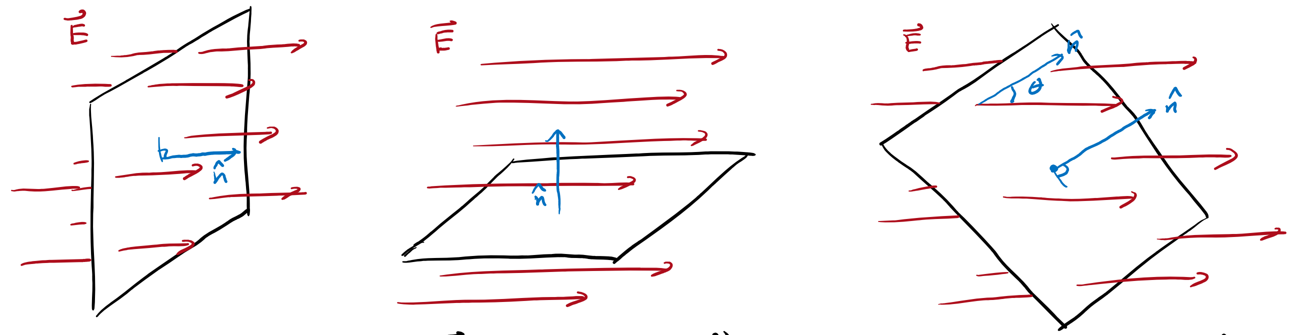
5.1 Electric Flux

Electric Flux (Φ) : a measure of how many electric field lines pass through a given area



$\vec{E} \parallel \hat{n}$ $(\theta = 0°)$ — maximum flux

$\vec{E} \perp \hat{n}$ $(\theta = 90°)$ — minimum flux

$\vec{E}$ is at angle θ to $\hat{n}$ — some flux

Therefore, the dot product of $\vec{E} \cdot \hat{n}$ will give us electric flux

$$\Phi_E = \vec{E} \cdot \hat{n} = |\vec{E}||\hat{n}| \cos\theta$$

but we want to give meaning to $\hat{n}$, so we use $d\vec{S}$ in the $\hat{n}$ direction instead

$$\boxed{d\Phi_E = \vec{E} \cdot d\vec{S} = |\vec{E}||d\vec{S}| \cos\theta}$$

DIFFERENTIAL ELECTRIC FLUX

$\Downarrow$ integrating gives us total flux

$$\boxed{\Phi_E = \int_S \vec{E} \cdot d\vec{S}}$$

We may want to use electric flux density (displacement vector) $\vec{D}$,

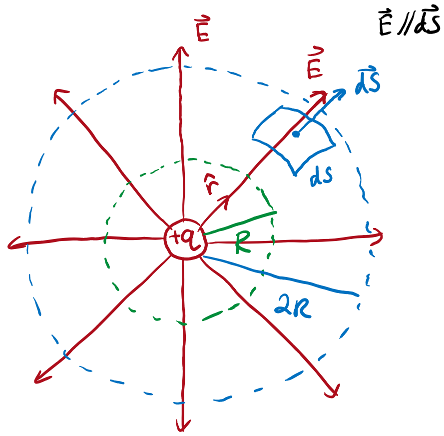
$$\vec{D} = \varepsilon_0 \vec{E}$$

$$\Phi_E = \int_S \vec{D} \cdot d\vec{S} = \int_S \varepsilon_0 \vec{E} \cdot d\vec{S}$$

5.2 Electric Flux over a Closed Surface

We can choose arbitrary closed surfaces of certain shapes to encompass our charged object → let's take a look at some examples!

① Closed Sphere Around a Point Charge



If we place a Gaussian sphere around the point charge, the $\vec{E}$-field decays as $1/r^2$

↳ there are less field lines for the $2R$ sphere than the R sphere

How can we find the total flux on the closed sphere?

