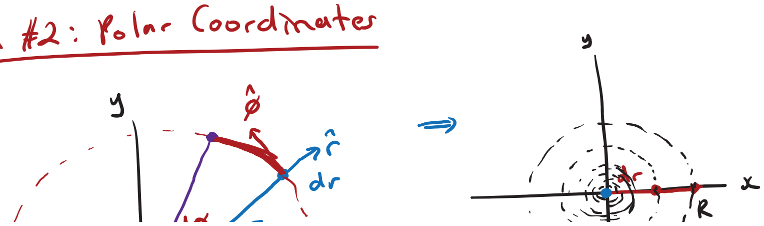
$$A_{circle} = 4\int_0^R \left[y\right]_0^{\sqrt{R^2 - x^2}} dx$$

$$A_{circle} = 4\int_0^R \sqrt{R^2 - x^2}\, dx$$

→ have to use trig substitution ($x = R\sin\theta$)

(takes too long...)

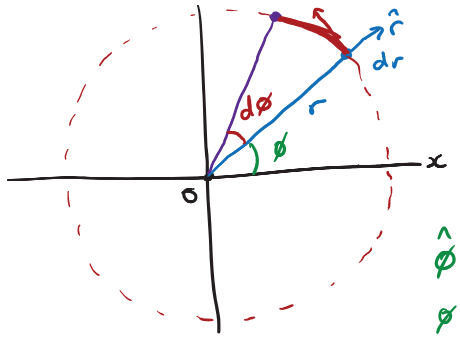
Method #2: Polar Coordinates

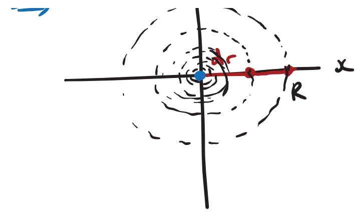


→ We need to find out how to define dS with dr and $d\phi$

↑

defined by differential length elements





defined by differential length elements

$$dS = (dr)(rd\phi)$$

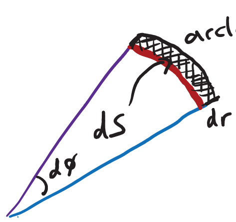
$\hat{\phi}$ = azimuthal vector

ϕ = azimuthal angle

→ Lets set some boundary conditions:

r-values: $0 \to R$ (radius of circle)

ϕ-values: $0 \to 2\pi$ (full revolution)

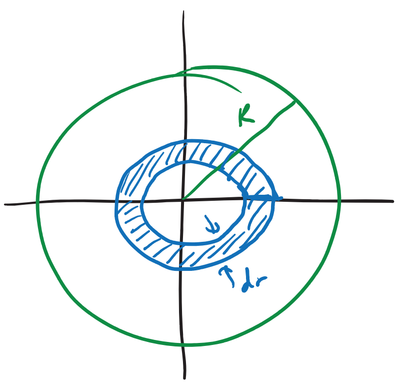


$$A_{circle} = \int_0^R \int_0^{2\pi} r\,d\phi\,dr$$

$$A_{circle} = \int_0^R r \int_0^{2\pi} d\phi\,dr = \int_0^R r\,[2\pi - 0]\,dr$$

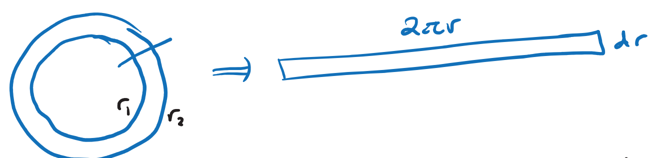
$$A_{circle} = 2\pi \int_0^R r\,dr = 2\pi\left[\frac{r^2}{\cancel{2}}\right]_0^R = \pi R^2$$

An Alternative Way of Thinking



→ change in radius is dr → very tiny

→ circumference is $2\pi r$



→ we can get a surface element $dS = 2\pi r\,dr$

$$A_{circle} = 2\pi \int_0^R r\,dr = 2\pi\left[\frac{r^2}{2}\right]_0^R = \frac{2\pi R^2}{2} = \pi R^2$$

