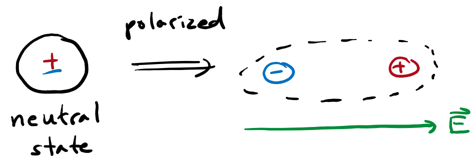
8.1 Introduction

Insulator: e^- are tightly bound to the atom

$\to$ negligible macroscopic movements in external applied $\vec{E}$-field

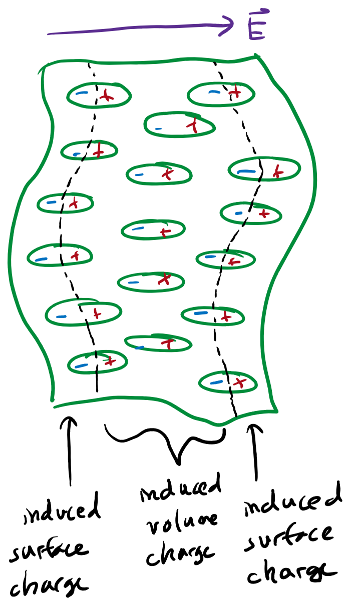
Dielectric: insulating material that can be polarized when an external $\vec{E}$-field is applied



bound e^- are displaced over subatomic/microscopic distances

8.2 Polarization Vector

Let's look at a polarized dielectric:



We use a polarization vector $\vec{P}$ to characterize the polarized states of a dielectric material

How? → construct small volume Δv around a point in the dielectric
→ add vectorially all the dipole moments $\vec{p}_i$ within Δv

$$\boxed{\vec{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=0}^{n\Delta v} \vec{p}_i \quad ; \quad \vec{p}_i = q\vec{d}}$$

Note:
→ n is # of atoms per unit volume
→ $\vec{d}$ is the direction vector from $-q$ to $+q$ within a dipole

What are the bound charge densities?

Surface Charge

$$\rho_s = \vec{P} \cdot \hat{n}$$

⇓

we just need to know the strength of $\vec{P}$

Volume Charge

$$\rho_v = -\nabla \cdot \vec{P}$$

⇓

spatial rate of change of $\vec{P}$ determines the volume-bound induced charge density

Aside: Divergence of a vector field at a point is the net outward flux of $\vec{P}$ per unit volume as the point tends to zero

$$\text{div } \vec{P} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{P} \cdot d\vec{s}}{\Delta v}$$

$$\text{div } \vec{P} = \vec{\nabla} \cdot \vec{P}$$

$$= \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$$

8.3a Electric Flux Density and Dielectric Constant

Let's update our expression for Gauss Law when looking at dielectrics.

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

We can rewrite as point-form; thinking about divergence of $\vec{E}$

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

But $\rho = \rho_{free} + \rho_v$

$$\nabla\cdot\vec{E} = \frac{\rho_f + \rho_v}{\varepsilon_0}$$

But $\rho_v = -\nabla\cdot\vec{P}$

$$\nabla\cdot\vec{E} = \frac{\rho_f - \nabla\cdot\vec{P}}{\varepsilon_0}$$

Move ε_0

$$\nabla\cdot\varepsilon_0\vec{E} = \rho_f - \nabla\cdot\vec{P}$$

Solve for ρ_f

$$\nabla\cdot\varepsilon_0\vec{E} + \nabla\cdot\vec{P} = \rho_f$$

$$\nabla\cdot(\varepsilon_0\vec{E} + \vec{P}) = \rho_f$$

We call $(\varepsilon_0\vec{E} + \vec{P})$ the electric flux density or electric displacement or displacement vector $\vec{D}$

$$\nabla\cdot\vec{D} = \rho_f$$

$$\oint_S \vec{D}\cdot d\vec{S} = Q_{encl}^{free}$$

$$\boxed{\vec{D} = \varepsilon_0\vec{E} + \vec{P}}$$

Integral Form (Gauss Law)

$$\boxed{\oint \vec{D}\cdot d\vec{S} = Q_{encl}^{free}}$$

Point Form (Gauss Law)

$$\boxed{\nabla\cdot\vec{D} = \rho_{free}}$$

$\rho_{free} = \rho_f \Rightarrow$ free charge density

$Q_{encl}^{free} \Rightarrow$ free charge enclosed within surface S

These are the 1st of Maxwell's Equations!