→ electric field at $\vec{dS}$ is:

$$\vec{E} = \frac{kQ}{r^2}\hat{r} \quad \text{or} \quad \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \quad \text{where } k = \frac{1}{4\pi\varepsilon_0}$$

→ find $\vec{dS}$ for a spherical surface:

$$\vec{dS} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

→ putting this all together,

$\vec{D}$ comes in handy when we start discussing materials →

$$d\Phi_E = \vec{D}\cdot\vec{dS} = \varepsilon_0 \vec{E}\cdot\vec{dS}$$

$$d\Phi_E = \cancel{\varepsilon_0}\left(\frac{Q}{4\pi\cancel{\varepsilon_0}\cancel{r^2}}\right)\hat{r}\cdot\cancel{r^2}\sin\theta\, d\theta\, d\phi\,\hat{r}$$

$\theta : 0 \to \pi$

$\phi : 0 \to 2\pi$

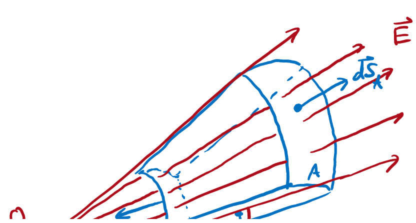
$$\int d\Phi_E = \int \frac{Q}{4\pi}\sin\theta\, d\theta\, d\phi$$

$$\Phi_E = \frac{Q}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\, d\theta$$

$$\Phi_E = \frac{Q}{4\pi}(2\pi)\left[-\cos\theta\right]_0^{\pi}$$

$$\Phi_E = Q$$ ← electric flux is just the enclosed charge within our surface!

② Charge Outside a Closed Surface

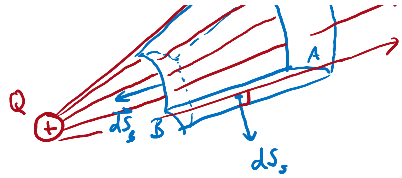


What if our charge was outside our closed surface?

↳ we note that $\vec{dS}$ is $//\ \hat{n}$ (surface normal vector)

The 4 faces/sides have $\vec{dS}_s \perp \vec{E}$, so $\Phi_s = 0$

↳ $d\Phi_E = |\vec{E}||\vec{dS}_s|\cos\theta = 0$



The 4 faces/sides have $dS_s \perp E$, so $\Phi_s = 0$

↳ $d\Phi_E = |\vec{E}||d\vec{S}_s| \cos\theta = 0$

The top face A has $d\vec{S}_A \parallel \vec{E}$ so:

$$d\Phi_A = |\vec{E}||d\vec{S}_A| \cos 0 = \underline{\underline{E\,dS_A}}$$

$$\Phi_A = \frac{Q}{4\pi\varepsilon_0} \int_S \sin\theta\, d\theta\, d\phi$$

The bottom face B has $d\vec{S}_B$ being $180°$ from $\vec{E}$, so:

$$d\Phi_B = |\vec{E}||d\vec{S}_B| \cos 180 = \underline{\underline{-E\,dS_B}}$$

$$\Phi_B = \frac{-Q}{4\pi\varepsilon_0} \int_S \sin\theta\, d\theta\, d\phi$$

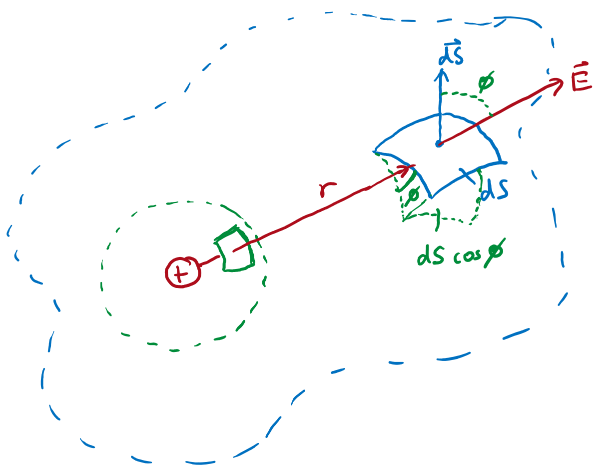
Total Flux: consider all 6 sides

$$\Phi_{TOT} = 4\Phi_s + \Phi_A + \Phi_B$$
$$= 0 + E\,dS_A - E\,dS_B$$
$$= E\,dS - E\,dS$$

$\underline{\underline{\Phi_{TOT} = 0}}$ Electric flux is $\underline{0}$ when the charge is outside the surface!

⚠ Electric flux is ... +ve when it exits a surface
-ve when it enters a surface

③ Charge Inside Arbitrary Closed Surface



→ If we look at a charge inside a nonspherical surface, we claim that Φ_E is the same as that of the inner sphere

→ Looking at this diagram, $d\vec{S} \parallel \vec{E}_\perp$ and this makes an angle ϕ to $\vec{E}$

↳ this means that the projection of dS onto the inner spherical surface is $dS\cos\phi$

→ But we note $dS \to 0$ and $\phi \to 0$ and as such they are the same!