SUMMARY

Unit Vectors

$\hat{r}$, $\hat{\phi}$

Differential Length Elements

dr in $\hat{r}$ direction

$rd\phi$ for the arclength

Differential Area Element

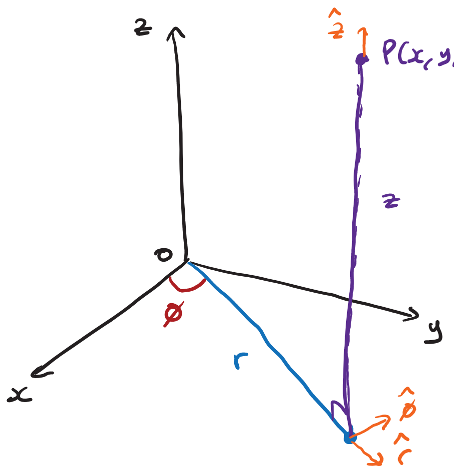
$dS = r\,dr\,d\phi$

1.8 Coordinate Systems: Cylindrical Coordinates

Overview

→ the cylindrical coordinate system is another way to represent point $P(x,y,z)$ in 3D space

→ we will use variables r, ϕ, z



$y = r\sin\phi$

$x = r\cos\phi$

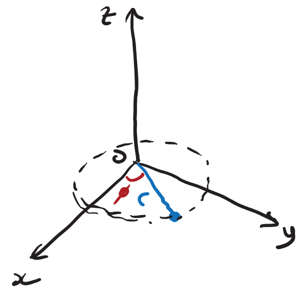
$\tan\phi = \frac{y}{x}$

$x^2 + y^2 = r^2$

$z = z$

When can we use this coordinate system?

→ symmetry about an axis



Differential Elements

Looking at point P, we define several unit vectors

→ $\hat{r}$ pointing radially outward

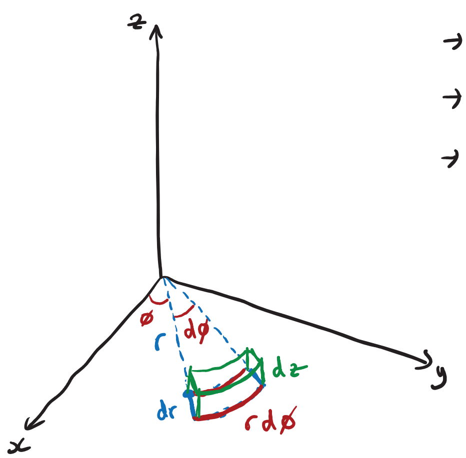
→ $\hat{\phi}$ that changes the angle r makes with the x-axis

→ $\hat{z}$ to change the height

$dr, d\phi, dz$

→ these are the variables that will change

Let's look at the differential length elements



→ dr along $\hat{r}$

→ dz along $\hat{z}$

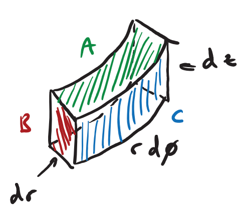
→ $r\,d\phi$ along $\hat{\phi}$

Differential Volume Element

$dv = dr\,dz\,r\,d\phi$

$dv = r\,dr\,d\phi\,dz$

Differential Surface Area Elements



$A_A = r\,dr\,d\phi$

$A_B = dr\,dz$

$A_C = r\,d\phi\,dz$

Calculate Surface Area of the Curved Part of Cylinder

→ We know that $ds = r\,d\phi\,dz$

→ Find boundaries for ϕ and z

... der of length L: z-values: $0 \to L$ φ-values: $0 \to 2\pi$

→ Find boundaries for ϕ and z

↳ For a cylinder of length L: z-values: $0 \to L$ $\quad$ ϕ-values: $0 \to 2\pi$

→ Solve using double integrals:

$$SA = \int_0^L \int_0^{2\pi} r\,d\phi\,dz$$

r must be R, the radius of the cylinder

$$SA = R\int_0^L \int_0^{2\pi} d\phi\,dz = R\int_0^L dz \int_0^{2\pi} d\phi$$

$$SA = R(L-0)(2\pi - 0) = 2\pi RL$$

→ surface area of cylinder's curved surface!