Linear Dielectrics:

Many dielectrics are highly linear → assume $\vec{P}$ is proportional to $\vec{E}$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

where ε_0 = permittivity of free space

χ_e = electric susceptibility

Then $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$

$$\vec{D} = \varepsilon_0\vec{E} + \varepsilon_0\chi_e\vec{E}$$

$$\vec{D} = \varepsilon_0\underbrace{(1+\chi_e)}_{\varepsilon_r}\vec{E}$$

measures how strongly a material polarizes with respect to free space

ε_r → relative permittivity / dielectric constant

$\vec{D} = \epsilon_0\underbrace{(1+\chi_e)}\vec{E}$

$\epsilon_r \rightarrow$ relative permittivity / dielectric constant

$\boxed{\vec{D} = \epsilon_0 \epsilon_r \vec{E}}$ If we define $\epsilon = \epsilon_0 \epsilon_r$ as the absolute permittivity, we get

$$\boxed{\vec{D} = \epsilon \vec{E}}$$

How does this help?

Once we have $\vec{D}$ and $\vec{E}$, we can easily find $\vec{P}$:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \Rightarrow \vec{P} = \vec{D} - \epsilon_0 \vec{E} \quad \text{but } \vec{D} = \epsilon_0 \epsilon_r \vec{E}$$

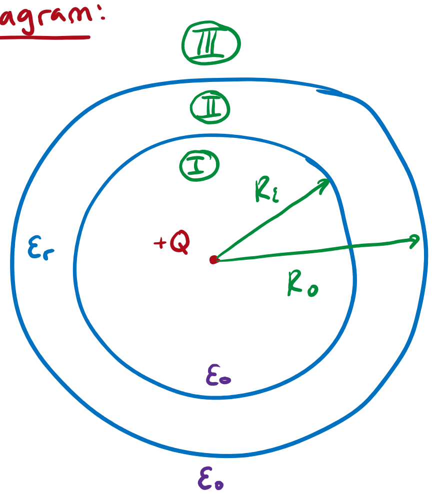
$$\vec{P} = \epsilon_0 \epsilon_r \vec{E} - \epsilon_0 \vec{E}$$

$$\boxed{\vec{P} = \epsilon_0 (\epsilon_r - 1) \vec{E} = (\epsilon - \epsilon_0) \vec{E}}$$

8.3b Example 1

A point charge is surrounded by a dielectric spherical shell of ε_r with an inner radius R_i and outer radius R_o. Calculate $\vec{D},\ \vec{E},\ \vec{P}$, and V as a function of r.

Diagram:



Determine Regions: I: $r < R_i$ II: $R_i < r < R_o$ III: $r > R_o$

Free Charges: Q at the origin, so if we were to calculate $\vec{D}$, the problem is that of a point charge → apply Gauss Law!

Analyze Regions: Due to spherical symmetry, we expect $\vec{E}$ and $\vec{D}$ to be radial

→ note that $\varepsilon = \varepsilon_o$ for air

I $r < R_i$

$$\oint \vec{D}\cdot d\vec{S} = Q_f = Q$$

$$D_I \oint dS = Q$$

$$D_I (4\pi r^2) = Q$$

$$\vec{D}_I = \frac{Q}{4\pi r^2}\hat{r}$$

But $\vec{D}_I = \varepsilon \vec{E}_I$

$$\vec{E}_I = \frac{\vec{D}_I}{\varepsilon}\hat{r}$$

$$\vec{E}_I = \frac{Q}{4\pi\varepsilon_o r^2}\hat{r}$$

And $\vec{P}_I = (\varepsilon - \varepsilon_o)\vec{E}_I$ but $\varepsilon = \varepsilon_o$

$$\vec{P}_I = \vec{0}$$

II $R_i < r < R_o$

$$\oint \vec{D}\cdot d\vec{S} = Q_f$$

$$D_{II}(4\pi r^2) = Q$$

$$\vec{D}_{II} = \frac{Q}{4\pi r^2}\hat{r}$$

But $\vec{D}_{II} = \varepsilon \vec{E}_{II}$

$$\vec{D}_{II} = \varepsilon_r\varepsilon_o \vec{E}_{II}$$

$$\vec{E}_{II} = \frac{Q}{4\pi\varepsilon_o\varepsilon_r r^2}\hat{r}$$

And $\vec{P}_{II} = (\varepsilon - \varepsilon_o)\vec{E}_{II}$ but $\varepsilon = \varepsilon_o\varepsilon_r$