∴ Electric flux is still just $Q_{enclosed}$ within any closed surface!

5.3 Gauss's Law

Gauss Law: For materials, we use: $\oint \vec{D}\cdot d\vec{S} = \oint \varepsilon_0 \vec{E}\cdot d\vec{S} = Q_{enclosed}$

In general, we have: $\boxed{\Phi_E = \oint \vec{E}\cdot d\vec{S} = \dfrac{Q_{enclosed}}{\varepsilon_0}}$ GAUSS LAW

Some things to remember:

1. Gauss law is always true and based on $1/r^2$ nature of the force law
2. Must apply Gauss law on a closed surface
3. $\Phi_E = 0$ within a closed surface, and the net flux is 0 outside the surface
4. $\Phi_E = Q_{enclosed}$
5. $d\vec{S}$ is the normal to the surface
6. If $\Phi_E > 0$, there is a net positive $Q_{enclosed}$ and more $\vec{E}$-field flows OUT

   If $\Phi_E < 0$, there is a net negative $Q_{enclosed}$ and more $\vec{E}$-field flows IN

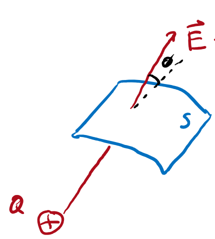
5.4 Calculating Electric Field using Gauss's Law

You may only apply Gauss's Law to calculate electric fields under certain conditions

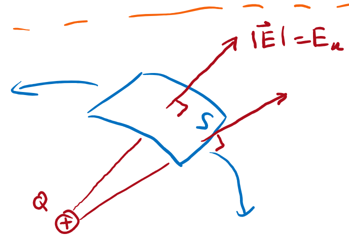
→ there must be a special symmetry called Gaussian symmetry where it is possible to apply Gauss's Law

→ for there to Gaussian symmetry, we need to analyze $\vec{E}$ around the surface:

$$\vec{E} = \underbrace{E_n \hat{n}}_{\text{normal component}} + \underbrace{E_t \hat{t}}_{\text{tangential component}}$$

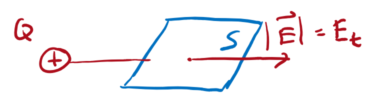


Can't apply Gauss's Law as there is a "mixture" of components



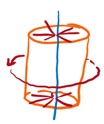
E_n must be constant around the entire surface and be parallel to $\vec{dS}$

→ We can apply Gauss's Law!



E_t is constant around the surface for areas where $\vec{E}$ is not normal to the surface

→ We can apply Gauss's Law!



When total $\vec{E}$ is either normal to the surface or only tangential where it is not normal, we can apply Gauss's Law!

→ Gauss's Law becomes:

$$\oint \vec{E} \cdot \vec{dS} = \oint E_n \, dS = E_n \oint dS = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$\boxed{E_n = \frac{Q_{enclosed}}{\varepsilon_0 \oint dS}}$$

When Can We Use Gauss's Law?

① When total $\vec{E}$ is derived from fully normal or fully tangential components with respect to each face or partial surface of your total shape

② Total charge should be the same / symmetrical from every point on the surface

③ We can pick a surface to be around where we want to find $\vec{E}$ so that it is symmetrical to all charges

What Surfaces can we Use?

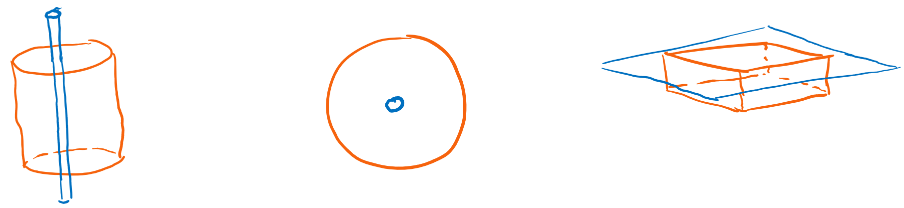
Gaussian Surface: In engineering, it is an arbitrary surface chosen to have Gaussian

Gaussian Surface: In engineering, it is an arbitrary surface chosen to have Gaussian symmetry so that we can apply Gauss's Law

