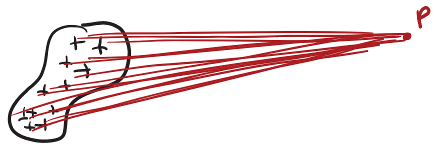
4.1 Charge Density

Before, we looked at individual charges and we now want to look at continuous charge distributions.

↳ what if there were millions of particles with a charge on or in an object

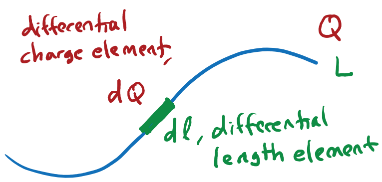


We must introduce CHARGE DENSITY so that we can calculate the total charge of an object or its electric field contributions

What types of charge densities are there?

LINEAR CHARGE DENSITY

$[\rho_\ell]$



$dQ = \rho_\ell \, d\ell$

$$\boxed{Q = \int_L \rho_L \, d\ell}$$

If there is uniform charge distribution, where $\rho_\ell = \rho_0$, then

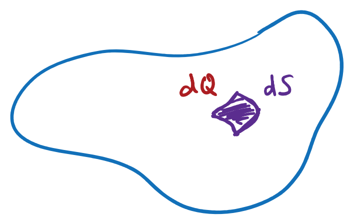
$$Q = \rho_0 \underbrace{\int_L d\ell}_{L,\ \text{total length}}$$

$Q = \rho_0 L$

↳ $\rho_0 = \dfrac{Q}{L}$

SURFACE CHARGE DENSITY

$[\rho_s]$



$dQ = \rho_s \, dS$

$$\boxed{Q = \int_S \rho_s \, dS}$$

If there is uniform charge distribution, where $\rho_s = \rho_0$, then

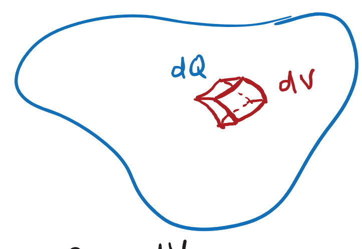
$$Q = \rho_0 \underbrace{\int_S dS}_{S,\ \text{total surface area}}$$

$Q = \rho_0 S$

↳ $\rho_0 = \dfrac{Q}{S}$

VOLUME CHARGE DENSITY

$[\rho_v]$



$dQ = \rho_v \, dV$

$$\boxed{Q = \int_V \rho_v \, dV}$$

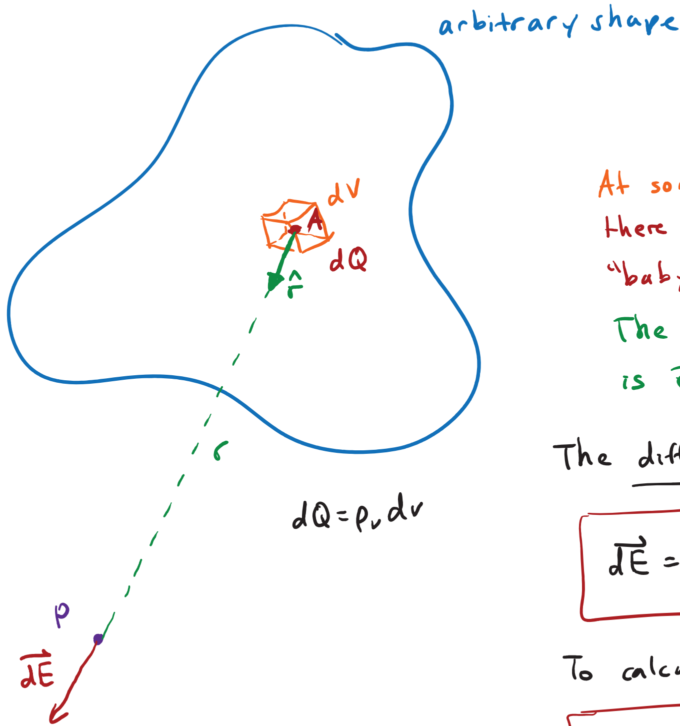
If there is uniform charge distribution, where $\rho_v = \rho_0$, then

$$Q = \rho_0 \underbrace{\int_V dV}_{V,\ \text{total volume}}$$

$Q = \rho_0 V$

↳ $\rho_0 = \dfrac{Q}{V}$

4.2 Calculation of Electric Field from Distributed Charges



We are trying to find the total electric field $\vec{E}$ at a point P away from this volume.