$$\vec{P}_{II} = \frac{\varepsilon_o(\varepsilon_r - 1)Q}{4\pi\varepsilon_o\varepsilon_r r^2}\hat{r}$$

$$\vec{P}_{II} = \frac{(\varepsilon_r - 1)Q}{4\pi\varepsilon_r r^2}\hat{r}$$

III $r > R_o$

$$\oint \vec{D}\cdot d\vec{S} = Q_f$$

$$D_{III}(4\pi r^2) = Q$$

$$\vec{D}_{III} = \frac{Q}{4\pi r^2}\hat{r}$$

But $\vec{D}_{III} = \varepsilon \vec{E}_{III}$

$$\vec{E}_{III} = \frac{Q}{4\pi\varepsilon_o r^2}\hat{r}$$

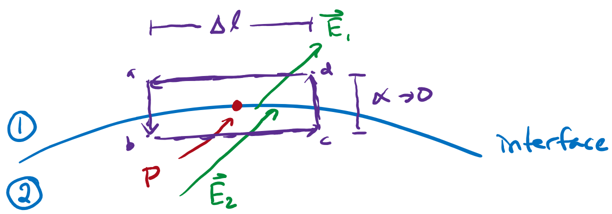
And $\vec{P}_{III} = (\varepsilon - \varepsilon_o)\vec{E}_{III}$ but $\varepsilon = \varepsilon_o(1)$

$$\vec{P}_{III} = \vec{0}$$

8.4a Boundary Conditions

Let's consider problems involving two or more media.

We take a look at point P lying on the interface / boundary between two arbitrary media 1 and 2.



Tangential Boundary Condition:

→ we have a line integral for closed path abcda

$$\oint \vec{E} \cdot \vec{d\ell}$$

↳ $\Delta \ell$ parallel to horizontal boundary when it is very small

→ Then looking at $\vec{E}_1$ and $\vec{E}_2$, we have:

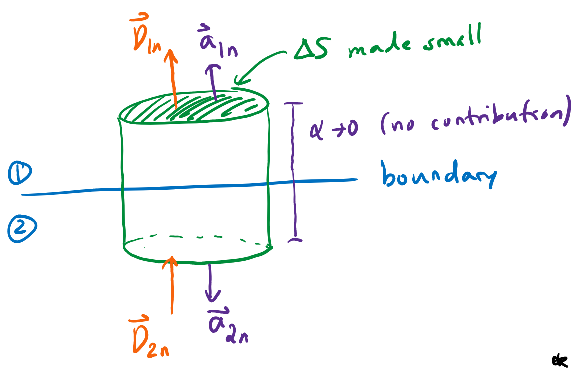
$$\oint \vec{E} \cdot \vec{d\ell} = (E_{1t} - E_{2t})\Delta \ell = 0$$

$$\boxed{E_{1t} = E_{2t}}$$

The tangential component of $\vec{E}$ is continuous across the boundary

Normal Boundary Condition:

→ Apply Gauss Law across the boundary:



$$\oint \vec{D} \cdot \vec{dS} = (D_{1n} - D_{2n})\Delta S = Q_f = \rho_s \Delta S$$

$$\rho_s = D_{1n} - D_{2n}$$

* In a dielectric interface, if we have **no** free charges on the boundary, then $\rho_s^{free} = 0$

Then $\boxed{D_{1n} = D_{2n}}$

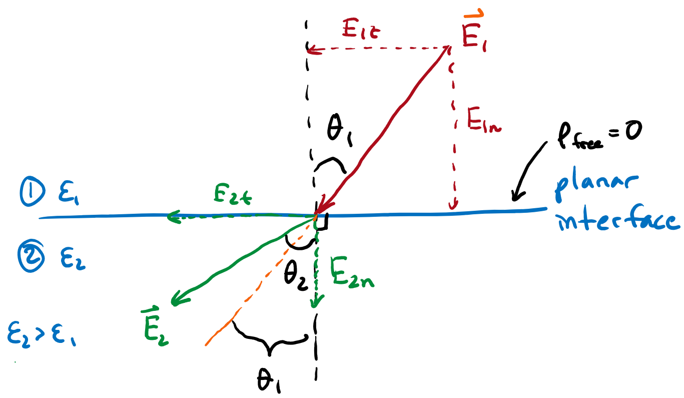
$\vec{D}$ is continuous at a dielectric-dielectric interface because there is no free charge ($\rho_{free} = 0$)