→ Cylindrical Gaussian Surface for an infinite cylinder of symmetrical charge distribution

→ Spherical Gaussian Surface for a sphere with symmetrical charge distribution

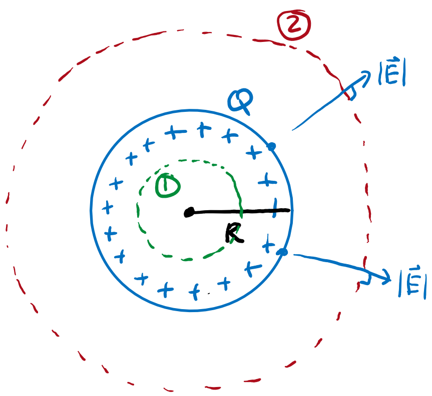
→ Pill Box Gaussian Surface for an infinite plane with symmetrical charge distribution



5.4.1 Examples

Example #1. Charge Q is distributed uniformly over the surface of a sphere with radius R. Calculate the electric field at a point P a distance r away from the center of the sphere for all values of r.

Diagram:



Region 1 $(r < R)$

→ $Q_{enclosed} = 0$

→ apply Gauss's Law

→ $\oint_{sphere} dS = 4\pi r^2$

$$\oint_{sphere} \vec{E} \cdot d\vec{S} = \frac{Q_{enclosed}}{\varepsilon_0}$$

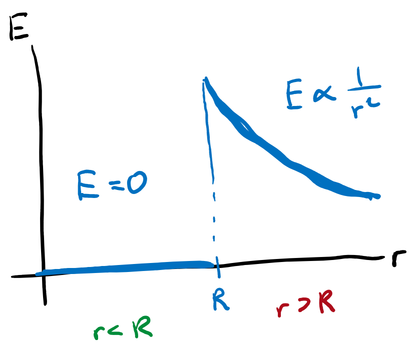
$$E \underbrace{\oint_{sphere} dS} = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$E(4\pi r^2) = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$E = \frac{Q_{enclosed}}{4\pi r^2 \varepsilon_0}$$

$$\boxed{E = 0}$$

Plot E vs. r



Region 2 $(r > R)$

→ $Q_{enclosed} = Q$

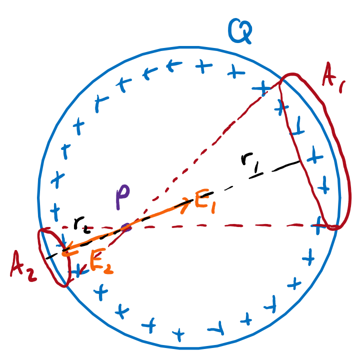
→ apply Gauss's Law

→ $\oint_{sphere} dS = 4\pi r^2$

$$E = \frac{Q_{enclosed}}{4\pi r^2 \varepsilon_0}$$

$$\boxed{\vec{E} = \frac{Q}{4\pi r^2 \varepsilon_0}\hat{r}}$$

Why is E = 0 Inside the Sphere?



$|\vec{E}_1| = |\vec{E}_2|$ due to:

→ they are proportional to $\frac{1}{r^2}$

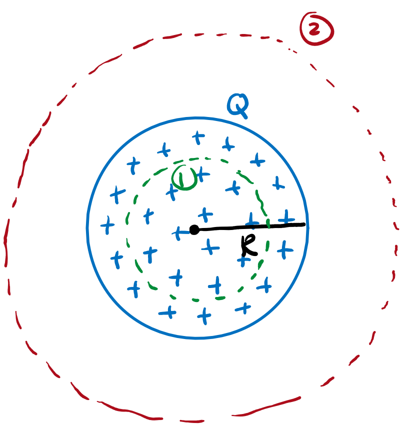
→ but $A_1 > A_2$ by the same amount that $\frac{1}{r_1^2} < \frac{1}{r_2^2}$

Only works if there is a uniform charge distribution!

5.4.2 Examples

Example #2. What if charge Q is uniformly distributed over the volume of a sphere of radius R? Repeat the analysis from the first question.