At some arbitrary point A with dV, there is a "baby charge" dQ that creates a "baby E-field" $\vec{dE}$ at point P

The distance between the dV and dE elements is $\vec{r}$ with $\hat{r}$ pointing towards point P

The <u>differential electric field element</u> is:

$$\boxed{\vec{dE} = \frac{k\,dQ}{r^2}\hat{r} = \frac{k\rho_v dV}{r^2}\hat{r}}$$

To calculate the <u>total electric field</u>:

$$\boxed{\vec{E} = \int_V \vec{dE} = \int_V \frac{k\rho_v dV}{r^2}\hat{r}}$$

→ be careful because $\vec{E}$ is a vector quantity so we must consider

dE_x, dE_y, dE_z

$\downarrow \quad \downarrow \quad \downarrow$

$\vec{E}_x \quad \vec{E}_y \quad \vec{E}_z$

$\hat{x} \quad \hat{y} \quad \hat{z}$

$$\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$$

4.3 Steps to Calculate Electric Field

Some steps to consider when calculating electric fields:

#0: Find Charge Density $\quad d\vec{E} = \frac{k\,dQ}{r^2}$ ← $dQ = \rho\, d?$

↳ given or ρ dependent on function or $\rho = \frac{Q}{L \text{ or } S \text{ or } V}$

#1: Choose Coordinate System

#2: Represent Arbitrary Point A and $dl/ds/dv$ in Terms of Coordinate System $\Big\}\, dQ$

#3: Write Expression for $d\vec{E}$

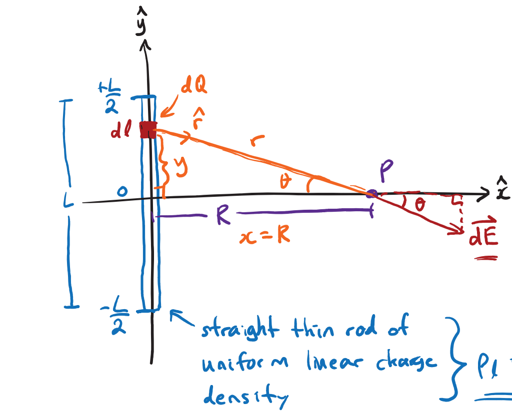
#4: Symmetry → do I need to integrate w.r.t the x-, y-, and z-directions, or do some of them cancel out?

#5: Find $\vec{E}$ by Integrating $d\vec{E}$ (dE_x, dE_y, and/or dE_z)

4.4.1 Examples

Example #1. Charge Q is uniformly distributed over length L. Calculate the electric field at a point P, a distance R away on an axis going through the middle of the line.

Draw and Label Diagram



straight thin rod of uniform linear charge density $\} \rho_l = \frac{Q}{L}$

Comments:

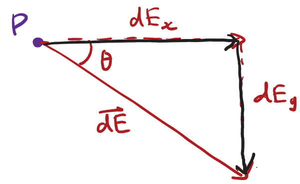
→ we divide L into baby lengths dl

Find dQ and $d\vec{E}$

$\rightarrow dQ = \rho_l \, dl$ but $\rho_l = \frac{Q}{L}$ and $dl = dy$

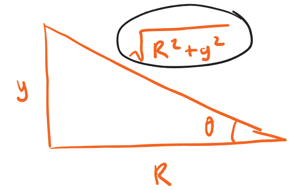
$$dQ = \frac{Q}{L} dy$$

→ component analysis for $d\vec{E}$



$$d\vec{E} = \frac{k\,dQ}{r^2}\hat{r}$$

$$\hat{r} = \frac{\langle x, -y\rangle}{\sqrt{x^2+y^2}}$$



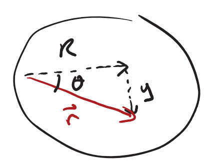
$$r_x = \frac{+x}{\sqrt{x^2+y^2}}\hat{x} \qquad r_y = \frac{-y}{\sqrt{x^2+y^2}}\hat{y}$$

$$r_x = \frac{R\ \hat{x}}{\sqrt{R^2+y^2}} \qquad r_y = \frac{-y\ \hat{y}}{\sqrt{R^2+y^2}}$$