8.4b Example 2

We have a large planar boundary between two different dielectrics with absolute permittivity of ϵ_1 and ϵ_2 where $\epsilon_2 > \epsilon_1$. Electric field is incident in Dielectric 1 with amplitude E_1 and angle of θ_1 with the normal. There is no surface charge on the boundary. Answer the following questions:

a. In Dielectric 2, is angle θ_2 larger or smaller than θ_1?
b. Calculate the value of E_2 and θ_2

Diagram:



(a) θ_2 larger/smaller than θ_1?

→ No free charge @ interface so:

$$D_{1n} = D_{2n}$$

→ $\vec{D} = \varepsilon \vec{E}$ so:

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

$$E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} \quad \text{but } \varepsilon_2 > \varepsilon_1$$

$$\therefore E_{2n} < E_{1n}$$

→ At boundary, $E_{1t} = E_{2t}$

→ From diagram, $\theta_2 > \theta_1$

(b) Find E_2 and θ_2

$$\Rightarrow \left.\begin{aligned} E_{1t} &= E_1 \sin\theta_1 \\ E_{2t} &= E_2 \sin\theta_2 \end{aligned}\right\} \quad \begin{aligned} E_{1t} &= E_{2t} \\ E_1 \sin\theta_1 &= E_2 \sin\theta_2 \quad (1) \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} E_{1n} &= E_1 \cos\theta_1 \\ E_{2n} &= E_2 \cos\theta_2 \end{aligned}\right\} \quad E_2 \cos\theta_2 = \frac{\varepsilon_1}{\varepsilon_2} E_1 \cos\theta_1 \quad (2)$$

⇒ Divide (1) by (2):

$$\frac{E_1 \sin\theta_1 = E_2 \sin\theta_2}{\frac{\varepsilon_1}{\varepsilon_2} E_1 \cos\theta_1 = E_2 \cos\theta_2}$$

$$\frac{\varepsilon_2}{\varepsilon_1} \tan\theta_1 = \tan\theta_2$$

$$\theta_2 = \arctan\left(\frac{\varepsilon_2}{\varepsilon_1} \tan\theta_1\right)$$

⇒ Square (1) and (2):

$$\left.\begin{aligned} E_1^2 \sin^2\theta_1 &= E_2^2 \sin^2\theta_2 \\ \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 E_1^2 \cos^2\theta_1 &= E_2^2 \cos^2\theta_2 \end{aligned}\right\} \text{add}$$

$$E_1^2\left(\sin^2\theta_1 + \left[\frac{\varepsilon_1}{\varepsilon_2}\right]^2 \cos^2\theta_1\right) = E_2^2 \left(\underbrace{\sin^2\theta_2 + \cos^2\theta_2}_{1}\right)$$

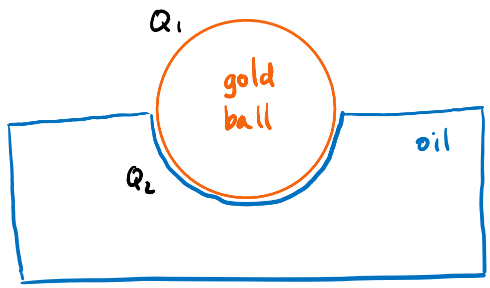
$$E_2 = E_1 \sqrt{\sin^2\theta_1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 \cos^2\theta_1}$$

8.4c Example 3

A solid gold ball is floated on an oil bath so that half the ball is above the oil and the other half is immersed in oil. The oil has a relative permittivity of ε_r. Charge Q is released on the ball.

a. Calculate the electric field in space
b. What is the charge density on the ball for the part above the oil and in the oil?
c. What is the charge density in the oil on the boundary with the conductor?