Calculate the Volume of a Cylinder

→ the differential volume element is $dv = r\,dr\,d\phi\,dz$ so a triple integral is required!

$$V = \int_0^L \int_0^{2\pi} \int_0^R r\,dr\,d\phi\,dz$$

$$V = \int_0^L dz \int_0^{2\pi} d\phi \int_0^R r\,dr$$

$$V = [L-0][2\pi - 0]\left[\frac{r^2}{2}\right]_0^R$$

$$V = 2\pi L\left[\frac{R^2}{2} - 0\right]$$

$$V = \pi L R^2$$

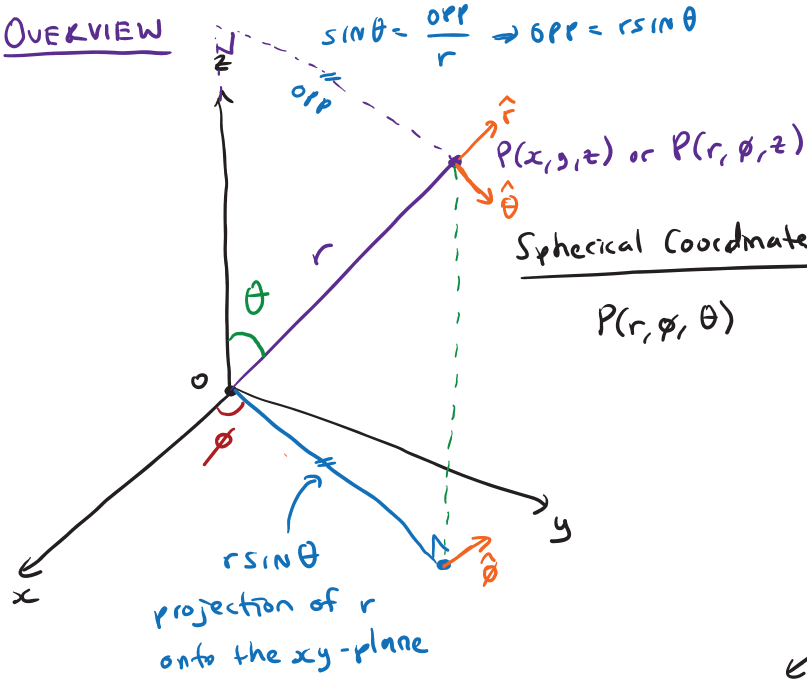
Note: If we have

$$\int_a^b \int_c^d \int_e^g f(x,y,z)\,dx\,dy\,dz,$$

if $f(x,y,z)$ can be separated into individual terms of x, y, z as products, such as: $f(x,y,z) = h(x) \cdot l(y) \cdot m(z)$, we can rewrite the integral:

$$\int_a^b dz \int_c^d dy \int_e^g dx$$

Overview

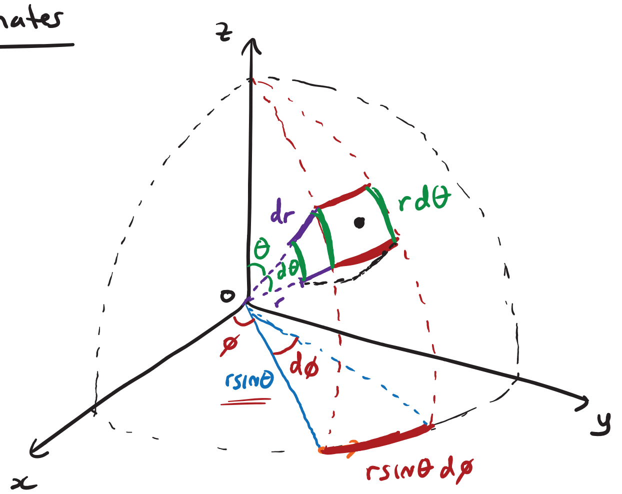


→ The unit vectors are: $\hat{r}, \hat{\phi}, \hat{\theta}$

Spherical Coordinates

$P(r, \phi, \theta)$

Differential Elements



Differential Length Elements

→ dr → $rd\theta$ → $r\sin\theta\, d\phi$

Differential Volume Element

$$dv = dr(rd\theta)(r\sin\theta\, d\phi)$$

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Surface Area of Sphere

→ we need dS, a differential surface area element

$$dS = (rd\theta)(r\sin\theta\, d\phi)$$

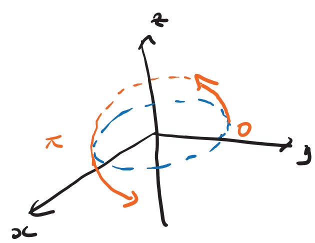
$$dS = r^2 \sin\theta\, d\theta\, d\phi$$

