## 6.2 Mathematical Definition

Let's consider a charge $Q$ in space

→ We want to bring in a test charge $q$ from $\infty$ to point $P$ in the presence of $Q$

→ This means we must do work to overcome the force exerted on $q$ from $Q$'s $\vec{E}$-field

→ We define work to be:

$$W_I = \int_{\infty}^{P} \vec{F}_I \cdot \vec{dl} \qquad \text{where} \quad \begin{array}{l} \vec{F}_I = \text{force I exert to move } q \\ \vec{dl} = \text{differential path element} \end{array}$$

Assuming zero-acceleration, the force I exert must be equal and opposite to the force from $Q$'s $\vec{E}$-field, which we know is $q\vec{E}$

$$\vec{F}_I = -q\vec{E}$$

$$W_I = -\int_{\infty}^{P} q\vec{E} \cdot \vec{dl}$$

Electric Potential: the work done per positive test charge when $q \to 0$

$$V_P = \lim_{q \to 0} \frac{W_I}{q} = -\int_{\infty}^{P} \vec{E} \cdot \vec{dl}$$

$$\boxed{V_P = -\int_{\infty}^{P} \vec{E} \cdot \vec{dl} \qquad [V] \text{ units of volts}}$$

Important Points:

1. $V_P$ is always defined assuming a positive test charge
2. In general, the reference point is considered to be infinity
3. Since $\vec{E}$-fields are conservative fields, we get the same electric potential no matter what path we take to get to $P$ (we choose the easiest path → straight line or along $\vec{E}$-field line)

## 6.3 Electric Potential due to a Point Charge

To find the electric potential at point P due to a point charge, we must integrate

Why? Because $\vec{E}$-field varies along the path so the work we do will vary

The simplest path would be a straight line from $\infty$ to P

How can we approach this?

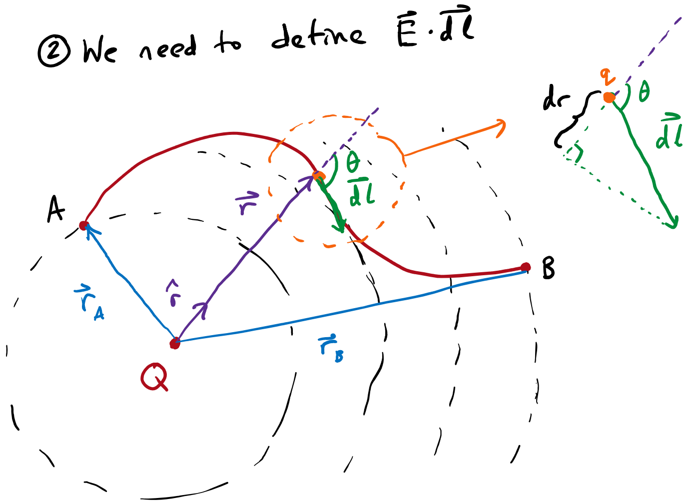

① We have already reached a point r' away from Q and we take a small step $\vec{dl}$ towards Q. It experiences an $\vec{E}$-field:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{(r')^2}\hat{r}'$$

$$dV = \vec{E}\cdot\vec{dl}$$

↳ what's this?

② We need to define $\vec{E}\cdot\vec{dl}$




Here, q moves along the red path from A to B in a general case, moving $\vec{dl}$ along the path and making angle $\theta$ with the electric field vector along $\hat{r}$.

So, we know
$$\vec{E}\cdot\vec{dl} = |\vec{E}||\vec{dl}|\cos\theta$$
$$\vec{E}\cdot\vec{dl} = |\vec{E}|\cancel{|\vec{dl}|}\left(\frac{dr}{\cancel{dl}}\right)$$
$$\vec{E}\cdot\vec{dl} = E\,dr \leftarrow \text{scalar!!}$$