Diagram:

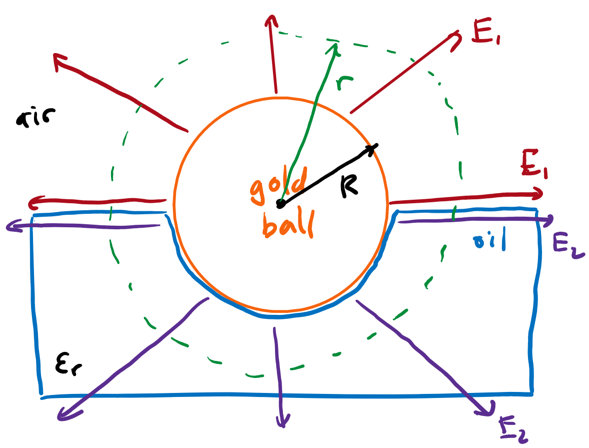


Analyze Charge Distribution:

Because oil can be polarized, the top and bottom half of the gold ball will have different charge:

$$Q = Q_1 + Q_2$$

(a) Electric Field in Space



At the boundary: E_1 is tangential to E_2

$$E_{1t} = E_{2t} \quad \text{but } E_{1t} = E_1 \;;\; E_{2t} = E_2$$

$$E_1 = E_2 = E$$

And the electric flux density is:

$$D_1 = \varepsilon_0 E_1 = \varepsilon_0 E \quad ① \qquad D_2 = \varepsilon_0 \varepsilon_r E_2 = \varepsilon_0 \varepsilon_r E \quad ②$$

Apply Gauss Law:

$$\underbrace{\oint \vec{D} \cdot d\vec{s}}_{\text{must split into air/water regions}} = Q_f \quad \text{with} \quad Q_f = Q$$

$$\int_{air} D_1 \, ds + \int_{oil} D_2 \, ds = Q$$

sub in ① and ② →

$$D_1 (2\pi r^2) + D_2 (2\pi r^2) = Q$$

$$\varepsilon_0 E (2\pi r^2) + \varepsilon_0 \varepsilon_r E (2\pi r^2) = Q$$

$$E \varepsilon_0 (2\pi r^2)(1 + \varepsilon_r) = Q$$

$$\vec{E} = \frac{Q}{2\pi \varepsilon_0 (1 + \varepsilon_r) r^2} \hat{r} \quad \leftarrow \text{by inspection}$$

(b) Charge Density on Ball

$\vec{E}$ on a conductor boundary is associated with charge density on the conductor

Gold Ball in Air

$$E_1(r=R) = \frac{\rho_1}{\varepsilon_0}$$

$$\rho_1 = \varepsilon_0 \frac{Q}{2\pi\varepsilon_0(1+\varepsilon_r)R^2}$$

$$\underline{\rho_1 = \frac{Q}{2\pi(1+\varepsilon_r)R^2}}$$

Gold Ball in Oil

$$E_2(r=R) = \frac{\rho_2}{\varepsilon_0\varepsilon_r}$$

$$\rho_2 = \frac{Q\varepsilon_0\varepsilon_r}{2\pi\varepsilon_0(1+\varepsilon_r)R^2}$$

$$\underline{\rho_2 = \frac{Q\varepsilon_r}{2\pi(1+\varepsilon_r)R^2}}$$

(c) Charge Density in Oil @ Boundary

→ oil is a dielectric so we need to find $\vec{P}$ first

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

$$\vec{P} = \vec{D} - \varepsilon_0\vec{E}$$

$$\vec{P} = \varepsilon_0\varepsilon_r\vec{E} - \varepsilon_0\vec{E} = \varepsilon_0(\varepsilon_r - 1)\vec{E}$$

$$\vec{P} = \frac{\varepsilon_0(\varepsilon_r-1)Q}{2\pi\varepsilon_0(1+\varepsilon_r)r^2} \rightarrow \vec{P} = \frac{Q(\varepsilon_r-1)}{2\pi(\varepsilon_r+1)r^2}$$

→ we can then find <u>surface</u> charge density:

$$\rho_s = \vec{P}\cdot\hat{n} \quad \text{when } r=R$$

$$\rho_s = \frac{Q(\varepsilon_r-1)}{2\pi(\varepsilon_r+1)R^2}$$

but if $+Q$ is on gold ball, it must attract $-Q$ in oil

$$\underline{\rho_s = \frac{-Q(\varepsilon_r-1)}{2\pi(\varepsilon_r+1)R^2}}$$