$$d\vec{E} = \frac{k\,dQ}{r^2}\hat{r} = \frac{k\,dQ}{r^2}\left\langle \frac{R}{\sqrt{R^2+y^2}}, \frac{-y}{\sqrt{R^2+y^2}}\right\rangle$$

$$d\vec{E} = \frac{k\,dQ}{\left(\sqrt{R^2+y^2}\right)^3}\langle R, -y\rangle$$

$$d\vec{E} = \frac{k\,dQ}{(R^2+y^2)^{3/2}}\langle R, -y\rangle$$



$$\hookrightarrow dE_x = \frac{k\,dQ\,R}{(R^2+y^2)^{3/2}}\hat{x}$$

By symmetry, $\int dE_y = 0$,

so we can say that

$$\vec{E} = \vec{E}_x$$

Solve for $\vec{E}$

$$\vec{E} = \vec{E}_x = \int dE_x$$

$$\vec{E} = \int_{-\frac{L}{2}}^{L/2} \frac{kR\left(\frac{Q}{L}dy\right)}{(R^2+y^2)^{3/2}}\ \hat{x}$$

$$\vec{E} = \frac{kRQ}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dy}{(R^2+y^2)^{3/2}}\ \hat{x}$$

$$\vec{E} = \frac{kRQ}{L}\int_{\theta_1}^{\theta_2} \frac{R\sec^2\theta\, d\theta}{R^3\sec^3\theta}\ \hat{x}$$

$$\vec{E} = \frac{kR^2Q}{LR^3}\int_{\theta_1}^{\theta_2} \frac{1}{\sec\theta}\, d\theta\ \hat{x}$$

Aside: $\tan\theta = \frac{y}{R} \Rightarrow y = R\tan\theta$

$$dy = R\sec^2\theta\, d\theta$$

$$\sec\theta = \frac{HYP}{ADJ} = \frac{\sqrt{R^2+y^2}}{R}$$

$$\sqrt{R^2+y^2} = R\sec\theta$$

$$\left(\sqrt{R^2+y^2}\right)^3 = R$$

$$-\frac{L}{2} \rightarrow \theta_1 \qquad \frac{L}{2} \rightarrow \theta_2$$

$$E = \frac{\cdots}{LR^3}\int_{\theta_1} \frac{\cdots}{\sec\theta}\, d\theta\ \hat{x}$$

$$\vec{E} = \frac{kQ}{RL}\int_{\theta_1}^{\theta_2} \cos\theta\, d\theta\ \hat{x}$$

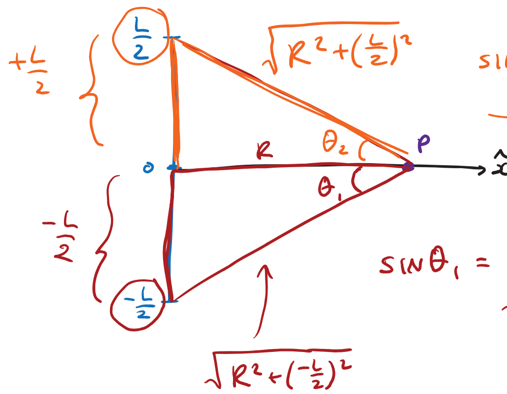
$$\vec{E} = \frac{kQ}{RL}\Big[\sin\theta\Big]_{\theta_1}^{\theta_2}\ \hat{x}$$

$$\vec{E} = \frac{kQ}{RL}\left[\sin\theta_2 - \sin\theta_1\right]\hat{x}$$

$$\vec{E} = \frac{kQ}{RL}\left[\frac{L/2 - (-L/2)}{\sqrt{R^2 + L^2/4}}\right]\hat{x}$$

$$\boxed{\vec{E} = \frac{kQ}{R\sqrt{R^2 + L^2/4}}\ \hat{x}}$$

$-\frac{L}{2} \rightarrow \theta_1$ $\qquad$ $\frac{L}{2} \rightarrow \theta_2$



$$\sin\theta_2 = \frac{L/2}{\sqrt{R^2 + \frac{L^2}{4}}}$$

$$\sin\theta_1 = \frac{-L/2}{\sqrt{R^2+(-\frac{L}{2})^2}} = \frac{-L/2}{\sqrt{R^2 + \frac{L^2}{4}}}$$

A few comments:

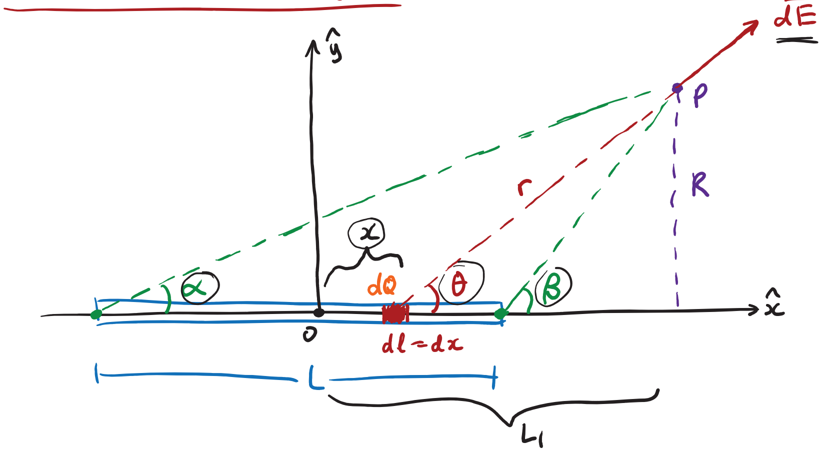
→ if $R \gg L$

$$\vec{E} \rightarrow \frac{2k\rho_\ell}{R}\,\hat{x} = \frac{\rho_\ell}{2\pi\varepsilon_0 R}\,\hat{x}$$

4.4.2 Examples

Example #2. A uniformly distributed charge sits on a horizontal line of length L with linear charge density $\rho_l = \rho_0$. Calculate the electric field at point P, where P is a perpendicular distance R away from the horizontal axis such that the line joining P with one end of the line makes an angle α and with the other end of the line makes an angle β.

Draw and Label Diagram



→ no symmetry, so let's just choose middle of L

→ dl is a small length element on the x-axis, so $\underline{dl = dx}$

→ this dx has a charge dQ:

$dQ = \rho_l \, dl$ but $\rho_l = \rho_0$, $dl = dx$

$$\underline{dQ = \rho_0 \, dx}$$

→ we must find $\vec{dE}$

$$\vec{dE} = \frac{k\,dQ}{r^2}\hat{r} = \frac{k\rho_0\,dx}{r^2}\hat{r}$$

Solve: must solve $\vec{E}_x$ and $\vec{E}_y$ separately:

x-direction

$dE_x = dE\cos\theta$

$dE_x = \frac{k\rho_0 \cos\theta \, dx}{r^2}$

→ sub $r = \frac{R}{\sin\theta}$

$dE_x = \frac{k\rho_0 \cos\theta\, dx}{\frac{R^2}{\sin^2\theta}}$

$dE_x = \frac{k\rho_0 \cos\theta\, dx}{R^2\csc^2\theta}$ ←

$dE_x = \frac{k\rho_0\cos\theta \cdot \cancel{R}\cancel{\csc^2\theta}\, d\theta}{R^{\cancel{2}}\cancel{\csc^2\theta}}$

$dE_x = \frac{k\rho_0}{R}\cos\theta\, d\theta$

$\vec{E}_x = \frac{k\rho_0}{R}\int_\alpha^\beta \cos\theta\, d\theta$

$\vec{E}_x = \frac{k\rho_0}{R}\left[\sin\theta\right]_\alpha^\beta$

$$\underline{\vec{E}_x = \frac{k\rho_0}{R}\left[\sin\beta - \sin\alpha\right]}$$

y-direction

$dE_y = dE\sin\theta$

→ sub $r = \frac{R}{\sin\theta}$

$dx = R\csc^2\theta\, d\theta$

$dE_y = \frac{k\rho_0 \sin\theta\, dx}{r^2}$

$dE_y = \frac{k\rho_0\sin\theta \cdot \cancel{R}\cancel{\csc^2\theta}\, d\theta}{R^{\cancel{2}}\cancel{\csc^2\theta}}$

$dE_y = \frac{k\rho_0}{R}\sin\theta\, d\theta$

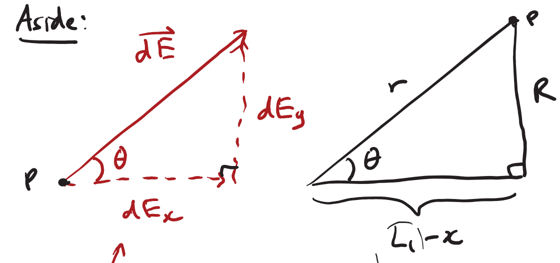
$\vec{E}_y = \frac{k\rho_0}{R}\int_\alpha^\beta \sin\theta\, d\theta$