③ We have a point $\vec{r}'$ away, so:
$$\vec{E}\cdot\vec{dl} = E\,dr'$$

Putting it all together: We have:

$$V_P = -\int_\infty^r \vec{E}\cdot\vec{dl} = -\int_\infty^r E\,dr = -\int_\infty^r \frac{Q}{4\pi\varepsilon_0 r'^2}dr' = \frac{Q}{4\pi\varepsilon_0 r}$$

$$\boxed{V_P = \frac{Q}{4\pi\varepsilon_0 r}}$$

ELECTRIC POTENTIAL DUE TO POINT CHARGE

↳ If $Q>0$, $V_P>0$ → repulsion means positive work

↳ If $Q>0$, $V_P > 0$ $\rightarrow$ repulsion means positive work

If $Q<0$, $V_P < 0$ $\rightarrow$ attraction means negative work

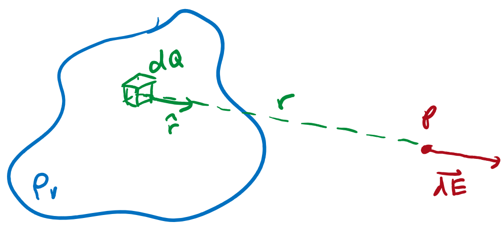
## 6.4 Electric Potential due to Distributed Charges

There are 2 main ways of calculating electric potential $V_P$:

**Method #1** → Find $\vec{E}$-field everywhere as a function of $r$ and solving $V_r = -\int_\infty^r E\,dr$

**Method #2** → Apply superposition and consider the charge distribution as a summation of individual charges $dQ$

How do we use this?




The differential potential $dV_P$ due to a point charge is:

$$dV_P = \frac{dQ}{4\pi\epsilon_0 r} = \frac{\rho_v \, dV}{4\pi\epsilon_0 r}$$

$$\boxed{V_P = \int \frac{\rho_v \, dV}{4\pi\epsilon_0 r}}$$

no need for symmetry!

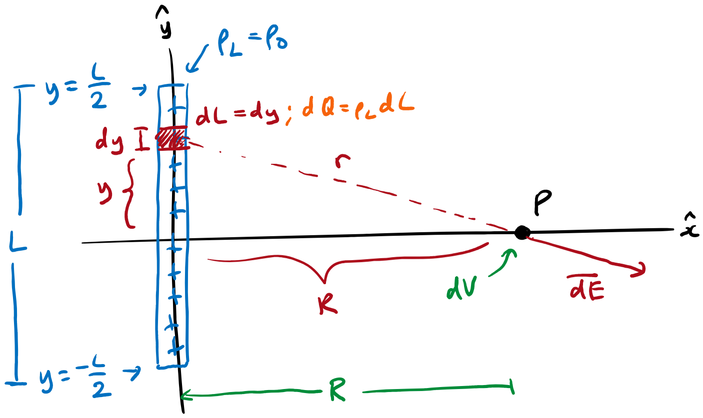
When to use which method?

- Gaussian Symmetry in Charge Distribution
  - YES → ① Find $\vec{E}$-field w/ Gauss Law ② Find $V_P = -\int_\infty^r E\,dr$
  - NO → Can Charge Distribution be broken into Individually Symmetric Cases?
    - YES → ① Find $\vec{E}$-field w/ Gauss Law ② Find $V_P = -\int_\infty^r E\,dr$
    - NO → ① Find $\vec{E}$-field w/ $\vec{E} = \int \frac{k\,dQ}{r^2}\hat{r}$ ② Find $V_P = \int \frac{k\,dQ}{r}$

## 6.4.1 Examples

**Example #1. Find the electric potential due to a charge Q uniformly distributed over a line L at a point on the axis bisecting the line.**

Diagram:




Pick Method:

→ we can't use Gauss Law for finite lengths

→ we must use superposition

Find dV:

$$dV = \frac{k\,dQ}{r} = \frac{k\rho_L dL}{r} = \frac{k\rho_L dy}{r}$$

→ must define r w.r.t. y:

$$r = \sqrt{R^2 + y^2}$$

$$dV = \frac{k\rho_L dy}{\sqrt{R^2+y^2}}$$

Find V:

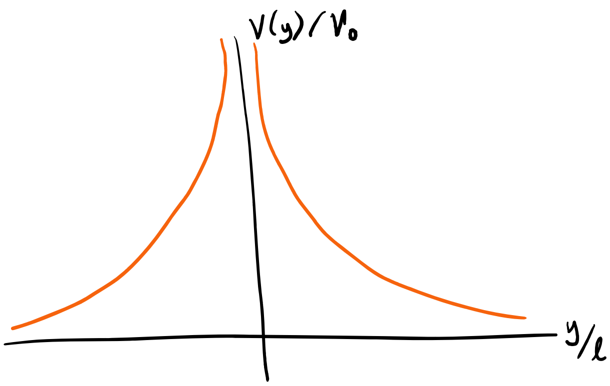
$$V = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{k\rho_L dy}{\sqrt{R^2+y^2}}$$

Aside:

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right)$$

$$V = k\rho_L \ln\left[\frac{L/2 + \sqrt{(L/2)^2 + R^2}}{-L/2 + \sqrt{(-L/2)^2 + R^2}}\right]$$

Plotting the Potential




Notes:

→ $V_0$ is $k\rho_L$ (a constant)

What happens for a +ve and -ve charge?

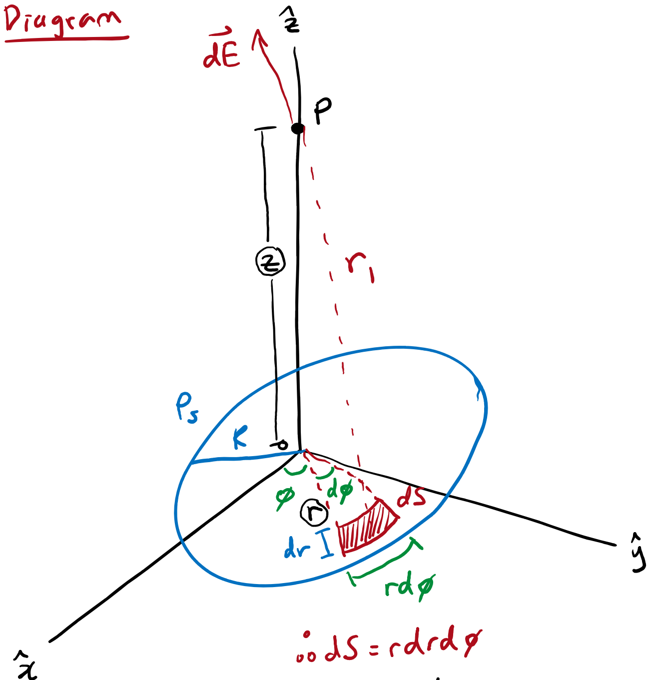
+ve: rolls down the hill and loses potential

-ve: climb the potential hill and rise to a higher potential

### 6.4.2 Examples

**Example #2. A charge Q is distributed over a disk of radius R, with uniform charge density. Calculate the electric potential at point P, a distance z above the center of the disk.**

Diagram




$\therefore dS = r\,dr\,d\phi$

$dQ = \rho_s\, dS$

Pick Method: → no Gaussian symmetry so we must use superposition

Find dV:

$$dV = \frac{k\,dQ}{r_1} = \frac{k\rho_s r\,dr\,d\phi}{r_1}$$

↳ dependent on r

$$r_1 = \sqrt{r^2+z^2}$$

$$dV = \frac{k\rho_s r\,dr\,d\phi}{\sqrt{r^2+z^2}} \qquad \phi : 0 \to 2\pi \qquad r : 0 \to R$$

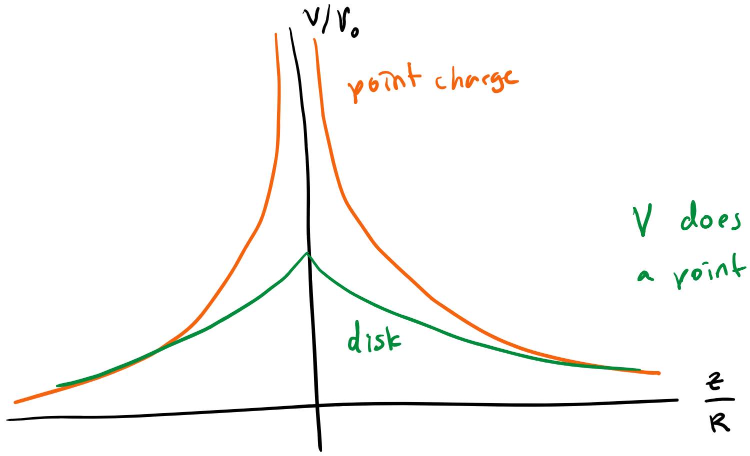
Find V:

$$V = k\rho_s \int_0^{2\pi} d\phi \int_0^{R} \frac{r\,dr}{\sqrt{r^2+z^2}}$$

$$V = k\rho_s (2\pi)\left[\sqrt{R^2+z^2} - z\right]$$

$$V = 2\pi k\rho_s\left[\sqrt{R^2+z^2} - z\right]$$

Plot the Potential

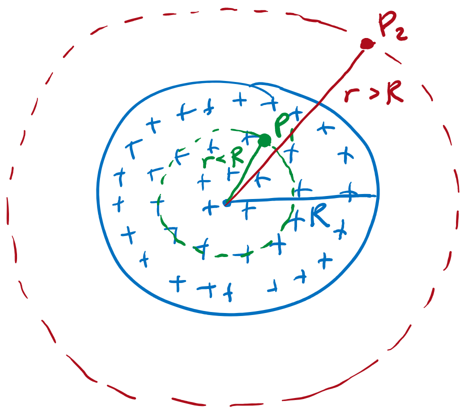



V does not go to infinity like it does with a point charge!

### 6.4.3 Examples

**Example #3. A charge Q is uniformly distributed over the volume of a sphere of radius R. Calculate the electric potential as a function of r from 0 to infinity.**

Diagram




Pick Method: We can use Gauss Law to find the $\vec{E}$-field by choosing a spherical Gaussian surface ... this means we use Method 1

→ We must remember there are 2 regions we must analyze: $r<R$, $r>R$

$r>R$ $(R \le r \le \infty)$

$$\vec{E}_2(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$V_2(r) = -\int_\infty^r E\,dr = -\frac{Q}{4\pi\varepsilon_0}\int_\infty^r \frac{1}{r^2}\,dr$$

$$V_2(r) = \frac{Q}{4\pi\varepsilon_0 r}$$

$\vec{E}$-field behaves like a point charge, so should $V$

$r<R$ $(0 \le r \le R)$

We know from before that the

$$\vec{E}_1(r) = \frac{\rho_v r}{3\varepsilon_0}\hat{r} \quad \text{where } \rho_v = \frac{Q}{\frac{4}{3}\pi R^3}$$

$$V_1(r<R) = -\int_\infty^{P_1} E\,dr$$

$$V_1(r<R) = -\int_\infty^R E_2\,dr + \left(-\int_R^r E_1\,dr\right)$$

$\vec{E}$-field changes with $r$, so we take an alternative approach and simplify our integral by splitting it up (outside → in)

$$V_1(r<R) = V_2(R) - \int_R^r \frac{\rho_v r}{3\varepsilon_0}\,dr$$

$$V_1(r<R) = \frac{kQ}{R} + \frac{\rho_v}{6\varepsilon_0}[R^2 - r^2]$$

$$V_1(r<R) = \frac{kQ}{R} + \frac{Q[R^2 - r^2]}{8\pi R^3\varepsilon_0}$$

## 6.5 Electric Potential and Electrostatic Potential Energy

Are electric potential ($V_p$) and electrostatic potential energy ($\mathcal{U}$), the same? NO!!

Analogy: → gravitational potential at a point in the gravitational field

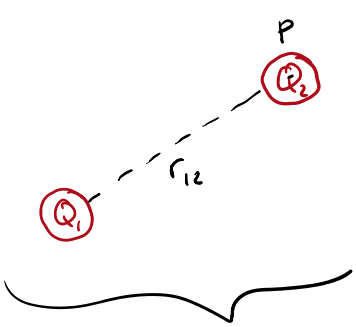
↳ this is the gravitational potential energy of a unit mass placed at that point

Similarly, the electric potential at any point in the $\vec{E}$-field is the electric potential energy of a unit positive charge at that point

→ electric potential is the amount of work done to bring $q$ from $\infty \rightarrow P$

→ electric potential energy is the energy needed to move a charge against the electric field

### Lets Take a Closer Look:




Once assembled, the total stored energy of the system is:

$$\mathcal{U}_{TOT} = \mathcal{U}_1 + \mathcal{U}_2$$

$$\mathcal{U}_{TOT} = 0 + \Delta\mathcal{U}_e \quad \text{(initial energy of } Q_1 \text{ is } 0)$$

$$\mathcal{U}_{TOT} = qV_P$$

→ $Q_1$ alone has no force acting on it, so no stored energy

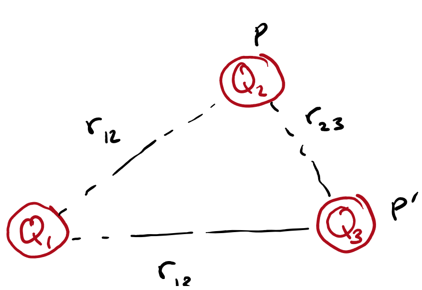
→ $Q_2$ at P will have electric potential of $\frac{kQ_1}{r_{12}}$

(work I have to do to bring +1C charge to P)

→ This means the work to bring $Q_2$ to P is given by $Q_2V_P$, stored as potential energy

$\therefore \Delta\mathcal{U}_e = qV_P$ where $q$ = charge we are bringing in
$V_P$ = V at that point P

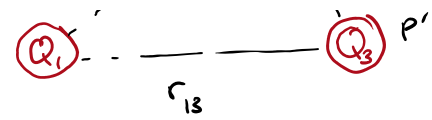
### What happens if we bring in a third charge?




→ If we add a 3rd charge, we know the energy that's already stored in the system is:

$$\mathcal{U}_i = \frac{kQ_1Q_2}{r_{12}} \Rightarrow \mathcal{U}_e = qV_P, \quad V_P = \frac{kQ_1}{r_{12}}$$

→ the electric potential at P' is:




→ the electric potential at $P'$ is:

$$V_{P'} = \frac{kQ_1}{r_{13}} + \frac{kQ_2}{r_{23}}$$

→ the change in potential energy is:

$$\Delta U_e = Q_3 V_{P'} = \frac{kQ_1Q_3}{r_{13}} + \frac{kQ_2Q_3}{r_{23}}$$

→ the total energy stored in the system will be:

$$U_f = U_i + \Delta U_e = \frac{kQ_1Q_2}{r_{12}} + \frac{kQ_1Q_3}{r_{13}} + \frac{kQ_2Q_3}{r_{23}}$$

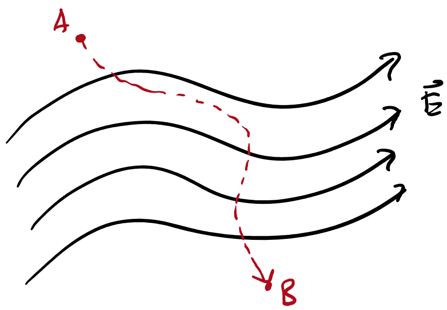
For a <u>general case</u>:

$$\boxed{U_{TOT} = k \sum_{i=1}^{N} \sum_{j>i}^{N} \frac{Q_iQ_j}{r_{ij}}}$$

### 6.6 Electric Potential Difference Between Points

In many cases, we are interested in finding the potential difference between 2 points

So how can we calculate this?



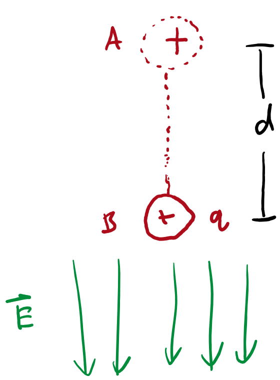

**Calculating Potential Difference**

$$\Delta V_{A\to B} = -\int_A^B \vec{E}\cdot\vec{dl}$$

$$\Delta V_{A\to B} = -\left(\int_\infty^B \vec{E}\cdot\vec{dl} - \int_\infty^A \vec{E}\cdot\vec{dl}\right)$$

$$\Delta V_{A\to B} = -\int_\infty^B \vec{E}\cdot\vec{dl} + \int_\infty^A \vec{E}\cdot\vec{dl} = \underline{\underline{V_B - V_A}}$$