Diagram



Region 1 $(r<R)$

→ we must introduce charge density ρ_v to find out what portion of Q is enveloped by the chosen Gaussian sphere or radius r

$$\rho_v = \frac{Q}{V} \quad \text{and} \quad V = \frac{4}{3}\pi R^3$$

$$\rho_v = \frac{Q}{\frac{4}{3}\pi r^3} \Rightarrow Q = \frac{4}{3}\rho_v \pi r^3$$

$Q_{enclosed} = \frac{4}{3}\rho_v \pi r^3$, apply Gauss's Law;

$$\oint \vec{E}\cdot d\vec{S} = \frac{Q_{enclosed}}{\varepsilon_o}$$

$$E\oint_{sphere} ds = \frac{\frac{4}{3}\rho_v \pi r^3}{\varepsilon_o} \quad \text{so} \quad \oint_{sphere} ds = 4\pi r^2$$

$$E(\cancel{4}\cancel{\pi}\cancel{r^2}) = \frac{\cancel{\frac{4}{3}}\rho_v \cancel{\pi} r^{\cancel{3}}}{\varepsilon_o}$$

$$\boxed{\vec{E} = \frac{\rho_v r}{3\varepsilon_o}\hat{r}}$$

If $\rho_v = \rho_o r$, we still get Gaussian symmetry

If $\rho_r = \rho_o \sin\theta$, we no longer have Gaussian symmetry

Region 2 $(r>R)$

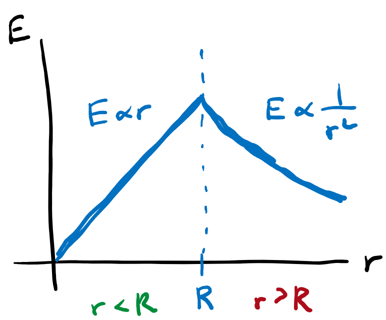
$Q_{enclosed} = Q$

$$\oint_{sphere} \vec{E}\cdot d\vec{S} = \frac{Q_{enclosed}}{\varepsilon_o}$$

$$E\underbrace{\oint_{sphere} ds} = \frac{Q}{\varepsilon_o}$$

$$\boxed{\vec{E} = \frac{Q}{4\pi\varepsilon_o r^2}\hat{r}}$$

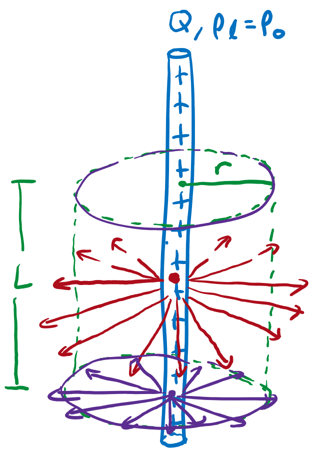
Plot E vs. r



5.4.3 Examples

Example #3. Charge is distributed over an infinite line with linear charge density $\rho_l = \rho_o$. Calculate the electric field at point P, which is at a distance r away from the line.

Diagram



→ We chose a cylindrical Gaussian surface of length L and radius r around the line charge

→ $\vec{E}$-field radiates out of the line charge and $\vec{E} \parallel \vec{ds}$, so E is constant on the curved surface (fully normal)

→ $\vec{E}$-field is fully tangential to the surface at the top and bottom caps of cylinder

→ We can apply Gauss's Law!

$$\underbrace{\oint \vec{E} \cdot ds}_{\text{total flux}} = \frac{Q_{enclosed}}{\varepsilon_o} \quad \text{where} \quad Q_{enclosed} = \rho_l (L) = \rho_o L$$

$$\hookrightarrow \oint \vec{E} \cdot \vec{ds} = \underbrace{\int_{top} \vec{E}_t \cdot \vec{ds}}_{0} + \underbrace{\int_{bottom} \vec{E}_b \cdot \vec{ds}}_{0} + \underbrace{\int_{curved} \vec{E}_c \cdot \vec{ds}}_{\text{solve}}$$

$$\oint \vec{E} \cdot \vec{ds} = \int \vec{E}_c \cdot \vec{ds} = \frac{Q_{enclosed}}{\varepsilon_o}$$

$$E_c \int ds = \frac{\rho_o L}{\varepsilon_o} \quad \rightarrow \quad \int ds = (2\pi r) L$$