$\vec{E}_y = \frac{k\rho_0}{R}\left[-\cos\theta\right]_\alpha^\beta$

$\vec{E}_y = \frac{k\rho_0}{R}\left[-\cos\beta + \cos\alpha\right]$

$$\underline{\vec{E}_y = \frac{k\rho_0}{R}\left[\cos\alpha - \cos\beta\right]}$$

Aside:



$dE_x = dE\cos\theta$

$dE_y = dE\sin\theta$

$\sin\theta = \frac{R}{r}$

$r = \frac{R}{\sin\theta}$

$\tan\theta = \frac{R}{L_1 - x}$

$L_1 - x = \frac{R}{\tan\theta} = R\cot\theta$

$\frac{d}{d\theta}(L_1 - x) = \frac{d}{d\theta}(R\cot\theta)$

$0 - \frac{dx}{d\theta} = R(-\csc^2\theta)$

$$\underline{dx = R\csc^2\theta\, d\theta}$$

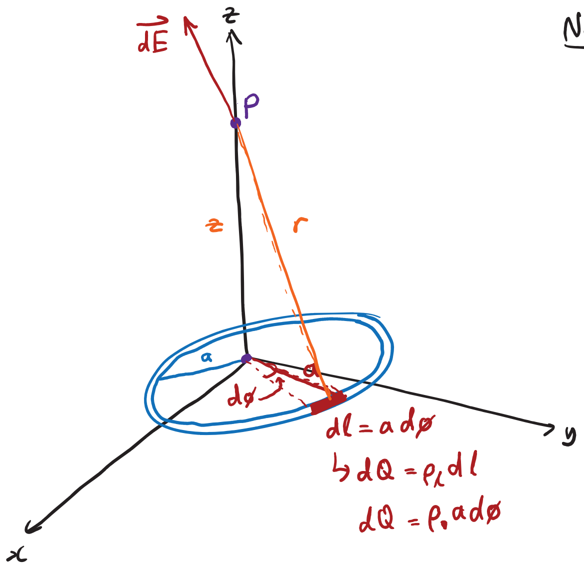
$$\therefore \vec{E} = \frac{k\rho_0}{R}\left[(\sin\beta - \sin\alpha)\,\hat{x} + (\cos\alpha - \cos\beta)\,\hat{y}\right]$$

4.4.3 Examples

Example #3. Charge is uniformly distributed over a ring of radius a. Calculate the E-field at a height z above the center of the ring.

↳ line in a circle

Draw and Label Diagram



$dl = a\,d\phi$

↳ $dQ = \rho_l\, dl$

$dQ = \rho_0 a\, d\phi$

Notes: → due to symmetry, all $\vec{dE}$'s perpendicular ($\perp$) to $\hat{z}$ will sum to 0

→ $\vec{dE}_x = \vec{dE}_y = \vec{0}$

→ so we just need to look at $\vec{dE}_z$

↳ integrate ϕ from $0 \to 2\pi$

→ find $\vec{dE}_z$

$$\hat{r} = \frac{\langle x, y, z\rangle}{|r|}$$

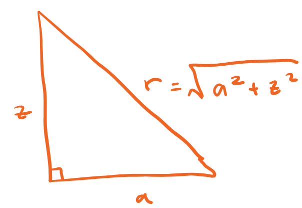
$$\vec{dE} = \frac{k\,dQ}{r^2}\hat{r}$$

$$\hat{z} \to \frac{z}{r}$$

$$\vec{dE}_z = \frac{k\,dQ}{r^2}\cdot\frac{z}{r}\,\hat{z}$$

$$\vec{dE}_z = \frac{k z \rho_0 a\, d\phi}{r^3}\,\hat{z}$$

$$\vec{dE}_z = \frac{k\rho_0 a z\, d\phi}{(a^2+z^2)^{3/2}}\,\hat{z}$$



Solve for $\vec{E}$

$$\vec{E} = \vec{E}_z = \int_0^{2\pi} \frac{k\rho_0 a z\, d\phi}{(a^2+z^2)^{3/2}}\,\hat{z}$$

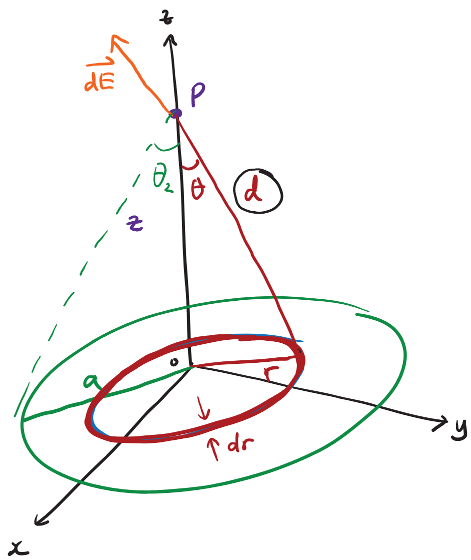
$$\vec{E} = \frac{k\rho_0 a z}{(a^2+z^2)^{3/2}} \int_0^{2\pi} d\phi\ \hat{z}$$

$$\boxed{\vec{E} = \frac{2\pi k \rho_0 a z}{(a^2+z^2)^{3/2}}\,\hat{z}}$$

4.4.4 Examples

Example #4. Charge is distributed over the surface of a disk of radius a. Calculate the electric field at a height z above the center of the disk.

Draw and Label Diagram



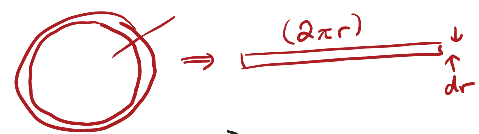
→ by symmetry, the E-field due to the ring is only along the z-axis

→ each mini-ring has a surface charge density of:

$\rho_s = \rho_o$ with radius r

$dQ = \rho_s \, dS$ and $dS = (2\pi r)dr$

$dQ = 2\pi \rho_o r \, dr$



→ we know $\vec{dE} = \vec{dE_z}$

$$\vec{dE_z} = \frac{k\,dQ}{d^2} \cdot \frac{z}{d} \hat{z} \qquad \hat{r} = \frac{\langle x, y, z \rangle}{d}$$

$\hookrightarrow d = \sqrt{r^2 + z^2}$

$$\vec{dE_z} = \frac{kz(2\pi\rho_o r\,dr)}{(r^2+z^2)^{3/2}} \hat{z}$$

Aside: $\tan\theta = \frac{r}{z}$ | $\sec\theta = \frac{HYP}{ADJ} = \frac{\sqrt{r^2+z^2}}{z}$

$r = z\tan\theta$ | $(r^2+z^2)^{3/2} = z^3\sec^3\theta$

$dr = z\sec^2\theta \, d\theta$

$0 \to \theta_1$ $\qquad a \to \theta_2$

$\cos\theta = 1$ $\qquad \cos\theta_2 = \frac{z}{\sqrt{a^2+z^2}}$

Solve for $\vec{E}$

$$\vec{E} = \vec{E_z} = 2\pi k z \rho_o \int_0^a \frac{r\,dr}{(r^2+z^2)^{3/2}} \hat{z}$$

$$\vec{E} = 2\pi k \cancel{z} \rho_o \int_{\theta_1}^{\theta_2} \frac{\cancel{z}\tan\theta \, \cancel{z}\cancel{\sec^2}\theta \, d\theta}{\cancel{z^3}\sec^{\cancel{3}}\theta} \hat{z}$$

$$\vec{E} = 2\pi k \rho_o \int_{\theta_1}^{\theta_2} \frac{\tan\theta}{\sec\theta} d\theta \ \hat{z}$$

$$\vec{E} = 2\pi k \rho_o \int_{\theta_1}^{\theta_2} \sin\theta \, d\theta \ \hat{z}$$

$$\vec{E} = 2\pi k \rho_o \left[-\cos\theta \right]_{\theta_1}^{\theta_2} \hat{z}$$

$$\vec{E} = 2\pi k \rho_o \left[\cos\theta_1 - \cos\theta_2 \right] \hat{z}$$

$$\boxed{\vec{E} = 2\pi k \rho_o \left[1 - \frac{z}{\sqrt{a^2+z^2}} \right] \hat{z}}$$

4.4.5 Examples

Example #5. Charge Q is uniformly distributed over the surface of a sphere. Calculate the E-field at points P_1 and P_2 where P_1 is inside the spherical shell and P_2 is outside the spherical shell.