$$SA = \int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta\, d\theta\, d\phi$$, but $r = R$ at the surface

$$SA = R^3 \int_0^{2\pi}\int_0^{\pi} \underline{\sin\theta\, d\theta}\, d\phi$$

Volume of Sphere

$$\int dv = V_{sphere}$$



Let's look at boundaries:

r-values: $0 \rightarrow R$

φ-values: $0 \rightarrow 2\pi$

θ-values: $0 \rightarrow \pi$

$$V_{sphere} = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \underbrace{r^2 \sin\theta}_{(r^2)(\sin\theta)}\, dr\, d\theta\, d\phi$$

$$V_{sphere} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{R} r^2\, dr$$

$$= (2\pi)(-\cos\theta)\Big|_0^{\pi} \left(\frac{r^3}{3}\right)\Big|_0^{R}$$

$$= 2\pi(1+1)\left(\frac{R^3}{3}\right)$$

$$= \frac{4\pi R^3}{3}$$

$$SA = R^3 \int_0^{2\pi}\int_0^{\pi} \underbrace{\sin\theta}\, d\theta\, d\phi$$

$$SA = R^3 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta = R^3 (2\pi) \underbrace{\left(-\cos\theta\right)\Big|_0^{\pi}}$$

$$SA = 2\pi R^3 (2) \quad \therefore \underline{\underline{SA = 4\pi R^3}}$$

1.10 Examples and Important Integrals

Example #1. A sphere of radius R has a mass M with density linearly changing with radius, such that the density is 0 at the center of the sphere. Calculate the density as a function of radius.

Given: sphere → $\underline{R}$, radius → $\underline{M}$, mass

→ density $\rho(r)$ is a linear function

$$\rho(r) = \underset{\hookrightarrow \rho_0}{a}r + \underset{\hookrightarrow \rho_1}{b}$$

$$\rho(r) = \rho_0 r + \rho_1$$

$$\underline{\underline{\rho(r) = \rho_0 r}}$$

↑ we need to define ρ_0

Find: $\rho(r)$

→ when $r = 0$, $\rho = 0$

$$\rho(0) = 0$$

$$\rho(0) = \rho_0(0) + \rho_1$$

$$0 = 0 + \rho_1$$

$$\underline{\underline{\rho_1 = 0}}$$

Approach: → relate ρ to $\underline{M}$ and R

$$\rho = \frac{m}{V} \Rightarrow \underline{m = \rho V}$$

$\hookrightarrow \rho(r)$ is dependent upon r

→ find the differential mass element dm

$$m = \rho V$$

$$dm = \rho(r)\,dV \leftarrow dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$dm = \rho(r) r^2 \sin\theta\, dr\, d\theta\, d\phi \leftarrow \rho(r) = \rho_0 r$$

$$dm = \rho_0 r^3 \sin\theta\, dr\, d\theta\, d\phi$$

Bounds: $r: 0 \to R$

$\theta: 0 \to \pi$

$\phi: 0 \to 2\pi$

→ solve for M using $\int dm$

$$M = \int dm = \iiint \rho_0 r^3 \sin\theta\, dr\, d\theta\, d\phi$$

$$M = \rho_0 \int_0^R \int_0^\pi \int_0^{2\pi} \underline{r^3} \sin\theta\, d\phi\, d\theta\, dr$$

$$M = \rho_0 \int_0^R r^3 dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = \rho_0 \left[\frac{r^4}{4}\right]_0^R \left[-\cos\theta\right]_0^\pi \left[\phi\right]_0^{2\pi}$$

$$M = \rho_0 \left[\frac{R^4}{4}\right]\left[\underbrace{-\cos\pi}_{1} + \underbrace{\cos 0}_{1}\right]\left[2\pi\right] = \frac{\rho_0 R^4}{\cancel{4}}(\cancel{4}\pi)$$

$$\underline{\underline{M = \pi \rho_0 R^4}}$$

↑

M $\Rightarrow \rho(r) = \rho_0 r$

$$M = \pi \rho_0 R^4$$

$$\rho_0 = \frac{M}{\pi R^4} \implies \rho(r) = \rho_0 r$$

$$\rho(r) = \frac{rM}{\pi R^4}$$

1.10 Examples and Important Integrals

Example #2. The density of a sphere varies as $\rho = \rho_0 \sin\theta$. Calculate the total mass.

Given: sphere $\rho(\theta) = \rho_0 \sin\theta$

Find: M, total mass

Approach: → we know that to find M, we need dm

$$dm = \rho(\theta)\, dv \quad \leftarrow dv = r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$dm = (\rho_0 \sin\theta)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$dm = \rho_0 r^2 \sin^2\theta\, dr\, d\theta\, d\phi$$

→ find M using $\int dm$

Boundaries: sphere $r: 0 \to R$, $\theta: 0 \to \pi$, $\phi: 0 \to 2\pi$

$$M = \int dm = \iiint \rho_0 r^2 \sin^2\theta\, dr\, d\theta\, d\phi$$

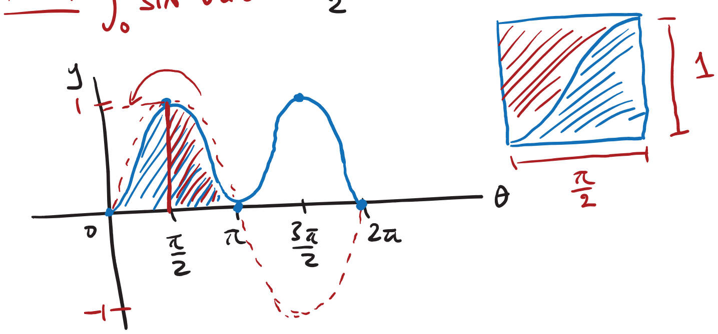
$$M = \rho_0 \int_0^R r^2 dr \int_0^\pi \sin^2\theta\, d\theta \int_0^{2\pi} d\phi$$

$$M = \rho_0 \left[\frac{r^3}{3}\right]_0^R \left[\frac{\pi}{2}\right] \left[\phi\right]_0^{2\pi}$$

$$M = \frac{\rho_0 \pi}{\cancel{2}} \left(\frac{R^3}{3}\right) (\cancel{2}\pi)$$

$$M = \frac{1}{3} \rho_0 \pi^2 R^3$$

Aside: $\int_0^\pi \sin^2\theta\, d\theta = \frac{\pi}{2}$



1.10 Examples and Important Integrals

Example #3. The density of a sphere is uniform a nd given by $\rho = \rho_0$. Calculate the total mass.

Given: sphere $\rho = \rho_0$ (uniform density)

Find: M, total mass

Approach: → in order to find M, we need dm

$$dm = \rho\, dv = \underline{\underline{\rho_0}}\, dv \leftarrow dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

→ solve for M using $\int dm$

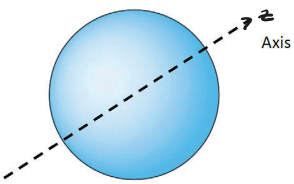
$$M = \int dm = \int \rho_0\, dv = \rho_0 \underbrace{\int dv}$$

↳ this is the volume of the sphere $= \frac{4}{3}\pi R^3$

$$M = \rho_0 \left(\frac{4}{3}\pi R^3\right)$$

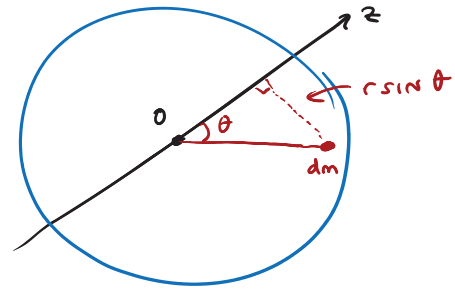
$$M = \frac{4}{3}\pi \rho_0 R^3$$

Example #4. Calculate the moment of inertia of a sphere of mass M and radius R rotating about the axis shown in the figure below. The sphere has uniform density.



Given: sphere, M, R, $\rho = \rho_0$ Find: I, total moment of inertia

From ECE 105, $dI = \text{"}mr^2\text{"} = dm\, r^2$, then integrate to get $I = \int dI$



Approach: → find an expression for dI

$dI = dm\, r^2$ → r is the distance dm is away from the axis of rotation
"r" $= r\sin\theta$

$$dI = dm(r\sin\theta)^2$$

→ let's solve for I using $I = \int dI$

$$I = \int dI = \int (r\sin\theta)^2\, dm$$

↳ find dm: $dm = \rho\, dv$ but $\rho = \rho_0$ (uniform density)

$dm = \rho_0\, dv \leftarrow dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$

→ because of uniform density:

$$\rho_0 = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

$$I = \iiint r^2 \sin^2\theta \left(\frac{3M}{4\pi R^3}\right) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$I = \frac{3M}{4\pi R^3} \iiint r^4 \sin^3\theta\, dr\, d\theta\, d\phi$$

$$I = \frac{3M}{4\pi R^3} \int_0^R r^4\, dr \int_0^\pi \sin^3\theta\, d\theta \int_0^{2\pi} d\phi$$

Bounds:
sphere: $r: 0 \to R$
$\theta: 0 \to \pi$
$\phi: 0 \to 2\pi$

$$I = \frac{\cancel{3}M}{\cancel{4}\pi R^3} \left[\frac{r^5}{5}\right]_0^R \left[\frac{\cancel{4}}{\cancel{3}}\right] \left[\phi\right]_0^{2\pi}$$

$$I = \frac{M}{\cancel{\pi} \cancel{R^3}} \left[\frac{R^{\cancel{5}\,2}}{5}\right] \left[2\cancel{\pi}\right]$$

$$\underline{\underline{I = \frac{2}{5} MR^2}}$$

Aside: $\int_0^\pi \sin^3\theta\, d\theta = \int_0^\pi \sin\theta \cdot \sin^2\theta\, d\theta$

$$= \int_0^\pi \sin\theta(1 - \cos^2\theta)\, d\theta$$
$(1-u^2)$ $-du$

Let $u = \cos\theta \to du = -\sin\theta\, d\theta$
$-du = \sin\theta\, d\theta$

$u(0) = \cos 0 = 1$
$u(\pi) = \cos\pi = -1$

$$= \int_1^{-1} (u^2 - 1)\, du$$

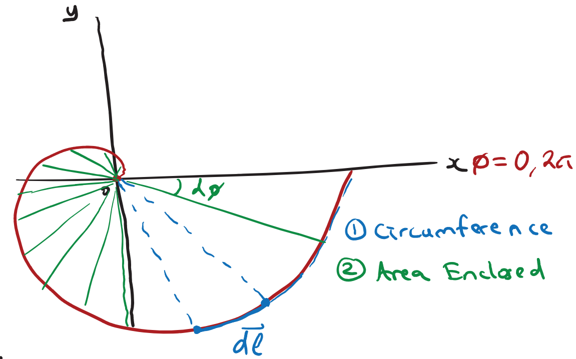
$$= \int_1^{-1} u^2\, du - \int_1^{-1} du$$

$$= \left[\frac{u^3}{3}\right]_1^{-1} - \left[u\right]_1^{-1} = \left(\frac{-1}{3} - \frac{1}{3}\right) - (-1-1)$$

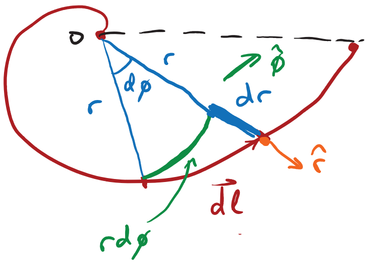
$$= \frac{-2}{3} + 2 = \underline{\underline{\frac{4}{3}}}$$

Example #5. An Archimedes' spiral is the trajectory of a point moving uniformly on a straight line of a plane while the line turns itself uniformly around one of its points. An example is the rotation of the stylus on a good old vinyl disk. The curve of one type of an Archimedes' spiral is given by the mathematical function $r = e^{\phi}$, where r is the radius from the origin and ϕ is the angle the line makes with the x-axis. Let us calculate the following parameters for one turn (that is ϕ changes from 0 to 2π):

1. **The circumference of the curve**
2. **The area enclosed within the curve**



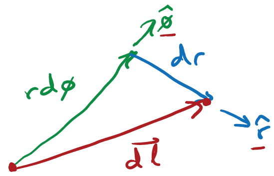
① CIRCUMFERENCE



→ How do we define $d\vec{l}$?

We must relate it to $dr, d\phi$

From the diagram, we see that:



$$d\vec{l} = (r\,d\phi)\,\hat{\phi} + (dr)\,\hat{r}$$

$$d\vec{l} \cdot d\vec{l} = \left[(r\,d\phi)\,\hat{\phi} + (dr)\,\hat{r}\right] \cdot \left[(r\,d\phi)\,\hat{\phi} + (dr)\,\hat{r}\right]$$

$$(dl)^2 = (r\,d\phi)^2 \underbrace{\hat{\phi}\cdot\hat{\phi}}_{(1)} + 2r\,d\phi\,dr \underbrace{\hat{\phi}\cdot\hat{r}}_{(0)} + (dr)^2 \underbrace{\hat{r}\cdot\hat{r}}_{(1)}$$

$$(dl)^2 = (r\,d\phi)^2 + (dr)^2$$

$$dl = \sqrt{(r\,d\phi)^2 + (dr)^2} \quad \rightarrow \text{must take out } d\phi$$

$$dl = \sqrt{(d\phi)^2\left[(r)^2 + \left(\frac{dr}{d\phi}\right)^2\right]}$$

$$dl = d\phi\sqrt{r^2 + \left(\frac{dr}{d\phi}\right)^2} \quad \rightarrow r = e^{\phi} \quad \frac{dr}{d\phi} = e^{\phi}$$

$$dl = d\phi\sqrt{e^{2\phi} + e^{2\phi}}$$

$$dl = d\phi\sqrt{2e^{2\phi}}$$

$$dl = \sqrt{2}\,e^{\phi}\,d\phi$$

→ find circumference C by integrating dl for $\phi: 0 \rightarrow 2\pi$

$$C = \int dl = \int_0^{2\pi} \sqrt{2}\,e^{\phi}\,d\phi$$

$$C = \sqrt{2}\left[e^{\phi}\right]_0^{2\pi}$$

$$C = \sqrt{2}\left[e^{2\pi} - 1\right]$$

② AREA ENCLOSED

→ need to find a differential area element, called dS

$$dS = r\,dr\,d\phi \qquad r = e^{\phi} \qquad r: 0 \rightarrow e^{\phi} \qquad \phi: 0 \rightarrow 2\pi$$

→ solve for A by integrating with respect to dS

$$A = \int dS = \iint r\,dr\,d\phi$$

$$A = \int_0^{2\pi}\int_0^{e^{\phi}} r\,dr\,d\phi$$

$$A = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{e^{\phi}} d\phi$$

$$A = \frac{1}{2}\int_0^{2\pi}\left[e^{2\phi}\right] d\phi$$

$$A = \frac{1}{2}\left[\frac{1}{2}e^{2\phi}\right]_0^{2\pi}$$

$$A = \frac{1}{4}\left[e^{4\pi} - 1\right]$$

$$A = \frac{1}{2}\int_0^{2\pi}\left[e^{2\phi}\right]d\phi$$

1.10 Examples and Important Integrals

Some Important Integrals

① $\int_a^b \sin^2\theta \, d\theta$ or $\int_a^b \cos^2\theta \, d\theta$

If $b-a = m\pi$ (where m is an integer)

↳ integral is always $\boxed{\frac{m\pi}{2}}$

② $\int_a^b \sin\theta \cos\theta \, d\theta$

If $b-a = m\pi$ (where m is an integer)

↳ integral is always $\boxed{0}$