**What About Uniform $\vec{E}$-Fields?**




$$\Delta V_{A\to B} = V_B - V_A = -\int_A^B \vec{E}\cdot\vec{dl} = -E\underbrace{\int_A^B dl}_{\text{total distance}}$$

$$\boxed{\Delta V_{A\to B} = -Ed}$$

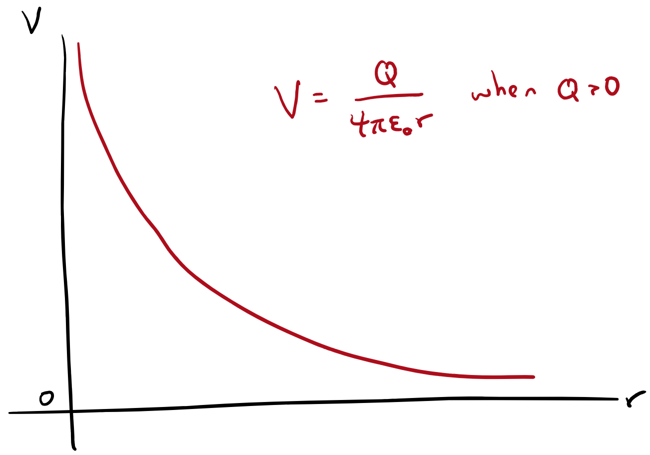
Or in general:

$$\Delta V_{A\to B} = V_B - V_A = \frac{kQ}{r_B} - \frac{kQ}{r_A}$$

$$\boxed{\Delta V_{A\to B} = kQ\left(\frac{1}{r_B} - \frac{1}{r_A}\right)}$$

### 6.7 Visualizing Electric Potential

How can we conceptualize electric potential?




When we place a positive test charge in the system ...

→ $+q$ will fall down the hill because it is repelled

When we place a negative test charge in the system...

→ $-q$ will ride up the hill since it is attracted to Q

* The greater the force, the steeper the hill *

## 6.8 Electric Field from Electric Potential

We know that $dV = \underbrace{-\vec{E}\cdot\vec{dl}}$

$\hookrightarrow$ In Cartesian coordinates, we can write this as follows:

$$\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} \quad ; \quad \vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$= \langle E_x, E_y, E_z \rangle \qquad\qquad = \langle dx, dy, dz \rangle$$

$$\underbrace{dV = -(\underline{E_x}dx + \underline{E_y}dy + \underline{E_z}dz)}$$

$$\underline{E_x} = \frac{-\partial V}{\partial x} \qquad \underline{E_y} = \frac{-\partial V}{\partial y} \qquad \underline{E_z} = \frac{-\partial V}{\partial z}$$

But the gradient or del operator is given as:

$$\nabla = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)$$

We can simplify and condense the equations above:

$$E = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right)$$

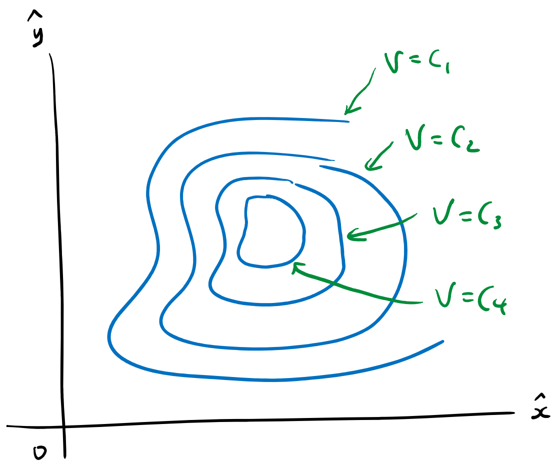
$$\boxed{E = -\nabla V}$$

## 6.9 Equipotential Surfaces

Equipotential Surface: a curve characterized by a constant potential V

We can look at contour maps!

↳ We can use $V(x,y)$ in 2D:




Since $\Delta V_{A\to B} = -\int_A^B \vec{E}\cdot d\vec{l}$, the only way $\Delta V_{A\to B} = 0$ is if $\vec{E}$ is $\perp$ to $d\vec{l}$ so $\cos 90° = 0$

$\therefore$ $\vec{E} \perp d\vec{l}$ for equipotential surfaces!

* no work is done to move the charges along these equipotentials *