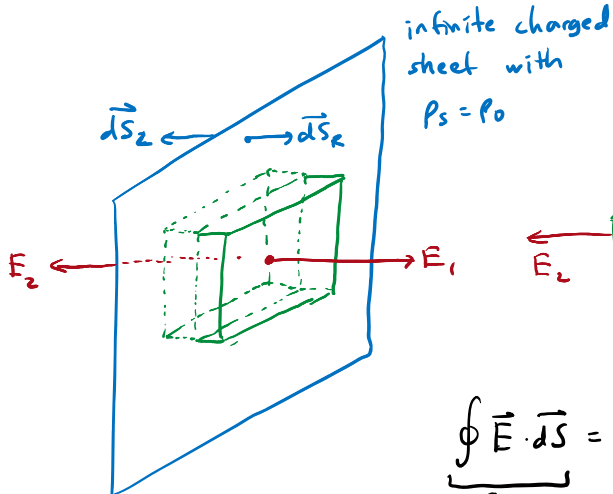
$$E_c (2\pi r L) = \frac{\rho_o L}{\varepsilon_o}$$

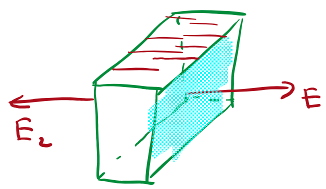
$$E_c = \frac{\rho_o \cancel{L}}{2\pi r \cancel{L} \varepsilon_o} \Rightarrow \boxed{\vec{E}_c = \frac{\rho_o}{2\pi r \varepsilon_o} \hat{r}}$$

5.4.4 Examples

Example #4. Charge is uniformly distributed over an infinite sheet with charge density given by $\rho_s = \rho_0$. Calculate the electric field at a height h above the sheet.

Diagram





→ We have planar symmetry and can choose a Gaussian surface that's rectangular

→ Looking at the shape, we have

↳ $|\vec{E}| \perp \vec{dS}$ at edges so $\Phi = 0$

↳ $|\vec{E}| \parallel \vec{dS}$ at large faces (left + right faces) of the rectangular box so $\Phi \neq 0$

$$\underbrace{\oint \vec{E}\cdot\vec{dS}}_{\text{total flux}} = \frac{Q_{enclosed}}{\varepsilon_0} \quad \text{with} \quad Q_{enclosed} = \rho_s A = \rho_0 A$$

$$\oint \vec{E}\cdot\vec{dS} = \underbrace{\int_L \vec{E}_L\cdot\vec{dS}_L + \int_R \vec{E}_R\cdot\vec{dS}_R}_{} + 0 = \frac{Q_{enclosed}}{\varepsilon_0} = \frac{\rho_0 A}{\varepsilon_0}$$

$$E_2\int_L dS_L + E_1\int_R dS_R = \frac{\rho_0 A}{\varepsilon_0}$$

→ $E_1 = E_2 = E$

→ $\int_L dS_L = \int_R dS_R = A$

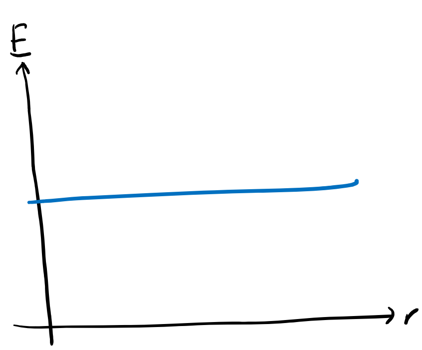
$$EA + EA = \frac{\rho_0 A}{\varepsilon_0}$$

$$2E\cancel{A} = \frac{\rho_0 \cancel{A}}{\varepsilon_0}$$

$$\boxed{\vec{E} = \frac{\rho_0}{2\varepsilon_0}\hat{r}}$$

Height or distance away from the sheet doesn't matter at all!

Plot E vs. r



Comparing All Options

Point Charge → $E \propto 1/r^2$

Infinite Line → $E \propto 1/r$

Infinite Sheet → E is the same

5.5 Other Cases of Gauss Law Applied to Electric Fields

Case I: Superposition

→ break up a problem into individually symmetric problems

↳ apply Gauss Law to each

→ then add up all answers vectorially to get the total electric field

Case II: Region where $E = 0$

→ If we know a region where $E = 0$, then we place a Gaussian surface in the region

→ this means $E = 0$, $\Phi = 0$

→ useful when asymmetry exists (+ve/−ve charges present)

Example #1: Two infinite charged sheets are separated by a distance d. The top sheet has a uniform surface charge density $+\rho_0$, while the bottom is $-\rho_0$. Calculate the electric field everywhere in space.

Diagram: